CONVERGENCE OF THE POINT INTEGRAL METHOD FOR THE POISSON EQUATION ON MANIFOLDS II: THE DIRICHLET BOUNDARY

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Abstract. The Poisson equation on manifolds plays an fundamental role in many applications. Recently, we proposed a novel numerical method called the Point Integral method (PIM) to solve the Poisson equations on manifolds from point clouds. In this paper, we prove the convergence of the point integral method for solving the Poisson equation with the Dirichlet boundary condition.

1. Introduction. This is a continuation of [6] in which the convergence of the PIM for solving the Poisson equation with the Neumann boundary is proven. The purpose of this paper is to prove the convergence of the PIM for the Dirichlet boundary. In particular, we consider the following Dirichlet problem for the Poisson equation on a smooth, compact $k$-dimensional submanifold $M$ in $\mathbb{R}^d$.

\begin{align}
\begin{cases}
-\Delta_M u(x) = f(x), & x \in M \\
u(x) = b(x), & x \in \partial M
\end{cases}
\end{align}

where $\Delta_M$ is the Laplace-Beltrami operator on $M$. Let $g$ be the Riemannian metric tensor of $M$, which is assumed to be inherited from the ambient space $\mathbb{R}^d$, that is, $M$ isometrically embedded in $\mathbb{R}^d$ with the standard Euclidean metric. If $M$ is an open set in $\mathbb{R}^d$, then $\Delta_M$ becomes standard Laplace operator, i.e., $\Delta_M = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$.

In [5] and [6], we have introduced the integral Laplace operator

\begin{align}
L_t u(x) = \frac{1}{t} \int_M R_t(x, y)(u(x) - u(y))d\mu_y,
\end{align}

and shown that the Poisson equation is well-approximated by the following integral equation

\begin{align}
- L_t u(x) + 2 \int_{\partial M} \bar{R}_t(x, y) \frac{\partial u}{\partial n}(y)d\tau_y = \int_M \bar{R}_t(x, y)f(y)d\mu_y,
\end{align}

where $n$ is the out normal of $M$, $R_t(x, y)$ and $\bar{R}_t(x, y)$ are kernel functions given as

\begin{align}
R_t(x, y) = C_t R \left( \frac{|x - y|^2}{4t} \right), \quad \bar{R}_t(x, y) = C_t \bar{R} \left( \frac{|x - y|^2}{4t} \right)
\end{align}

where $C_t = \frac{1}{(4\pi t)^{\frac{d+1}{2}}}$ is the normalizing factor. $R \in C^2(\mathbb{R}^+)$ be a positive function which is integrable over $[0, +\infty)$. In this integral equation, the Neumann boundary is natural. To enforce the Dirichlet boundary, we use the Robin boundary to bridge the Neumann boundary and the Dirichlet boundary. In particular, we consider the following Robin problem

\begin{align}
\begin{cases}
-\Delta_M u(x) = f(x), & x \in M \\
u(x) + \beta \frac{\partial u}{\partial n} = b(x), & x \in \partial M,
\end{cases}
\end{align}

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which on one hand approximates the Dirichlet problem (D1.a) when the parameter $\beta$ is small, and at the same time can be approximated by the following integral equation

\[(D2.a) \quad -L_t u(x) + \frac{2}{\beta} \int_{\partial M} \bar{R}_t(x, y) \left(b(y) - u(y)\right) d\tau_y = \int_M \bar{R}_t(x, y) f(y) d\mu_y \]

when the parameter $t$ is small. This integral equation is obtained from the equation (1.2) by substituting $\frac{\partial u}{\partial n}$ with $((b(y) - u(y))/\beta$.

In the PIM, we only need a point cloud sampling the underlying manifold $\mathcal{M}$ and its boundary $\partial \mathcal{M}$. Assume a set of sample points $P$ samples the submanifold $\mathcal{M}$ and a subset $S \subset P$ samples the boundary of $\mathcal{M}$. Let the points in $P$ respectively $S$ in a fixed order $P = (p_1, \cdots, p_n)$ where $p_i \in \mathbb{R}^d$, $1 \leq i \leq n$, respectively $S = (s_1, \cdots, s_m)$ where $s_i \in P$. In addition, assume we are given two vectors $V = (V_1, \cdots, V_n)^t$ where $V_i$ is a volume weight of $p_i$ in $\mathcal{M}$, and $A = (A_1, \cdots, A_m)^t$ where $A_i$ is an area weight of $s_i$ in $\partial \mathcal{M}$, so that for any Lipschitz function $f$ on $\mathcal{M}$ respectively $\partial \mathcal{M}$, $\int_{\mathcal{M}} f(x) d\mu_x$ respectively $\int_{\partial \mathcal{M}} f(x) d\tau_x$ can be approximated by $\sum_{i=1}^n f(p_i) V_i$ respectively $\sum_{i=1}^m f(s_i) A_i$. Here $d\mu_x$ and $d\tau_x$ are the volume form of $\mathcal{M}$ and $\partial \mathcal{M}$, respectively.

The integral equation (D2.a) is easy to be discretized over the point cloud $(P, S, V, A)$ to obtain the following linear system of $u = (u_1, \cdots, u_n)^t$. Let

\[(1.4) \quad L\mathbf{u}(p_i) = \frac{1}{t} \sum_{p_j \in P} \bar{R}_t(p_i, p_j)(u_i - u_j)V_j \]

be the discrete Laplace operator, which becomes the weighted graph Laplacian if set $V_j = \frac{1}{n}$ (e.g. [1]). We have for any $1 \leq i \leq n$

\[(D3.a) \quad -L\mathbf{u}(p_i) + \frac{2}{\beta} \sum_{s_j \in S} \bar{R}_t(p_i, s_j)(b(s_j) - u(s_j))A_j = \sum_{p_j \in P} \bar{R}_t(p_i, p_j)f(p_j)V_j. \]

We use the solution $u = (u_1, \cdots, u_n)^t$ of the above linear system to approximate the solution to the problem (D1.a).

The purpose of this paper is to show the convergence of the above numerical method in $H^1$ norm. Following the standard strategy in the numerical analysis, we prove the convergence by showing the consistency and the stability of the PIM for the Dirichlet problem. Comparing to the Neumann boundary problem considered in [6], the unknown variables $u_i$ not only appear in the discrete Laplace operator, but also appear in an integral over the boundary. Therefore, instead of showing the stability for the integral Laplace operator $L_t$ as in [6], we need to consider the stability for the following intermediate operator

\[(1.5) \quad K_t u(x) = \frac{1}{t} \int_{\mathcal{M}} R_t(x, y)(u(x) - u(y)) d\mu_y + \frac{2}{\beta} \int_{\partial \mathcal{M}} \bar{R}_t(x, y) u(y) d\tau_y. \]

This is the most difficult part in this paper. The main idea is to treat the integral over the boundary as the perturbation to the integral Laplace operator $L_t$ and use the stability of $L_t$ to control the boundary term (see Theorem 3.4 and 3.5).

1.1. Related work. Much of research has been done on solving PDEs on manifolds. The interested readers are referred to [6] and the references therein for the related work. Here we emphasize one point related to the harmonic function, i.e.,
the solution to the Laplace equation $\Delta_M u = 0$. The discrete harmonicity $L u = 0$ is extensively studied in the graph theory [2], and is closed related to random walks and electric networks on graphs [3]. Comparing the equation (D1.a) to the equation (D3.a), we notice that the smooth harmonicity $\Delta_M u = 0$ may not be well approximated by the discrete harmonicity $L u = 0$ as there is an extra integral over the boundary. Du et al. [4] also noticed this phenomenon in their study of nonlocal diffusion problems where the nonlocal operator takes the same form as the integral Laplace operator $L_t$, and proposed the so-called volume constraints to enforce the Dirichlet boundary. In a subsequent paper [], we show the volume constraint approach also produces a convergent solution to the problem (D1.a).

2. Assumptions and Results. We use the same set of assumptions as in [6], which is stated below for completeness.

Assumption 2.1.

- Assumptions on the manifold: $M, \partial M$ are both compact and $C^\infty$ smooth submanifolds isometrically embedded in a Euclidean space $\mathbb{R}^d$.
- Assumptions on the sample points $(P, S, V, A)$: $(P, S, V, A)$ is $h$-integrable approximation of $M$ and $\partial M$, i.e.
  (a) For any function $f \in C^1(M)$, there is a constant $C$ independent of $h$ and $f$ so that
  \[
  \left| \int_M f(y) dy - \sum_{p_i \in M} f(p_i) V_i \right| < Ch|\text{supp}(f)||f|_{C^1(M)}.
  \]
  where $|f|_{C^1} = \|f\|_\infty + \|\nabla f\|_\infty$ and $|\text{supp}(f)|$ is the volume of the support of $f$.
  (b) For any function $f \in \partial M$, there is a constant $C$ independent of $h$ and $f$ so that
  \[
  \left| \int_{\partial M} f(y) d\tau - \sum_{i=1}^n f(s_i) A_i \right| < Ch|\text{supp}(f)||f|_{C^1}.
  \]
- Assumptions on the kernel function $R(r)$:
  (a) $R \in C^2(\mathbb{R}^+)$;
  (b) $R(r) \geq 0$ and $R(r) = 0$ for $\forall r > 1$;
  (c) $\exists \delta_0 > 0$ so that $R(r) \geq \delta_0$ for $0 \leq r \leq \frac{1}{2}$.

