

Q -COMPLETE DOMAINS WITH CORNERS IN \mathbb{P}^n AND EXTENSION OF LINE BUNDLES

JOHN ERIK FORNÆSS, NESSIM SIBONY AND ERLEND F. WOLD

Dedicated to T. Ohsawa and T. Ueda for their sixtieth birthday

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ABSTRACT. We show that if a compact set X in \mathbb{P}^n is laminated by holomorphic submanifolds of dimension q , then $\mathbb{P}^n \setminus X$ is $(q+1)$ -complete with corners. Consider a manifold U , q -complete with corners. Let \mathcal{N} be a holomorphic line bundle in the complement of a compact in U . We study when \mathcal{N} extends as a holomorphic line bundle in U . We give applications to the non existence of some Levi-flat foliations in open sets in \mathbb{P}^n . The results apply in particular when U is a Stein manifold of dimension $n \geq 3$.

1. INTRODUCTION

Let \mathbb{P}^n be the complex projective space of dimension n . Let X be a compact set laminated by nonsingular varieties of dimension $1 \leq q \leq n - 1$. We prove that $\Omega := \mathbb{P}^n \setminus X$ is $(q + 1)$ -complete with corners. We then study the extension of holomorphic line bundles defined out of a compact set of a manifold U , q -complete with corners. This requires q to be large enough with respect to the dimension of U .

We recall a few definitions.

Definition 1.1. A real \mathcal{C}^2 function u defined in a complex manifold U is strictly q -convex if the complex Hessian (Levi form) has at least q strictly positive eigenvalues at every point. A complex manifold U is q -complete if it admits a strictly q -convex exhaustion function.

By smoothing, this is equivalent to saying that U admits a \mathcal{C}^∞ q -convex exhaustion function.

Definition 1.2. A function ρ is strictly q -convex with corners on U if for every point $p \in U$ there is a neighborhood U_p and finitely many strictly q -convex \mathcal{C}^2 functions $\{\rho_{p,j}\}_{j \leq \ell_p}$ on U_p such that $\rho|_{U_p} = \max_{j \leq \ell_p} \{\rho_{p,j}\}$.

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Definition 1.3. The manifold U is strictly q -convex with corners if it admits an exhaustion function which is strictly q -convex with corners outside a compact set.

Definition 1.4. The manifold U is strictly q -complete with corners if it admits an exhaustion function which is strictly q -convex with corners.

The notion of q -convexity with corners was introduced by Grauert [1]. The regularization of strictly q -convex functions with corners were studied by Diederich-Fornæss [4]. We will also use the following result by M. Peternell [12].

Theorem 1.5. *If U is an open set in \mathbb{P}^n which is strictly q -convex with corners, then it is strictly q -complete with corners.*

M. Peternell proved also that the complement in \mathbb{P}^n of a (possibly singular) analytic set of dimension q is $(q + 1)$ -complete with corners.

We recall the notion of compact laminated set. A compact set X in a complex manifold U is laminated by complex manifolds of dimension q if there is an atlas \mathcal{L} on X with charts $\Phi_V : V \rightarrow \mathbb{B} \times T_V$ where \mathbb{B} is a q dimensional ball in \mathbb{C}^q , T_V is a topological space, Φ_V is a homeomorphism. The change of coordinates are of the form $(z, t) \rightarrow (z', t')$ with $z' = h(z, t), t' = t'(t)$. The map h is holomorphic in z . If $X = U$ and the change of coordinates h, t' are holomorphic we say that we have a holomorphic foliation of dimension q . In \mathbb{P}^n it is a classical result that holomorphic foliations are always singular. More precisely, there is a nonempty analytic set E of codimension at least two so that $U := \mathbb{P}^n \setminus E$ is foliated as described above. One can also consider a covering (U_i) of \mathbb{P}^n and in each (U_i) we have $(w_i^1, \dots, w_i^{n-q})$ holomorphic $(1, 0)$ forms defining the leaves, satisfying the Frobenius condition and such that on $U_i \cap U_j$ there is an invertible holomorphic matrix (g_{ij}) with $(w_i) = (g_{ij})(w_j)$.