To simplify the notation and make the proof concise, we only consider the homogeneous boundary conditions, i.e. $b = 0$ in (D1.a). The analysis can be easily generalized to the non-homogeneous boundary conditions.

To compare the discrete numerical solution with the continuous exact solution, we interpolate the discrete solution $u = (u_1, \cdots, u_n)^t$ of the problem (D3.a) onto the smooth manifold using following interpolation operator:

\[
I_f(u)(x) = \frac{\sum_{p_j \in P} R_t(x, p_j) u_j V_j - \frac{2t}{\delta^2} \sum_{s_j \in S} \tilde{R}_t(x, s_j) u_j A_j - t \sum_{p_j \in P} \tilde{R}_t(x, p_j) f_j V_j}{\sum_{p_j \in P} R_t(x, p_j) V_j},
\]

where $f = [f_1, \cdots, f_n]^t = [f(p_1), \cdots, f(p_n)]^n$. It is easy to verify that $I_f(u)$ interpolates $u$ at the sample points $P$, i.e., $I_f(u)(p_j) = u_j$ for any $j$. In the analysis, $I_f(u)$ is used as the numerical solution of (D1.a) instead of the discrete solution $u$. 

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Now, we can state the main result.

**Theorem 2.1.** Under the assumptions in Assumption 2.1, let \( u \) is the solution to Problem \((D1.a)\) with \( f \in C^2(\mathcal{M}) \) and \( b = 0 \). Set \( f = (f(p_1), \ldots, f(p_n)) \). If the vector \( u \) is the solution to the problem \((D3.a)\) with \( b = 0 \). Then there exists constants \( C, T_0 \) and \( r_0 \) only depend on \( M \) and \( \partial M \), so that for any \( t \leq T_0 \),

\[
\|u - I_f(u)\|_{H^1(\mathcal{M})} \leq C \left( \frac{ht^{3/2}}{t^{1/2} + \beta^{1/2}} \right) \|f\|_{C(\mathcal{M})},
\]

as long as \( \frac{h}{t^{3/2}} \leq r_0 \) and \( \frac{\sqrt{t}}{\beta} \leq r_0 \).

3. **Structure of the Proof.** Since \( M \) is a \( C^\infty \) smooth submanifolds isometrically embedded in \( \mathbb{R}^d \), it can be locally parametrized as follows.

\[
x = \Phi(\gamma) : \Omega \subset \mathbb{R}^k \rightarrow \mathcal{M} \subset \mathbb{R}^d
\]

where \( \gamma = (\gamma^1, \ldots, \gamma^k)^t \in \mathbb{R}^k \) and \( x = (x^1, \ldots, x^d)^t \in M \). In what follows, we use the index with prime (e.g., \( \gamma' \)), respectively without prime (e.g., \( \gamma \)) to represent the coordinate component of \( \gamma \) in the parameter domain, respectively the coordinate component of \( x \) in the ambient space. We also use Einstein convention for the brevity of notation.

We introduce the following notations which will be used later. Let \( \partial_i = \frac{\partial}{\partial \gamma^i} \) be the tangent vector along the direction \( \gamma^i \). Since \( M \) is a submanifold in \( \mathbb{R}^d \) with induced metric, \( \partial_i = (\partial_i \Phi^1, \ldots, \partial_i \Phi^d) \) and the metric tensor

\[
g_{ij} = <\partial_i, \partial_j> = \partial_i \Phi^l \partial_j \Phi^l.
\]

Let \( g^{ij} \) denote the inverse of \( g_{ij} \), i.e.,

\[
g_{ij} g^{ij} = \delta_{ij} = \begin{cases} 
1, & i = j, \\
0, & i \neq j.
\end{cases}
\]

For any function \( f \) on \( M \), \( \nabla f = g^{ij} \partial_j f \partial_i \) denotes the gradient of \( f \). We can view the gradient \( \nabla f \) as a vector in the ambient space and let \( \nabla^j f \) denote the \( j \) component of the gradient \( \nabla f \) in the ambient coordinates, i.e.,

\[
\nabla^j f = \partial_i \Phi^j g^{ij} \partial_j f \quad \text{and} \quad \partial^j f = \partial_i \Phi^j \nabla^j f.
\]

In the rest of this section, we show the convergence of the numerical solution of the problem \((D3.a)\) to the true solution of the Dirichlet problem \((D1.a)\) as \( \beta, t, h \to 0 \). This is achieved in three steps. In the first step, we show that the true solution of the Dirichlet problem can be approximated by the true solution of the Robin problem, which is given in Theorem 3.1. The next two steps are similar to those for proving the convergence of the Neumann problem. In the second step, we show the truncation error is small by measuring the extent to which the numerical solution and the true solution of the Robin problem satisfy the integral equation \((D2.a)\) (Theorem 3.2 and Theorem 3.3). In the third step, we show that as is \( \Delta_M \), the operator \( K_t \) is stable (Theorem 3.4 and Theorem 3.5).

First, we show that the true solution of the Dirichlet problem can be approximated by the true solution of the Robin problem.
Theorem 3.1. Suppose $u$ is the solution of the Dirichlet problem (D1.a) and $u_{R,\beta}$ is the solution of the Robin problem (R1.a), then

$$\|u - u_{R,\beta}\|_{H^1(M)} \leq C\beta^{1/2}\|u\|_{H^2(M)}.$$ 

Proof. Let $w = u - u_{R,\beta}$, then $w$ satisfies

$$\begin{cases}
\Delta_M w = 0, & \text{on } M, \\
w + \beta \frac{\partial w}{\partial n} = \beta \frac{\partial u}{\partial n}, & \text{on } \partial M.
\end{cases}$$

By multiplying $w$ on both sides of the equation and integrating by parts, we can get

$$0 = \int_M w \Delta_M w d\mu_x$$

$$= -\int_M |\nabla w|^2 d\mu_x + \int_{\partial M} w \frac{\partial w}{\partial n} d\tau_x$$

$$= -\int_M |\nabla w|^2 d\mu_x - \frac{1}{\beta} \int_{\partial M} u^2 d\tau_x + \int_{\partial M} w \frac{\partial u}{\partial n} d\tau_x$$

$$\leq -\int_M |\nabla w|^2 d\mu_x - \frac{1}{2\beta} \int_{\partial M} u^2 d\tau_x + 2\beta \int_{\partial M} \left|\frac{\partial u}{\partial n}\right|^2 d\tau_x,$$

which implies that

$$\int_M |\nabla w|^2 d\mu_x + \frac{1}{2\beta} \int_{\partial M} u^2 d\tau_x \leq 2\beta \int_{\partial M} \left|\frac{\partial u}{\partial n}\right|^2 d\tau_x.$$

Moreover, we have

$$\|w\|_{L^2(M)} \leq C \left(\int_M |\nabla w|^2 d\mu_x + \frac{1}{2\beta} \int_{\partial M} u^2 d\tau_x\right) \leq C\beta \int_{\partial M} \left|\frac{\partial u}{\partial n}\right|^2 d\tau_x.$$

Combining above two inequalities and using the trace theorem, we get

$$\|u - u_{R,\beta}\|_{H^1(M)} \leq C\beta^{1/2} \left\|\frac{\partial u}{\partial n}\right\|_{L^2(\partial M)} \leq C\beta^{1/2}\|u\|_{H^2(M)}.$$ 

Now we show the true solution of the Robin problem can be approximated using the PIM. Theorem 3.4 and 3.3 show the truncation error of the PIM is bounded, Theorem 3.4 and 3.5 show the stability of the operator $K_t$.

Theorem 3.2. Assume both the submanifolds $M$ and $\partial M$ are $C^\infty$ smooth. Let $u(x)$ be the solution of the problem (R1.a) and $u_t(x)$ be the solution of the corresponding problem (D2.a). If $u \in C^4(M)$, then there exist constants $C, T_0$ depending only on $M$ and $\partial M$, so that for any $t \leq T_0$

$$\|K_t(u - u_t) - \varphi\|_{L^2(M)} \leq C t^{1/2}\|u\|_{C^2(M)},$$

$$\|K_t(u - u_t) - \varphi\|_{H^1(M)} \leq C\|u\|_{C^4(M)}.$$ 

where $\varphi(x) = \int_{\partial M} n^i(y)n^j(x, y)\nabla^i\nabla^j u(y)\tilde{R}_i(x, y) dy$ with $\alpha = \Phi^{-1}(x), \xi(x, y) = \Phi^{-1}(x) - \Phi^{-1}(y)$ and $\eta(x, y) = \xi^i(x, y)\partial_i\Phi(\alpha)$.
The proof of this theorem is exactly same as the proof of Theorem in [6], and thus is omitted here. Similar to the Neumann problem, the reason that we single out the term \( \varphi \) in the truncation error is that the operator \( K_t \) has a better stability for the truncation error like \( \varphi \) as shown in Theorem 3.5, which leads to a better convergence rate for \( u_t \).

**Theorem 3.3.** Under the assumptions in Assumption 2.1, let \( u_t(x) \) be the solution of the problem (D2.a) with \( b = 0 \) and \( u \) be the solution of the problem (D3.a) with \( b = 0 \). If \( f \in C^1(\mathcal{M}) \) in both problems, then there exists constants \( C, T_0, r_0 \) depending only on \( \mathcal{M} \) and \( \partial \mathcal{M} \) so that

\[
\left\| K_t \left( I_f^D(u) - u_t \right) \right\|_{L^2(\mathcal{M})} \leq \frac{Ch}{\sqrt{t}} \|f\|_{C^1},
\]

\[
\left\| K_t \left( I_f^D(u) - u_t \right) \right\|_{H^1(\mathcal{M})} \leq \frac{Ch}{t^{3/2}} \|f\|_{C^1}.
\]

as long as \( t \leq T_0, \frac{\sqrt{t}}{\beta} \leq r_0 \) and \( \frac{h}{\sqrt{t}r^2} \leq r_0 \).

**Theorem 3.4.** Assume \( \mathcal{M} \) and \( \partial \mathcal{M} \) are \( C^\infty \), and \( u(x) \) solves the following equation

\[-K_t u = r,\]

for any \( r \in H^1(\mathcal{M}) \). Then, there exist constants \( C, T_0, r_0 > 0 \) independent on \( t \), such that

\[\left\| u \right\|_{H^1(\mathcal{M})} \leq C \left( \left\| r \right\|_{L^2(\mathcal{M})} + \frac{t}{\sqrt{\beta}} \left\| r \right\|_{H^1(\mathcal{M})} \right),\]

as long as \( t \leq T_0 \) and \( \frac{\sqrt{t}}{\beta} \leq r_0 \).

**Theorem 3.5.** Assume \( \mathcal{M} \) and \( \partial \mathcal{M} \) are \( C^\infty \). Let \( r = \int_{\partial \mathcal{M}} b^i(y)\eta^iR_t(x,y)dy \) where \( b^i(y) \in L^\infty \). Assume \( u(x) \) solves the following equation

\[-K_t u = r.\]

Then, there exist constants \( C, T_0, r_0 > 0 \) independent on \( t \), such that

\[\left\| u \right\|_{H^1(\mathcal{M})} \leq C \sqrt{t} \max_i \left( \left\| b_i \right\|_{\infty} \right),\]

as long as \( t \leq T_0 \) and \( \frac{\sqrt{t}}{\beta} \leq r_0 \).

Theorem 2.1 follows easily from Theorem 3.1, 3.2, 3.3, 3.4 and 3.5.

**Proof of Theorem 2.1**

Set \( r = -K_t(u_{\beta} - u_{\beta,t}) + \varphi \), where \( \varphi = \int_{\partial \mathcal{M}} b^i(y)\eta^i(x,y)R_t(x,y)dy \) and \( b^i(y) = n^i(y)\nabla^i\nabla^j u(y) \). \( u_{\beta} \) solves the Robin problem (R1.a), and \( u_{\beta,t} \) solves the integral equation (D2.a).

Assume \( v \) solves the equation \( K_t v = \varphi \). By Theorem 3.2 and Theorem 3.4, we have

\[\left\| u_{\beta} - u_{\beta,t} - v \right\|_{H^1(\mathcal{M})} \leq Ct^{1/2} \left\| u_{\beta} \right\|_{C^3(\mathcal{M})} + Ct \left\| u_{\beta} \right\|_{C^4(\mathcal{M})}\]

for some constant \( C \) and any small enough \( t \). On the other hand, for any \( 1 \leq i \leq d \), we have \( \left\| b^i \right\|_{\infty} = C \left\| u_{\beta} \right\|_{C^2(\mathcal{M})} \) for some constant \( C \). By Theorem 3.5,

\[\left\| v \right\|_{H^1(\mathcal{M})} \leq C \sqrt{t} \left\| u_{\beta} \right\|_{C^2(\mathcal{M})}\]
for some constant $C$ and any small enough $t$. Thus we have
\[ \|u_\beta - u_{\beta,t}\|_{H^1(M)} \leq C t^{1/2} \|u_\beta\|_{C^2(M)} + C t^{1/2} \|u_\beta\|_{C^1(M)} + C t \|u_\beta\|_{C^4(M)} \leq C t^{1/2} \|u_\beta\|_{C^4(M)}. \]

Moreover, using Theorem 3.3 and Theorem 3.4, we obtain
\[ \|u_\beta - u_{\beta,t,h}\|_{H^1(M)} \leq C h t^{3/2} \|f\|_{C^1(M)}, \]
where $u_{\beta,t,h} = I_f(u)$ and $u = [u_1, \ldots, u_n]^t$ solves (D3.a).

Combining the above two estimates, we obtain
\[ \|u_\beta - u_{\beta,t,h}\|_{H^1(M)} \leq C t^{1/2} \|u_\beta\|_{C^4(M)} + C h t^{3/2} \|f\|_{C^1(M)}. \]

Then, using the following well known results on the regularity of the solution to the Poisson equation, we have
\[ \|u_\beta\|_{C^4(M)} \leq C \|f\|_{C^2(M)}. \]
Then the proof is completed by using Theorem 3.1. \(\square\)

4. Stability of $K_t$ (Theorem 3.4 and 3.5). In this section, we will prove Theorem 3.4 and 3.5. Both these two theorems are concerned with the stability of $K_t$, which plays an important role in our proof. We first consider the operator $L_t$ which emerges in solving the Neumann problem. We turn to consider the stability of $K_t$. Comparing with the proof of the stability of $L_t$, the stability of $K_t$ is more difficult to analyze due to the appearance of the boundary term in $K_t$.

In the proof, we need following theorem which has been proved in [].

**Theorem 4.1.** For any function $u \in L^2(M)$, there exists a constant $C > 0$ independent on $t$ and $u$, such that
\[ \langle u, L_t u \rangle_M \geq C \int_M \left| \nabla v \right|^2 d\mu_x \]
where $\langle f, g \rangle_M = \int_M f(x)g(x)d\mu_x$ for any $f, g \in L^2(M)$, and
\[ v(x) = \frac{C_t}{w_t(x)} \int_M R \left( \frac{|x-y|^2}{4t} \right) u(y)d\mu_y, \]
and $w_t(x) = C_t \int_M R \left( \frac{|x-y|^2}{4t} \right) d\mu_y.$

**Proof.** of Theorem 3.4
Using Theorem 4.1, we have
\[ \|\nabla v\|_{L^2(M)}^2 \leq C \langle u, L_t u \rangle = \int_M u(x)r(x)d\mu_x - \frac{2}{\beta} \int_M u(x) \left( \int_{\partial M} \bar{R}_t(x,y)u(y)d\tau_y \right) d\mu_x. \]
where $v$ is the same as defined in Theorem 4.1. We control the second term on the
right hand side of (4.2) as follows.