The main results are the following:

Theorem 1.6. *Let X be a compact set laminated by complex manifolds of dimension $1 \leq q \leq n - 1$ in \mathbb{P}^n . Then $\Omega := \mathbb{P}^n \setminus X$ is strictly $(q + 1)$ -complete with corners.*

Theorem 1.7. *Let X be a complex manifold with a proper Morse function $\rho : X \rightarrow (a, b)$ whose Levi form has at least three strictly positive eigenvalues at each point of X . Then every holomorphic line bundle over a set*

$$\{x \in X : \rho(x) > c\} \text{ for some } c \in (a, b)$$

extends to a holomorphic line bundle over X . The same holds if (a, b) is replaced by $[a, b)$.

The smoothing techniques by the first author and Diederich [4] imply then the following:

Theorem 1.8. *Let U be a complex manifold of dimension $n \geq 3$. Assume U is q -complete with corners with a corresponding exhaustion function ρ . Let \mathcal{N} be a holomorphic line bundle defined in $\{\rho > c\}$. Assume $q \geq \frac{2n+3}{3}$. Then the line bundle \mathcal{N} can be uniquely holomorphically extended to U .*

Our interest in the problem of extending a holomorphic line bundle has its origin in a paper by T. Ohsawa [11] on the nonexistence of real analytic Levi flat hypersurfaces in \mathbb{P}^2 , although he considered the extension of roots of globally defined bundles. If it were possible to extend roots of line bundles in dimension 2, a strategy by T. Ohsawa would give the nonexistence of real analytic Levi flat hypersurfaces in \mathbb{P}^2 .

Indeed if M is a real analytic Levi flat hypersurface in \mathbb{P}^2 , then the foliation on M extends to a foliation in a neighborhood of M and by pseudoconvexity of the complement we will obtain a foliation \mathcal{F} of \mathbb{P}^2 . If \mathcal{N} denotes the normal bundle of the foliation it is easy to show that it has roots \mathcal{N}_k for all $k \in \mathcal{N}$ in a given neighborhood of M . If it were possible to extend them to \mathbb{P}^2 , the Chern class of \mathcal{N} would be zero, a contradiction.

After the above results were obtained, S. Ivashkovich in an arXiv preprint [11] has obtained Hartogs' extension of roots of a line bundle in dimension $n \geq 3$, assuming that ρ is a strictly plurisubharmonic exhaustion function.

In the last section we give an application of the above results to the nonexistence of certain Levi-flat manifolds in open subset of projective spaces. There are however C.R.-manifolds in \mathbb{P}^n which are laminated by complex manifolds. Several examples are discussed in Ghys [7] and in Fornæss-Sibony-Wold [5].

The following question seems open. Let \mathcal{F} be a holomorphic foliation by Riemann surfaces in \mathbb{P}^n . It always has singularities. Is there a closed nonempty \mathcal{F} invariant set which does not intersect the singularities of \mathcal{F} ? Equivalently, does every leaf cluster on $\text{Sing}(\mathcal{F})$? The question is discussed for $n = 2$ in Camacho-Lins Neto-Sad [3]. But the answer seems unknown even for all $n \geq 3$.

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2. EXTENSION OF LINE BUNDLES ACROSS C.R. MANIFOLDS.

We recall some basics about line bundles. A topological (resp. holomorphic) line bundle $L \rightarrow X$ over a complex manifold X is an element of $H^1(X, \mathcal{C}^*)$ (resp. $H^1(X, \mathcal{O}^*)$), i.e., L is represented by a cover $\{U_j\}$ of X and functions $f_{ij} \in \mathcal{C}^*(U_i \cap U_j)$ (resp. $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$), satisfying the cocycle condition $f_{ij} \cdot f_{jk} \cdot f_{ki} = 1$. The bundle L is trivial if there exist $f_i \in \mathcal{C}^*(U_i)$ (resp. $f_i \in \mathcal{O}^*(U_i)$) such that $f_{ij} = f_j/f_i$. A way to show that L is trivial is to choose branches $g_{ij} = \log f_{ij}$ such that the cocycle condition $g_{ij} + g_{jk} + g_{ki} = 0$ is satisfied, solve $g_{ij} = g_j - g_i$ (which is always possible

in the topological category), and define $f_i := e^{g_i}$. Of course, branches of logarithms can be chosen in accordance with the cocycle condition only if the bundle is trivial. Recall that the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{C} \xrightarrow{\text{exp}} \mathcal{C}^* \rightarrow 0$$

induces an isomorphism $H^1(X, \mathcal{C}^*) \approx H^2(X, \mathbb{Z})$.