\[
\left| \int_{\mathcal{M}} u(x) \left( \int_{\partial \mathcal{M}} \left( \tilde{R}_t(x, y) - \frac{\tilde{w}_t(y)}{w_t(y)} R_t(x, y) \right) u(y) d\tau_y \right) d\mu_x \right|
= \int_{\partial \mathcal{M}} u(y) \left( \int_{\mathcal{M}} \left( \tilde{R}_t(x, y) - \frac{\tilde{w}_t(y)}{w_t(y)} R_t(x, y) \right) u(x) d\mu_x \right) d\tau_y
= \left| \int_{\partial \mathcal{M}} \frac{1}{w_t(y)} u(y) \left( \int_{\mathcal{M}} (w_t(y) \tilde{R}_t(x, y) - \tilde{w}_t(y) R_t(x, y)) u(x) d\mu_x \right) d\tau_y \right|
\leq C \|u\|_{L^2(\partial \mathcal{M})} \left( \int_{\partial \mathcal{M}} \left( \int_{\mathcal{M}} (w_t(y) \tilde{R}_t(x, y) - \tilde{w}_t(y) R_t(x, y)) u(x) d\mu_x \right)^2 d\tau_y \right)^{1/2},
\]

where \(\tilde{w}_t(x) = \int_{\mathcal{M}} \tilde{R}_t(x, y) d\mu_y\). Noticing that

\[
\int_{\mathcal{M}} (w_t(y) \tilde{R}_t(x, y) - \tilde{w}_t(y) R_t(x, y)) u(x) d\mu_x
= \int_{\mathcal{M}} \int_{\mathcal{M}} R_t(y, z) \tilde{R}_t(x, y) (u(x) - u(z)) d\mu_x d\mu_z,
\]

we have

\[
\int_{\partial \mathcal{M}} \left( \int_{\mathcal{M}} (w_t(y) \tilde{R}_t(x, y) - \tilde{w}_t(y) R_t(x, y)) u(x) d\mu_x \right)^2 d\tau_y
\leq \int_{\partial \mathcal{M}} \left( \int_{\mathcal{M}} \int_{\mathcal{M}} R_t(y, z) \tilde{R}_t(x, y) (u(x) - u(z)) d\mu_x d\mu_z \right)^2 d\tau_y
\leq \int_{\partial \mathcal{M}} \left( \int_{\mathcal{M}} \int_{\mathcal{M}} R_t(y, z) \tilde{R}_t(x, y) d\mu_x d\mu_z \right) \left( \int_{\mathcal{M}} \int_{\mathcal{M}} R_t(y, z) \tilde{R}_t(x, y) (u(x) - u(z))^2 d\mu_x d\mu_z \right) d\tau_y
\leq C \left( \int_{\mathcal{M}} \int_{\mathcal{M}} R_t(y, z) \tilde{R}_t(x, y) d\tau_y \right) \left( \int_{\mathcal{M}} \int_{\mathcal{M}} Q(x, z) (u(x) - u(z))^2 d\mu_x d\mu_z \right)
\]

where

\[
Q(x, z) = \int_{\partial \mathcal{M}} R_t(y, z) \tilde{R}_t(x, y) d\tau_y.
\]

Notice that \(Q(x, z) = 0\) if \(\|y - z\| \geq 16t\), and \(|Q(x, z)| \leq CC_t/\sqrt{t}\). We have

\[
|Q(x, z)| \leq \frac{CC_t}{\sqrt{t}} \left( \frac{\|x - z\|^2}{32t} \right).
\]
Then, we obtain the following estimate,

\begin{align*}
&\left| \int_{\mathcal{M}} \int_{\mathcal{M}} Q(x, z) (u(x) - u(z))^2 \, d\mu_x \, d\mu_z \right| \\
&\leq \frac{C}{\sqrt{t}} \left( \int_{\mathcal{M}} \int_{\mathcal{M}} c_t R \left( \frac{\|x - z\|^2}{32t} \right) (u(x) - u(z))^2 \, d\mu_x \, d\mu_z \right) \\
&\leq \frac{C}{\sqrt{t}} \left( \int_{\mathcal{M}} \int_{\mathcal{M}} c_t R \left( \frac{\|x - z\|^2}{4t} \right) (u(x) - u(z))^2 \, d\mu_x \, d\mu_z \right) \\
&\leq C \sqrt{t} \left( \int_{\mathcal{M}} u(x)r(x) \, d\mu_x \right) + \frac{1}{\beta} \left( \int_{\mathcal{M}} u(x) \left( \int_{\partial\mathcal{M}} \bar{R}_t(x, y) u(y) \, d\tau_y \right) \, d\mu_x \right) \\
&\leq C \sqrt{t} \left( \int_{\mathcal{M}} u(x) \left( \int_{\partial\mathcal{M}} \bar{R}_t(x, y) u(y) \, d\tau_y \right) \, d\mu_x \right).
\end{align*}

On the other hand,

\begin{align*}
\int_{\mathcal{M}} u(x) \left( \int_{\partial\mathcal{M}} \bar{w}_t(y) R_t(x, y) u(y) \, d\tau_y \right) \, d\mu_x \\
= \int_{\partial\mathcal{M}} \bar{w}_t(y) u(y) \left( \int_{\mathcal{M}} R_t(x, y) (u(x) - u(y)) \, d\mu_x \right) \, d\tau_y + \int_{\partial\mathcal{M}} \bar{w}_t(y) u^2(y) \, d\tau_y \\
= \int_{\partial\mathcal{M}} \bar{w}_t(y) u(y) (v(y) - u(y)) \, d\tau_y + \int_{\partial\mathcal{M}} \bar{w}_t(y) u^2(y) \, d\tau_y,
\end{align*}

where \(v\) is the same as defined in (4.1). Since \(u\) solves \(-K_t u = r(x)\), we have

\begin{equation}
\tag{4.4}
\bar{w}_t(x) u(x) = w_t(x) v(x) - \frac{2t}{\beta} \int_{\partial\mathcal{M}} R_t(x, y) u(y) \, d\tau_y - t \, r(x).
\end{equation}

Then, we obtain

\begin{align*}
\int_{\partial\mathcal{M}} \bar{w}_t(y) u(y) (v(y) - u(y)) \, d\tau_y \\
= \int_{\partial\mathcal{M}} \bar{w}_t(y) u(y) \left( \frac{2t}{\beta} \int_{\partial\mathcal{M}} R_t(x, y) u(x) \, d\tau_x + t \, r(y) \right) \, d\tau_y \\
\leq \frac{C \sqrt{t}}{\beta} \left\| u \right\|^2_{L^2(\partial\mathcal{M})} + C t \left\| u \right\|_{L^2(\partial\mathcal{M})} \left\| r \right\|_{L^2(\partial\mathcal{M})} \\
\leq \frac{C \sqrt{t}}{\beta} \left\| u \right\|^2_{L^2(\partial\mathcal{M})} + C t \left\| u \right\|_{L^2(\partial\mathcal{M})} \left\| r \right\|_{H^1(\mathcal{M})}.
\end{align*}

Combining the above estimates together, we have

\begin{align*}
\int_{\mathcal{M}} u(x) \left( \int_{\partial\mathcal{M}} \bar{R}_t(x, y) u(y) \, d\tau_y \right) \, d\mu_x \\
\geq \int_{\partial\mathcal{M}} \bar{w}_t(y) u^2(y) \, d\tau_y - \frac{C \sqrt{t}}{\beta} \left\| u \right\|^2_{L^2(\partial\mathcal{M})} - C t \left\| u \right\|_{L^2(\partial\mathcal{M})} \left\| r \right\|_{H^1(\mathcal{M})} \\
- C \sqrt{t} \left\| u \right\|_{L^2(\mathcal{M})} \left\| r \right\|_{L^2(\mathcal{M})} - \frac{C \sqrt{t}}{\beta} \left( \int_{\mathcal{M}} u(x) \left( \int_{\partial\mathcal{M}} \bar{R}_t(x, y) u(y) \, d\tau_y \right) \, d\mu_x \right).
\end{align*}
We can choose $\frac{C\sqrt{T}}{\beta}$ small enough such that $\frac{C\sqrt{T}}{\beta} \leq \min\{\frac{1}{2}, \frac{w_{\text{min}}}{6}\}$, which gives us

\[
\int_{\mathcal{M}} u(x) \left( \int_{\partial \mathcal{M}} R_t(x, y) u(y) \, d\mu_y \right) \, d\mu_x
\]

\[
\geq \frac{2}{3} \int_{\partial \mathcal{M}} w_t(y) u^2(y) \, d\mu_y - \frac{C\sqrt{T}}{\beta} \|u\|_{L^2(\partial \mathcal{M})}^2 - C t \|u\|_{L^2(\partial \mathcal{M})} \|r\|_{H^1(\mathcal{M})} - C \sqrt{T} \|r\|_{L^2(\mathcal{M})} \|r\|_{L^2(\mathcal{M})}
\]

\[
\geq \frac{w_{\text{min}}}{2} \|u\|_{L^2(\partial \mathcal{M})}^2 - C t \|u\|_{L^2(\partial \mathcal{M})} \|r\|_{H^1(\mathcal{M})} - C \sqrt{T} \|r\|_{L^2(\mathcal{M})} \|r\|_{L^2(\mathcal{M})}
\]

\[
\geq \frac{w_{\text{min}}}{4} \|u\|_{L^2(\partial \mathcal{M})}^2 - C t^2 \|r\|_{H^1(\mathcal{M})}^2 - C \sqrt{T} \|r\|_{L^2(\mathcal{M})} \|r\|_{L^2(\mathcal{M})}
\]

Substituting the above estimate to the first inequality (4.2), we obtain

\[
\|\nabla v\|_{L^2(\mathcal{M})} + \frac{w_{\text{min}}}{4\beta} \|u\|_{L^2(\partial \mathcal{M})}^2
\]

\[
\leq -C \int_{\mathcal{M}} u(x) r(x) \, d\mu_x + \frac{Cl^2}{\beta} \|r\|_{H^1(\mathcal{M})}^2 + \frac{C\sqrt{T}}{\beta} \|u\|_{L^2(\mathcal{M})} \|r\|_{L^2(\mathcal{M})}
\]

\[
\leq C \|u\|_{L^2(\partial \mathcal{M})} \|r\|_{L^2(\mathcal{M})} + \frac{Cl^2}{\beta} \|r\|_{H^1(\mathcal{M})}^2.
\]

Here we require that $\frac{C\sqrt{T}}{\beta}$ is bounded by a constant independent on $\beta$ and $t$. Now, using the representation of $u$ given in (4.4), we obtain

\[
\|\nabla u\|_{L^2(\mathcal{M})}^2 + \frac{w_{\text{min}}}{8\beta} \|u\|_{L^2(\partial \mathcal{M})}^2
\]

\[
\leq C \|\nabla v\|_{L^2(\mathcal{M})}^2 + \frac{Cl^2}{\beta^2} \left( \frac{1}{w_t(x)} \int_{\partial \mathcal{M}} R_t(x, y) u(y) \, d\mu_y \right)^2
\]

\[
+ C t^2 \left( \frac{r(x)}{w_t(x)} \right)^2 \|u\|_{L^2(\mathcal{M})}^2 + \frac{w_{\text{min}}}{8\beta} \|u\|_{L^2(\partial \mathcal{M})}^2
\]

\[
\leq C \|\nabla v\|_{L^2(\mathcal{M})}^2 + \left( \frac{C\sqrt{T}}{\beta^2} + \frac{w_{\text{min}}}{8\beta} \right) \|u\|_{L^2(\partial \mathcal{M})}^2 + C t \|r\|_{L^2(\mathcal{M})}^2 + C t^2 \|r\|_{H^1(\mathcal{M})}^2
\]

\[
\leq C \|\nabla v\|_{L^2(\mathcal{M})}^2 + \frac{w_{\text{min}}}{4\beta} \|u\|_{L^2(\partial \mathcal{M})}^2 + C t \|r\|_{L^2(\mathcal{M})}^2 + C t^2 \|r\|_{H^1(\mathcal{M})}^2
\]

\[
\leq C \|u\|_{L^2(\mathcal{M})} \|r\|_{L^2(\mathcal{M})} + C t \|r\|_{L^2(\mathcal{M})} + \frac{Cl^2}{\beta} \|r\|_{H^1(\mathcal{M})}^2.
\]

Here we require that $\frac{C\sqrt{T}}{\beta} \leq \frac{w_{\text{min}}}{8}$ in the third inequality. Furthermore, we have

\[
\|u\|_{L^2(\mathcal{M})}^2 \leq C \left( \|\nabla u\|_{L^2(\mathcal{M})}^2 + \frac{w_{\text{min}}}{8\beta} \|u\|_{L^2(\partial \mathcal{M})}^2 \right)
\]

\[
\leq C \|u\|_{L^2(\mathcal{M})} \|r\|_{L^2(\mathcal{M})} + C t \|r\|_{L^2(\mathcal{M})}^2 + \frac{Cl^2}{\beta} \|r\|_{H^1(\mathcal{M})}^2
\]

\[
\leq \frac{1}{2} \|u\|_{L^2(\mathcal{M})}^2 + C \|r\|_{L^2(\mathcal{M})}^2 + \frac{Cl^2}{\beta} \|r\|_{H^1(\mathcal{M})}^2,
\]

which implies that

\[
\|u\|_{L^2(\mathcal{M})} \leq C \left( \|r\|_{L^2(\mathcal{M})} + \frac{t}{\sqrt{\beta}} \|r\|_{H^1(\mathcal{M})} \right).
\]
Finally, we obtain
\[
\|\nabla u\|_{L^2(\Omega)}^2 \leq C \|u\|_{L^2(\Omega)} \|r\|_{L^2(\Omega)} + \frac{Ct^2}{\beta} \|r\|_{H^1(\Omega)}^2
\]
\[
\leq C \left( \|r\|_{L^2(\Omega)} + \frac{t}{\sqrt{\beta}} \|r\|_{H^1(\Omega)} \right)^2,
\]
which completes the proof. □

Proof of Theorem 3.5
In the proof of Theorem 3.4, let \(r(x) = \int_{\partial \Omega} b^i(y) \eta^j \dot{R}_i(x, y) dy\), and bound \(\int_{\Omega} u(x)r(x)dx\) by \(C \sqrt{t} \max_i (\|b^i\|_\infty) \|u\|_{H^1(\Omega)}\) instead of \(\|u\|_{L^2(\Omega)} \|r\|_{L^2(\Omega)}\) in (4.3) and (4.5), we obtain
\[
\|\nabla u\|_{L^2(\Omega)}^2 + \frac{u_{\min}}{8\beta} \|u\|_{L^2(\partial \Omega)}^2
\]
\[
\leq C \sqrt{t} \max_i (\|b^i\|_\infty) \|u\|_{H^1(\Omega)} + Ct \|r\|_{L^2(\Omega)}^2 + \frac{Ct^2}{\beta} \|r\|_{H^1(\Omega)}^2
\]
\[
\leq C \max_i (\|b^i\|_\infty) \left( \sqrt{t} \|u\|_{H^1(\Omega)} + t \right)
\]
where we use the estimates that
\[
\|r(x)\|_{L^2(\Omega)} \leq Ct^{1/4} \max_i (\|b^i\|_\infty), \quad \text{and}
\]
\[
\|r(x)\|_{H^1(\Omega)} \leq Ct^{-1/4} \max_i (\|b^i\|_\infty).
\]
Then, using the fact that
\[
\|u\|_{L^2(\Omega)}^2 \leq C \left( \|\nabla u\|_{L^2(\Omega)}^2 + \frac{u_{\min}}{8\beta} \|u\|_{L^2(\partial \Omega)}^2 \right),
\]
we have
\[
\|u\|_{H^1(\Omega)}^2 \leq C \max_i (\|b^i\|_\infty) \left( \sqrt{t} \|u\|_{H^1(\Omega)} + t \right),
\]
which completes the proof. □

5. Error analysis of the discretization (Theorem 3.3). In this section, we estimate the discretization error introduced by approximating the integrals in (D2.a), that is to prove Theorem 3.3. To simplify the notation, we introduce two intermediate operators defined as follows,

\( L_{t,h} u(x) = \sum_{p_j \in P} R_t(x, p_j)(u(x) - u(p_j))V_j \), and

\( K_{t,h} u(x) = \sum_{p_j \in P} R_t(x, p_j)(u(x) - u(p_j))V_j + \frac{2}{\beta} \sum_{s_j \in S} \bar{R}_t(x, s_j)u(s_j)A_j \).

If \(u_{t,h} = I_{t}(u)\) with \(u\) satisfying Equation (D3.a) with \(b = 0\). One can verify that the following two equations are satisfied,

\( -K_{t,h} u_{t,h}(x) = \sum_{p_j \in P} \bar{R}_t(x, p_j) f(p_j)V_j \).
The following lemma is needed for proving Theorem 3.3. Its proof is deferred to appendix.

**Lemma 5.1.** Suppose \( u = (u_1, \cdots, u_n)^t \) satisfies equation (D3.a) with \( b = 0 \), there exist constants \( C, T, r_0 \) only depend on \( \mathcal{M} \) and \( \partial \mathcal{M} \), such that

\[
(5.4) \quad \left( \sum_{i=1}^{n} u_i^2 V_i \right)^{1/2} + t^{1/4} \left( \sum_{i \in I_S} u_i^2 A_i \right)^{1/2} \leq C\|I_{\mathcal{M}}(u)\|_{H^1(\mathcal{M})} + C\sqrt{\mathcal{M}} t^{3/4}\|f\|_{\infty}
\]

as long as \( t \leq T, x_1^2 \leq r_0, y_1^2 \leq r_0 \). And \( I_S = \{1 \leq l \leq n : p_l \in S\} \).

**Proof of Theorem 3.3**

Denote

\[
u_{t,h}(x) = I_{\mathcal{M}}(u) = \frac{1}{w_{t,h}(x)} \left( \sum_{p_j \in P} R_t(x, p_j) u_j V_j - \frac{2t}{\beta} \sum_{s_j \in S} \tilde{R}_t(x, s_j) u_j A_j - t \sum_{p_j \in P} \tilde{R}_t(x, p_j) f_j V_j \right),
\]

where \( u = (u_1, \cdots, u_N)^t \) solves Equation (D3.a), \( f_j = f(p_j) \) and \( w_{t,h}(x) = \sum_{p_j \in P} R_t(x, p_j) V_j \).