If X deformation retracts to a closed subset $X' \subset X$ with $L|_{X'}$ trivial and L is a topological line bundle, then L is trivial, since branches of logarithms may then be chosen on X' .

If $\{U_j\}$ is a cover of X with all line bundles trivial over U_j , then any line bundle has a representation with respect to this cover. Assuming that a holomorphic bundle L is topologically trivial, we may split $f_{ij} = f_j/f_i$ with smooth f_j 's. Define $g_j := \log f_j$ (choose any branch). Then $g_j - g_i$ is holomorphic on $U_i \cap U_j$, hence $\omega := \bar{\partial}g_j$ is a well defined $(0, 1)$ -form. So if we can solve $\bar{\partial}$ at the level of $(0, 1)$ -forms, we solve $\bar{\partial}f = \omega$, and define $\tilde{g}_j = g_j - f$ and note that $f_j := e^{\tilde{g}_j}$ defines a holomorphic splitting of the 1-cocycle, *i.e.*, L is holomorphically trivial.

Our approach to proving Hartogs' extension for line bundles on a 3-convex n -dimensional manifold X , will be to choose a suitable 3-convex exhaustion ρ of X with Morse critical points, start with a line bundle $L \rightarrow \{\rho > c\}$, and extend locally across the level sets of ρ until we cover X . This basically reduces to showing that L is locally trivial near any point $p \in \{\rho = c\}$. In all cases except for the case where p is a critical point with index $2n - 3$, we observe that any *topological* line bundle is trivial near p . Then we use the fact, see Theorem 2.3 [1] below, that we can solve the $\bar{\partial}$ -equation at the level of $(0, 1)$ -forms near p . Near a critical point with index $2n - 3$, non-trivial topological bundles do exist. So in this case the triviality is an analytic phenomenon. The key difficulty is to prove Hartogs' extension phenomenon for line bundles across real analytic totally real manifolds of dimension three, and so we state this as a separate result.

Theorem 2.1. *Let X be a complex manifold of dimension at least three and let $M \subset X$ be a closed real analytic CR submanifold of CR-dimension less than or equal to $n-3$. Then the restriction map $H^1(X, \mathcal{O}^*) \rightarrow H^1(X \setminus M, \mathcal{O}^*)$ is surjective.*

Remark 2.2. It follows from the previous theorem that an analytic set M , of codimension at least three, is removable for holomorphic line bundles. We can first extend through the smooth points of M and then use the stratification of M into smooth manifolds.

Remark 2.3. Ivashkovich [11] has given a counter example to extension for a two dimensional real analytic totally real manifold in \mathbb{C}^2 , even under the assumption that the bundle is topologically trivial. We give other examples at the end of this section and in Section 5.

We rely on the following result by Andreotti-Grauert [1, Proposition 12]. We have translated their result using our definition of q -convexity.

Theorem 2.4. *Let V be a domain in \mathbb{C}^n , $n \geq 3$ and let $p \in V$. Let ρ be a smooth strictly q -convex function on V , with $q \geq 3$. There is a basis Q of Stein neighborhoods of p such that on the intersection D of Q with $\{\rho > c = \rho(p)\}$, the $\bar{\partial}$ -equation is solvable for $(0,1)$ -forms. More precisely, for any $\bar{\partial}$ -closed smooth $(0,1)$ form f in D there is a smooth function u in D such that $\bar{\partial}u = f$.*

To prove Theorem 2.1 it is enough to show the following: for any point $x \in M$ there exists an open neighborhood U_x of x in X such that any holomorphic line bundle on $U_x \cap (X \setminus M)$ is holomorphically trivial. In that case, given a line bundle $L \rightarrow X \setminus M$, we may find an open cover $\{U_\alpha\}_{\alpha \in I}$ of X such that L is trivial over $U_\alpha \setminus M$ for all α . Then L is represented by a cocycle $f_{\alpha,\beta} \in \mathcal{O}^*(U_{\alpha,\beta} \setminus M)$. But any holomorphic function in $U_\alpha \setminus M$ extends holomorphically across M , and so by the identity theorem $f_{\alpha,\beta}$ extends to a cocycle with respect to the cover $\{U_\alpha\}_{\alpha \in I}$ of X .

At any point of M we may choose local holomorphic coordinates $(z, w) \in \mathbb{C}^r \times \mathbb{C}^s$ on X , $z_j = x_j + iy_j$, such that $M = \{z_j = F_j, j = 1, \dots, k, x_j = G_j, j = 1, \dots, r\}$ where the F_j, G_j are CR functions of $y_{k+1}, \dots, y_r, w_1, \dots, w_s$, holomorphic in w and, vanishing to second order. The CR-dimension of M is s , and the real codimension $r \geq 3$. The function $\sum_j |z_j - F_j|^2 + \sum_j |x_j - G_j|^2$ is q -convex.

In the case $r \geq 4$ or if $r = 3$ and $k > 0$, the proof is short: near any point $x \in M$ we have that L is topologically trivial; $H^2((X \setminus M) \cap U_p, \mathbb{Z})$ is zero because $X \setminus M$ retracts on a ball B^* of dimension > 3 . By Theorem 2.4 we may also solve $\bar{\partial}$ near x , and so L is holomorphically trivial.

The main difficulty is to prove the result if $s = 0$, $r = 3$, and $k = 0$, *i.e.*, if M is a three dimensional totally real manifold in complex three dimensional space. If s were greater than zero then $\mathbb{C}^{r+s} \setminus M$ would (locally) retract onto $\Omega := \mathbb{C}^3 \setminus \{M \cap \{z_{r+1} = \dots = z_{r+s} = 0\}\}$. A line bundle on $\mathbb{C}^{r+s} \setminus M$ restricted to Ω , is locally topologically trivial by the extension result for $s = 0$, hence it is locally topologically trivial on $\mathbb{C}^{r+s} \setminus M$.

So we need to prove Theorem 2.1 in the case where $\dim(X) = 3$, and M is a totally real three dimensional manifold. Let $x \in M$. Since M is real analytic there exists an open neighborhood U_x of x in X and a biholomorphism $\psi : U_x \rightarrow V_0 \subset \mathbb{C}_0^3$ such that $\psi(M) = \tilde{M} = \{z \in V_0 : \operatorname{Re}(z_j) = 0 \text{ for } j = 1, \dots, 3\}$. So to prove Theorem 2.1 it suffices to prove the following:

Proposition 2.5. *Let $L \rightarrow V_0 \setminus \tilde{M}$ be a holomorphic line bundle. Then there exists an $\epsilon > 0$ such that L is holomorphically trivial over $(V_0 \setminus \tilde{M}) \cap \mathbb{B}_\epsilon^3$.*

Let $B \subset \mathbb{R}^3$ denote the open unit ball, and for $j = 1, 2$ let B_j denote the ball of radius one centered at $((-1)^{j+1}, 0, 0)$. Let $\tilde{U}_j := B \setminus \overline{B}_j$. Let D

denote the disk $D := \{(x_1, \dots, x_3) \in B : x_1 = 0\}$, and note that $\tilde{U}_1 \cap \tilde{U}_2$ retracts onto the punctured disk D^* . Let S^1 denote the circle of radius one half contained in D^* .

For $j = 1, 2$ let $U_j = (\tilde{U}_j \times i \cdot \mathbb{R}^3) \cap \mathbb{B}^3 \subset \mathbb{C}^3$, and let $D^* \hookrightarrow U_1 \cap U_2$ be the natural inclusion. Then $U_1 \cap U_2$ retracts onto D^* . Now (U_1, U_2) is an open cover of $\mathbb{B}^3 \setminus \tilde{M}$. Since each U_j retracts to a point, any topological line bundle $L \rightarrow \mathbb{B}^3 \setminus \tilde{M}$ is represented by a function $f_L \in \mathcal{C}^*(U_1 \cap U_2)$.

Lemma 2.6. *The line bundle L is trivial if and only if the winding number of f_L restricted to S^1 is zero.*

Proof. Since $U_1 \cap U_2$ retracts onto D^* , the function f_L has a continuous logarithm $\log f_L$ if the winding number is zero. Since $H^1(\mathbb{B}^3 \setminus \tilde{M}, \mathcal{C}) = 0$ we may write $\log f_L = g_2 - g_1$ for $f_j \in \mathcal{C}(U_j)$. So $f_j := e^{g_j}$ defines a multiplicative splitting of f_L .