For convenience, we set

\[
(5.5) \quad a_{t,h}(x) = \frac{1}{w_{t,h}(x)} \sum_{p_j \in P} R_t(x, p_j) u_j V_j, \quad \text{and}
\]

\[
(5.6) \quad c_{t,h}(x) = -\frac{t}{w_{t,h}(x)} \sum_{p_j \in P} \tilde{R}_t(x, p_j) f_j V_j, \quad \text{and}
\]

\[
(5.7) \quad d_{t,h}(x) = -\frac{2t}{\beta w_{t,h}(x)} \sum_{p_j \in P} \tilde{R}_t(x, p_j) f_j V_j.
\]

Next we upper bound the approximation error \( \|K_t(u_{t,h}) - K_{t,h}(u_{t,h})\| \). Since \( u_{t,h} = a_{t,h} + c_{t,h} + d_{t,h} \), we only need to upper bound the approximation error for \( a_{t,h}, c_{t,h} \) and \( d_{t,h} \) separately. For \( c_{t,h} \),

\[
\|K_t c_{t,h} - K_{t,h} c_{t,h}\| (x) \leq \frac{1}{t} \|c_{t,h}(x)\| \left| \int_{\mathcal{M}} R_t(x, y) d\mu_y - \sum_j R_t(x, p_j) V_j \right|
\]

\[
+ \frac{1}{t} \left| \int_{\mathcal{M}} R_t(x, y) c_{t,h}(y) d\mu_y - \sum_j R_t(x, p_j) c_{t,h}(p_j) V_j \right|
\]

\[
+ \frac{2}{\beta} \left| \int_{\partial \mathcal{M}} \tilde{R}_t(x, y) c_{t,h}(y) d\tau_y - \sum_{j \in S} \tilde{R}_t(x, x_j) c_{t,h}(x_j) A_j \right|
\]

\[
\leq \frac{Ch}{t^{3/2}} \|c_{t,h}(x)\| + \frac{Ch}{t^{3/2}} \|c_{t,h}\|_{\infty} + \frac{Ch}{t} \|\nabla c_{t,h}\|_{\infty} + \frac{Ch}{\beta} \left( t^{-1/2} |c_{t,h}|_{\infty} + t^{-1/2} \|\nabla c_{t,h}\|_{\infty} \right)
\]

\[
\leq \frac{Ch}{t^{3/2}} t \|f\|_{\infty} + \frac{Ch}{t} t^{1/2} \|f\|_{\infty} + \frac{Ch}{\beta} \|f\|_{\infty}
\]

\[
\leq \frac{Ch}{\sqrt{t}} \left( 1 + \frac{\sqrt{t}}{\beta} \right) \|f\|_{\infty}.
\]

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Now we upper bound $\|K_t a_{t,h} - K_{t,h} a_{t,h}\|_{L_2(M)}$. First, we have

\begin{equation}
(5.8) \quad \int_M (a_{t,h}(x))^2 \left| \int_M R_t(x, y) \, d\mu_y - \sum_j R_t(x, p_j) V_j \right|^2 \, d\mu_x
\end{equation}

\begin{align*}
&\leq \frac{Ch^2}{t} \int_M (a_{t,h}(x))^2 \, d\mu_x \\
&\leq \frac{Ch^2}{t} \int_M \left( \frac{1}{w_{t,h}(x)} \sum_j R_t(x, p_j) u_j V_j \right)^2 \, d\mu_x \\
&\leq \frac{Ch^2}{t} \int_M \left( \sum_j R_t(x, p_j) u_j^2 V_j \right) \left( \sum_j R_t(x, p_j) V_j \right) \, d\mu_x \\
&\leq \frac{Ch^2}{t} \left( \sum_j u_j^2 V_j \int_M R_t(x, p_j) \, d\mu_x \right) \leq \frac{Ch^2}{t} \sum_j u_j^2 V_j.
\end{align*}

Let

\begin{equation}
K_1 = C_t \int_M \frac{1}{w_{t,h}(y)} R \left( \frac{|x - y|^2}{4t} \right) R \left( \frac{|p_i - y|^2}{4t} \right) \, d\mu_y \\
- C_t \sum_j \frac{1}{w_{t,h}(p_j)} R \left( \frac{|x - p_j|^2}{4t} \right) R \left( \frac{|p_i - p_j|^2}{4t} \right) V_j.
\end{equation}

We have $|K_1| < \frac{Ch^2}{t^2}$ for some constant $C$ independent of $t$. In addition, notice that only when $|x - p_i|^2 \leq 16t$ is $K_1 \neq 0$, which implies

\begin{equation}
|K_1| \leq \frac{1}{\delta_0} |K_1| R \left( \frac{|x - p_i|^2}{32t} \right).
\end{equation}

Then we have

\begin{align*}
(5.9) \quad \int_M \left| \int_M R_t(x, y) a_{t,h}(y) \, d\mu_y - \sum_j R_t(x, p_j) a_{t,h}(p_j) V_j \right|^2 \, d\mu_x
&= \int_M \left( \sum_i C_i u_i V_i K_1 \right)^2 \, d\mu_x \\
&\leq \frac{Ch^2}{t} \int_M \left( \sum_i C_i u_i V_i R \left( \frac{|x - p_i|^2}{32t} \right) \right)^2 \, d\mu_x \\
&\leq \frac{Ch^2}{t} \int_M \left( \sum_i C_i R \left( \frac{|x - p_i|^2}{32t} \right) u_i^2 V_i \right) \left( \sum_i C_i R \left( \frac{|x - p_i|^2}{32t} \right) V_i \right) \, d\mu_x \\
&\leq \frac{Ch^2}{t} \sum_i \left( \int_M C_i R \left( \frac{|x - p_i|^2}{32t} \right) \, d\mu_x (u_i^2 V_i) \right) \leq \frac{Ch^2}{t} \left( \sum_i u_i^2 V_i \right).
\end{align*}
Let
\[
K_2 = C_t \int_{\partial M} \frac{1}{w_t,h(y)} R \left( \frac{|x - y|^2}{4t} \right) R \left( \frac{|p_i - y|^2}{4t} \right) d\tau_y - C_t \sum_j \frac{1}{w_t,h(p_j)} R \left( \frac{|x - p_j|^2}{4t} \right) R \left( \frac{|p_i - p_j|^2}{4t} \right) A_j.
\]

We have $|K_2| < \frac{C}{\delta_0}$ for some constant $C$ independent of $t$. In addition, notice that only when $|x - p_i|^2 \leq 16t$ is $K_2 \neq 0$, which implies
\[
|K_2| \leq \frac{1}{\delta_0} |K_2| R \left( \frac{|x - p_i|^2}{32t} \right).
\]

Then
\[
\int_{M} \left| \int_{\partial M} \tilde{R}_t(x, y) a_{t,h}(y) d\tau_y - \sum_j \tilde{R}_t(x, p_j) a_{t,h}(p_j) A_j \right|^2 d\mu_x
\]
\[
= \int_{M} \left( \sum_{\ell} C_t u_{\ell} V_k K_2 \right)^2 d\mu_x
\]
\[
\leq \frac{Ch^2}{t^2} \int_{M} \left( \sum_{\ell} C_t |u_{\ell}| V_k R \left( \frac{|x - p_i|^2}{32t} \right) \right)^2 d\mu_x
\]
\[
\leq \frac{Ch^2}{t^2} \int_{M} \left( \sum_{\ell} C_t R \left( \frac{|x - p_i|^2}{32t} \right) u_{\ell}^2 V_k \right) \left( \sum_{\ell} C_t R \left( \frac{|x - p_i|^2}{32t} \right) V_k \right) d\mu_x
\]
\[
\leq \frac{Ch^2}{t^2} \sum_{\ell} \left( \int_{M} C_t R \left( \frac{|x - p_i|^2}{32t} \right) d\mu_x (u_{\ell}^2 V_k) \right) \leq \frac{Ch^2}{t^2} \left( \sum_{\ell} u_{\ell}^2 V_k \right).
\]

Combining Equation (5.8), (5.9) and (5.10),
\[
\| K a_{t,h} - K_{t,h} a_{t,h} \|_{L^2(M)} \leq \frac{C}{t^{3/2}} \left( 1 + \frac{\sqrt{t}}{\beta} \right) \left( \sum_{\ell} u_{\ell}^2 V_k \right)^{1/2}
\]