On the other hand, assume that $f_L = f_2/f_1$ for $f_j \in \mathcal{C}^*(U_j)$. Since each U_j is star-shaped it follows that each f_j is homotopic through non-zero maps to a constant map, hence f_L is homotopic through non-zero maps to a constant map. \square

Lemma 2.7. *There exists an $\epsilon > 0$ such that if $L \rightarrow U_j \cap \mathbb{B}^3$ is a holomorphic line bundle, then $L|_{U_j \cap \mathbb{B}_\epsilon^3}$ is holomorphically trivial.*

Proof. By Theorem 2.4 there is an open neighborhood $\Omega_j \subset \mathbb{B}^3$ of the origin, such that the $\bar{\partial}$ -equation for $(0, 1)$ -forms is solvable on $\Omega_j \cap U_j$. Since $U_j \cap \mathbb{B}^3$ is star-shaped, the obstruction to L being trivial on $U_j \cap \mathbb{B}^3$ is represented by a $\bar{\partial}$ -closed $(0, 1)$ -form ω . Since ω is $\bar{\partial}$ -exact on $\Omega_j \cap U_j$ we have that $L|_{\Omega_j \cap U_j}$ is holomorphically trivial. Choose $\epsilon > 0$ small enough such that $\mathbb{B}_\epsilon^3 \subset \Omega_j$ for $j = 1, 2$. \square

Lemma 2.8. *There exists a neighborhood U of the origin such that the following holds: let ω be a $\bar{\partial}$ -closed $(0, 1)$ -form on $\mathbb{B}^3 \setminus \tilde{M}$. Then ω is $\bar{\partial}$ -exact on $U \setminus \tilde{M}$.*

Proof. This follows from Theorem 2.4 and using the function

$$\rho(z) := \sum_j (Re(z_j))^2.$$

Proof of Proposition 2.5: In the case $\dim(X) = k = 3$.

Assume to get a contradiction that $L \rightarrow \mathbb{B}^3 \setminus \tilde{M}$ is a holomorphic line bundle, which is not trivial in any neighborhood of the origin. From Lemma 2.8 we see that L is not topologically trivial, otherwise L would be holomorphically trivial on $\mathbb{B}_{\epsilon_0}^3 \setminus \tilde{M}$. By Lemma 2.7 there exists an $\epsilon > 0$ such that $L|_{\mathbb{B}_\epsilon^3 \setminus \tilde{M}}$ is represented by a non-vanishing holomorphic function $f \in \mathcal{O}^*((U_1 \cap U_2) \cap \mathbb{B}_\epsilon)$.

Since L is topologically nontrivial we get from Lemma 2.6 (and scaling) that f has a non-zero winding number when restricted to S^1 . Restricting f to $\{z_1 = 0\} \cap (\mathbb{B}_\epsilon^3 \setminus \tilde{M})$ we get by Hartogs' extension phenomenon that f extends to $\{z_1 = 0\} \cap \mathbb{B}_\epsilon^3$. Then non-trivial winding implies that the extended f must have zeroes. On the other hand these zeroes must be contained in \tilde{M} , which is a contradiction. \square

Example 2.9. We give an example of a topologically trivial line bundles with roots in the complement of a real line in \mathbb{C}^2 , which is not holomorphically trivial:

Cover the complement of the real z -axis by two open sets: $U = \{(z, w); w \neq 0\} \cup \{(x+iy, 0); y < 0\}$ and $V = \{(z, w); y > 0\}$. We define the line bundle by using the transition function $e^{1/w}$ on $U \cap V = \{(z, w); y > 0, w \neq 0\}$. Since we can write $1/w$ as the difference between smooth functions on U and V , it follows that the line bundle is topologically trivial. If the line bundle were holomorphically trivial, then $e^{1/w} = \frac{f_U}{f_V}$. But f_U extends holomorphically to a neighborhood of 0 so neither f_U nor f_V can be singular at $w = 0$, a contradiction.

3. PROOF OF THEOREM 1.7

Let ρ be a smooth exhaustion of a domain U in \mathbb{P}^n . We want to extend a holomorphic line bundle defined in $(\rho > c)$ past p with $\rho(p) = c$. We have to show that locally in $(\rho > c)$ the line bundle is holomorphically trivial and then extend the chart near p . We reduce to a $\bar{\partial}$ -problem for $(0, 1)$ -forms in the intersection of a neighborhood of p with $(\rho > c)$. For that we need that the Levi form at p has at least three strictly positive eigenvalues and we need to construct an appropriate neighborhood.