Now we upper bound $\| K d_{t,h} - K_{t,h} d_{t,h} \|_{L^2}$. We have
\[
\int_{M} (d_{t,h}(x))^2 \left| \int_{M} \tilde{R}_t(x, y) d\tau_y - \sum_j \tilde{R}_t(x, p_j) V_j \right|^2 d\mu_x
\]
\[
\leq \frac{Ch^2}{t^2} \int_{M} (d_{t,h}(x))^2 d\mu_x
\]
\[
\leq \frac{Ch^2}{\beta^2} \int_{M} \left( \frac{1}{w_{t,h}(x)} \sum_j \tilde{R}_t(x, p_j) u_j A_j \right)^2 d\mu_x
\]
\[
\leq \frac{Ch^2}{\beta^2} \int_{M} \left( \sum_j \tilde{R}_t(x, p_j) u_j^2 A_j \right) \left( \sum_j \tilde{R}_t(x, p_j) A_j \right) d\mu_x
\]
\[
\leq \frac{Ch^2}{\beta^2} \left( \sum_j u_j^2 A_j \int_{M} \tilde{R}_t(x, p_j) d\mu_x \right) \leq \frac{Ch^2}{\beta^2} \sum_j u_j^2 A_j.
\]
Let

\[
K_3 = C_t \int_{\mathcal{M}} \frac{1}{w_{t,h}(y)} R \left( \frac{|x-y|^2}{4t} \right) \tilde{R} \left( \frac{|p_i-y|^2}{4t} \right) d\mu_y
- C_t \sum_j \frac{1}{w_{t,h}(p_j)} R \left( \frac{|x-p_j|^2}{4t} \right) \tilde{R} \left( \frac{|p_i-p_j|^2}{4t} \right) V_j.
\]

We have \(|K_3| < \frac{Ch}{t^2}\) for some constant \(K_3\) independent of \(t\). In addition, notice that only when \(|x-p_i|^2 \leq 16t\) is \(K_3 \neq 0\), which implies

\[
|K_3| \leq \frac{1}{\delta_0} |C|R \left( \frac{|x-p_i|^2}{4t} \right).
\]

Then we have

\[
\begin{align*}
&\int_{\mathcal{M}} \left| \int_{\mathcal{M}} R_i(x,y) dt_{i,h}(y) d\mu_y - \sum_j R_i(x,p_j) dt_{i,h}(p_j) V_j \right|^2 d\mu_x \\
&= \frac{4t^2}{\beta^2} \int_{\mathcal{M}} \left( \sum_i C_t u_i A_i K_3 \right)^2 d\mu_x \\
&\leq \frac{C_t}{\beta^2} \int_{\mathcal{M}} \left( \sum_i C_t |u_i| A_i R \left( \frac{|x-p_i|^2}{32t} \right) \right)^2 d\mu_x \\
&\leq \frac{C_t}{\beta^2} \int_{\mathcal{M}} \left( \sum_i C_t R \left( \frac{|x-p_i|^2}{32t} \right) u_i A_i \right) \left( \sum_i C_t \tilde{R} \left( \frac{|x-p_i|^2}{32t} \right) A_i \right) d\mu_x \\
&\leq \frac{C_t \sqrt{t}}{\beta^2} \sum_i \left( \int_{\mathcal{M}} C_t R \left( \frac{|x-p_i|^2}{32t} \right) d\mu_x (u_i^2 A_i) \right) \leq \frac{C t \sqrt{t}}{\beta^2} \left( \sum_i u_i^2 A_i \right).
\end{align*}
\]

Let

\[
K_4 = C_t \int_{\partial \mathcal{M}} \frac{1}{w_{t,h}(y)} \tilde{R} \left( \frac{|x-y|^2}{4t} \right) \tilde{R} \left( \frac{|p_i-y|^2}{4t} \right) d\tau_y \\
- C_t \sum_j \frac{1}{w_{t,h}(p_j)} \tilde{R} \left( \frac{|x-p_j|^2}{4t} \right) \tilde{R} \left( \frac{|p_i-p_j|^2}{4t} \right) A_j.
\]

We have \(|K_4| < \frac{C h}{t^2}\) for some constant \(C\) independent of \(t\). In addition, notice that only when \(|x-p_i|^2 \leq 16t\) is \(K_4 \neq 0\), which implies

\[
|K_4| \leq \frac{1}{\delta_0} |K_4| R \left( \frac{|x-p_i|^2}{32t} \right).
\]
$$\int_{\mathcal{M}} \left| \int_{\beta \mathcal{M}} \bar{R}_t(x, y) d_{t,h}(y) d\tau_y - \sum_j \bar{R}_t(x, p_j) d_{t,h}(p_j) A_j \right|^2 d\mu_x$$

$$= \frac{4t^2}{\beta^2} \int_{\mathcal{M}} \left( \sum_i C_i u_i A_i K_i \right)^2 d\mu_x$$

$$\leq \frac{Ch^2}{\beta^2} \int_{\mathcal{M}} \left( \sum_i C_i |u_i| A_i R \left( \frac{|x - p_i|^2}{32t} \right) \right)^2 d\mu_x$$

$$\leq \frac{Ch^2}{\beta^2} \int_{\mathcal{M}} \left( \sum_i C_i R \left( \frac{|x - p_i|^2}{32t} \right) u_i A_i \right) \left( \sum_i C_i R \left( \frac{|x - p_i|^2}{32t} \right) A_i \right) d\mu_x$$

$$\leq \frac{Ch^2}{\beta^2 \sqrt{t}} \sum_i \left( \int_{\mathcal{M}} C_i R \left( \frac{|x - p_i|^2}{32t} \right) d\mu_x \left( u_i^2 V_i \right) \right) \leq \frac{Ch^2}{\beta^2 \sqrt{t}} \left( \sum_i u_i^2 A_i \right).$$

Combining Equation (5.11), (5.12) and (5.13),

$$\| K_t d_{t,h} - K_{t,h} d_{t,h} \|_{L^2(\mathcal{M})} \leq \frac{Ch}{\beta t^{3/4}} \left( 1 + \sqrt{\frac{t}{\beta}} \right) \left( \sum_i u_i^2 A_i \right)^{1/2}.$$

Now assembling the parts together, we have the following upper bound.

$$\| K_t u_{t,h} - K_{t,h} u_{t,h} \|_{L^2(\mathcal{M})} \leq \frac{Ch}{\beta t^{3/2}} \left( \| g \|_{\infty} + t \| f \|_{\infty} + \left( \sum_i u_i^2 V_i \right)^{1/2} + t^{1/4} \left( \sum_i u_i^2 A_i \right)^{1/2} \right).$$

At the same time, since $u_t$ respectively $u_{t,h}$ solves Problem (D2.a) respectively Problem (D3.a), we have

$$\| K_t (u_t) - K_{t,h} (u_{t,h}) \|_{L^2(\mathcal{M})} \leq \frac{Ch}{t^{1/2}} \| f \|_{\infty}.$$

From Equation (5.14) and (5.15), we get

$$\| K_t u_t - L_t u_{t,h} \|_{L^2(\mathcal{M})} \leq \frac{Ch}{t^{3/2}} \left( \sum_i u_i^2 V_i \right)^{1/2} + t^{1/4} \left( \sum_i u_i^2 A_i \right)^{1/2} + t \| f \|_{\infty}. $$
Using the similar techniques, we can get the upper bound of \(\|\nabla(K_t u_t - L_t u_{t,h})\|_{L^2(M)}\) as following.

\[
\|\nabla (K_t u_t - L_t u_{t,h})\|_{L^2(M)} \leq \frac{C h}{t^{3/2}} \left( t\|f\|_{C^1(M)} + \left( \sum_{i=1}^{n} u_i^2 V_i \right)^{1/2} + t^{1/4} \left( \sum_{l \in I_S} u_l^2 A_l \right)^{1/2} \right).
\]

In the remaining of the proof, we only need to get a prior estimate of \(\left( \sum_{i} u_i^2 V_i \right)^{1/2} + t^{1/4} \left( \sum_{i} u_i^2 A_i \right)^{1/2}\). First, using the estimate (5.16) and (5.17) and the Theorem 3.4, we have

\[
\|u_{t,h}\|_{H^1(M)} \leq \frac{C h}{t^{3/2}} \left( \left( \sum_{i} u_i^2 V_i \right)^{1/2} + t^{1/4} \left( \sum_{i} u_i^2 A_i \right)^{1/2} + \|f\|_{\infty} \right) + C\|K_t u_t\|_{L^2(M)} + C t^{3/4}\|K_t u_t\|_{H^1(M)}.
\]

Using the relation that \(K_t u_t = - \int_M \bar{R}_t(x, y) f(y) \mu_y\), it is easy to get that

\[
\|K_t u_t\|_{L^2(M)} \leq C\|f\|_{\infty},
\]
\[
\|\nabla(K_t u_t)\|_{L^2(M)} \leq \frac{C}{t^{3/2}} \|f\|_{\infty}.
\]

Substituting above estimates in (5.18), we have

\[
\|u_{t,h}\|_{H^1(M)} \leq \frac{C h}{t^{3/2}} \left( \left( \sum_{i} u_i^2 V_i \right)^{1/2} + t^{1/4} \left( \sum_{i} u_i^2 A_i \right)^{1/2} + \|f\|_{\infty} \right) + C\|f\|_{\infty}.
\]

Using Lemma 5.1, we have

\[
\left( \sum_{i} u_i^2 V_i \right)^{1/2} + t^{1/4} \left( \sum_{i} u_i^2 A_i \right)^{1/2} \\
\leq C\|u_{t,h}\|_{H^1(M)} + C\sqrt{h} \left( t^{3/4}\|f\|_{\infty} + \|g\|_{\infty} \right) \\
\leq \frac{C h}{t^{3/2}} \left( t\|f\|_{\infty} + \left( \sum_{i} u_i^2 V_i \right)^{1/2} + t^{1/4} \left( \sum_{i} u_i^2 A_i \right)^{1/2} \right) \\
+ C\|f\|_{\infty} + C\sqrt{h} t^{3/4}\|f\|_{\infty}
\]

Using the assumption that \(\frac{h}{t^{3/2}}\) is small enough such that \(\frac{C h}{t^{3/2}} \leq \frac{1}{2}\), we have

\[
\left( \sum_{i} u_i^2 V_i \right)^{1/2} + t^{1/4} \left( \sum_{i} u_i^2 A_i \right)^{1/2} \leq C\|f\|_{\infty}
\]

Then the proof is complete by substituting above estimate (5.22) in (5.16) and (5.17).

\[\Box\]

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**Appendix A. Proof of Lemma 5.1.**
Proof. First, denote

\[
u_{t,h}(x) = I_t(u) = \frac{1}{w_{t,h}(x)} \left( \sum_{j=1}^{n} R_t(x, p_j) u_j V_j - \frac{2t}{\beta} \sum_{s_j \in I_S} \tilde{R}_t(x, s_j) u_j A_j - t \sum_{j=1}^{n} \tilde{R}_t(x, p_j) f_j V_j \right),
\]

where \( f_j = f(p_j) \) and \( w_{t,h}(x) = \sum_{j=1}^{n} R_t(x, p_j) V_j \) and \( u = (u_j) \) solves (D3.a) with \( b = 0 \). Let

\[
v_1(x) = \frac{1}{w_{t,h}(x)} \sum_{j=1}^{n} R_t(x, p_j) u_j V_j, \quad v_2(x) = -\frac{2t}{\beta w_{t,h}(x)} \sum_{s_j \in I_S} \tilde{R}_t(x, s_j) u_j A_j, \quad v_3(x) = -\frac{t}{\beta w_{t,h}(x)} \sum_{j=1}^{n} \tilde{R}_t(x, p_j) f_j V_j,
\]

and then \( u_{t,h} = v_1 + v_2 + v_3 \) and

\[
\left\| u_{t,h} \right\|_{L^2(M)}^2 - \sum_{j=1}^{n} u_j^2 V_j = \left| \sum_{m,n=1}^{3} \left( \int_M v_m(x) v_n(x) d\mu_x - \sum_{j=1}^{n} v_m(x_j) v_n(x_j) V_j \right) \right| \\
\leq \sum_{m,n=1}^{3} \left| \int_M v_m(x) v_n(x) d\mu_x - \sum_{j=1}^{n} v_m(x_j) v_n(x_j) V_j \right|.
\]

We now estimate these six terms in the above summation one by one. First, we consider the term with \( m = n = 1 \). Denote

\[
A = \int_M \frac{C_t}{w_{t,h}^2(x)} R \left( \frac{|x - p_i|^2}{4t} \right) R \left( \frac{|x - p_i|^2}{4t} \right) d\mu_x - \\
\sum_{j=1}^{n} \frac{C_t}{w_{t,h}^2(p_j)} R \left( \frac{|p_j - p_i|^2}{4t} \right) R \left( \frac{|p_j - p_i|^2}{4t} \right) V_j,
\]

and then \( |A| \leq \frac{Ch}{t^{1/2}} \). At the same time, notice that only when \( |p_i - p_j|^2 < 16t \) is \( A \neq 0 \). Thus we have

\[
|A| \leq \frac{1}{\delta_0} |A| R \left( \frac{|p_i - p_j|^2}{32t} \right).
\]
Assembling all the above estimates together, we obtain

\[
\left| \int_{\mathcal{M}} v_2^2(x) d\mu_x - \sum_{j=1}^{n} v_2^2(p_j)V_j \right| \leq \sum_{i,l=1}^{n} |C_i u_i u_l V_i||A| \leq \frac{Ch}{t^{1/2}} \sum_{i=1}^{n} \left( \sum_{l=1}^{n} C_i R \left( \frac{|p_i - p_l|^2}{32t} \right) u_i u_l V_i \right) \leq \frac{Ch}{t^{1/2}} \left( \sum_{i=1}^{n} \left( \sum_{l=1}^{n} C_i R \left( \frac{|p_i - p_l|^2}{32t} \right) u_i^2 V_i \right) \right)^{1/2} \left( \sum_{i=1}^{n} u_i^2 V_i \right)^{1/2} \leq \frac{Ch}{t^{1/2}} \left( \sum_{i=1}^{n} u_i^2 V_i \right).\]

Using a similar argument, we can obtain the following estimates for the remaining terms,

\[
\left| \int_{\mathcal{M}} v_1(x)v_2(x) d\mu_x - \sum_{j=1}^{n} v_1(p_j) v_2(p_j)V_j \right| \leq \frac{Ch}{\beta} \left( \sum_{i=1}^{n} u_i^2 V_i \right)^{1/2} \left( \sum_{j=1}^{n} f_j^2 V_j \right)^{1/2}, \text{ and } \\
\left| \int_{\mathcal{M}} v_1(x)v_3(x) d\mu_x - \sum_{j=1}^{n} v_1(p_j) v_3(p_j)V_j \right| \leq \frac{Ch}{\beta^2} \left( \sum_{j=1}^{n} u_i^2 V_i \right)^{1/2} \left( \sum_{j=1}^{n} f_j^2 V_j \right)^{1/2}, \text{ and } \\
\left| \int_{\mathcal{M}} v_2^2(x) d\mu_x - \sum_{j=1}^{n} v_2^2(p_j)V_j \right| \leq \frac{Ch}{\beta^2} \sum_{l \in I_S} u_l^2 A_l, \text{ and } \\
\left| \int_{\mathcal{M}} v_2(x)v_3(x) d\mu_x - \sum_{j=1}^{n} v_2(p_j) v_3(p_j)V_j \right| \leq \frac{Ch}{\beta^4} \left( \sum_{j=1}^{n} u_i^2 A_l \right)^{1/2} \left( \sum_{j=1}^{n} f_j^2 V_j \right)^{1/2}, \text{ and } \\
\left| \int_{\mathcal{M}} v_3^2(x) d\mu_x - \sum_{j=1}^{n} v_3^2(p_j)V_j \right| \leq \frac{Ch}{\beta^3} \sum_{j=1}^{n} f_j^2 V_j.\]

Assembling all the above estimates together, we obtain

\[
\left\| u_{t,h} \right\|_{L^2(\mathcal{M})}^2 - \sum_{i=1}^{n} u_i^2 V_i \left\| \frac{Ch}{t^{1/2}} \left( \sum_{i=1}^{n} u_i^2 V_i + t^{1/2} \sum_{l \in I_S} u_l^2 A_l + t^2 \| f \|_\infty^2 \right) \right. 
\]
Similarly, we have

\[
\left\| u_{t,h} \right\|_{L^2(\partial M)}^2 - \sum_{l \in I_S} u_l^2 A_l \leq \frac{C h}{t} \left( \sum_{i=1}^n u_i^2 V_i + t^{1/2} \sum_{l \in I_S} u_l^2 A_l + t^2 \| f \|_\infty^2 \right).
\]

Using the assumption that \( h t^{3/2} \) is small enough such that \( \frac{C h}{h t^{3/2}} \leq \frac{1}{2} \), we obtain

\[
\sum_{i=1}^n u_i^2 V_i + t^{1/2} \sum_{l \in I_S} u_l^2 A_l \leq 2 \left( \left\| u_{t,h} \right\|_{L^2(\partial M)}^2 + t^{1/2} \left\| u_{t,h} \right\|_{L^2(\partial M)}^2 \right) + C h \left( t^{3/2} \| f \|_\infty^2 \right)
\]

\[
\leq C \left\| u_{t,h} \right\|_{H^1(M)}^2 + C h t^{3/2} \| f \|_\infty^2,
\]

which implies that

\[
\left( \sum_{i=1}^n u_i^2 V_i \right)^{1/2} + t^{1/4} \left( \sum_{l \in I_S} u_l^2 A_l \right)^{1/2} \leq C \left\| u_{t,h} \right\|_{H^1(M)} + C \sqrt{h} t^{3/4} \| f \|_\infty.
\]

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