We will be interested in extending holomorphic line bundles across certain (singular) hypersurfaces. The following lemma gives some conditions that easily imply extension:

Lemma 3.1. *Let K be a closed subset of a complex manifold X with $K = \overline{K^\circ}$. Assume that there exist open covers $\{U'_j\} \ll \{U_j\}$ of bK such that the following holds:*

- (1) *Any holomorphic line bundle $L \rightarrow K^\circ$ is trivial over $U_j \cap K^\circ$,*
- (2) *any $f \in \mathcal{O}(U_i \cap U_j \cap K^\circ)$ extends to a holomorphic function on $U'_i \cap U'_j$ for $i \neq j$, and*
- (3) *each connected component of $U'_i \cap U'_j \cap U'_k$ intersects K° for all i, j, k .*

Then any holomorphic line bundle on K° extends to a holomorphic line bundle on $K \cup (\cup_j U'_j)$.

Proof. Let $L \rightarrow K^\circ$ be a holomorphic line bundle. Let $\{V_j\}$ be a collection of open subsets of K° with $H^1(V_j, \mathcal{O}^*) = 0$, and such that $\{V_j\} \cup \{U'_j \cap K^\circ\}$ is an open cover of K° . Then by (1) the line bundle L is represented by transition functions on the intersections of all pairs of open sets in this cover. By (2) each transition function $f'_{ij} \in U'_i \cap U'_j \cap K^\circ$ extends to a function f_{ij} on $U'_j \cap U_i$, and since $1/f'_{ij}$ also extends, f_{ij} is non vanishing. Now we are done if

$$(3.1) \quad f_{ij} \cdot f_{jk} \cdot f_{ki} = 1 \text{ on } U'_i \cap U'_j \cap U'_k \text{ for all } i, j, k.$$

But this follows from (3) and the identity principle, since the cocycle condition (3.1) is satisfied on $U'_i \cap U'_j \cap U'_k \cap K^\circ$. \square

Lemma 3.2. *Let X be a complex manifold, let ρ be a nice strongly q -convex exhaustion of X , $q \geq 3$, set $\Omega_c := \{\rho > c\}$, and assume that $b\Omega$ does not contain any critical points for ρ . Then there exists $c' < c$ such that any holomorphic line bundle on Ω_c extends to a holomorphic line bundle on $\Omega_{c'}$.*

Proof. Construct a cover as in Lemma 3.1 by using Hartogs' extension phenomenon; Theorem 13.8 in [10], and Theorem 2.4. \square

We start with a brief discussion about the local structure of Morse exhaustion functions. If $p \in X$ is a Morse critical point for a strictly q -convex function ρ , then there are local holomorphic coordinates

$$z = (z_1, \dots, z_q, z_{q+1}, \dots, z_n) : U_p \rightarrow \mathbb{C}^n,$$

with $z(p) = 0$, such that the following holds (see [7], Lemma 3.9.4, or [5])

$$\rho(z', 0) = \sum_{j=1}^{q-r} x_j^2 + \delta_j y_j^2 + \sum_{j=q-r+1}^q x_j^2 - \delta_j y_j^2 + o(|z'|^2),$$

with $(z', z'') \in \mathbb{C}^q \times \mathbb{C}^{n-q}$, $\delta_j > 0$, $\delta_j < 1$ for $j = q - r + 1, \dots, q$, and r is the Morse index of the function $\rho(z', 0)$.

Furthermore there is a real coordinate change $\psi(z', z'')$, holomorphic for each fixed z'' , such that $\tilde{\rho}(z) = \rho(\psi(z))$ is of the form $\rho(\psi(z)) = Q(z) + o(|\psi(z)|^2)$ with

$$Q(z) = \sum_{j=1}^{q-r} x_j^2 + \delta_j y_j^2 + \sum_{j=q-r+1}^q x_j^2 - \delta_j y_j^2 - \sum_{j=1}^m u_j^2 + \sum_{j=m+1}^{2(n-q)} u_j^2,$$

where the u_j 's are real coordinates on \mathbb{C}^{n-q} , and $r + m$ is the Morse index of ρ at p .

We may further modify $\tilde{\rho}$ by choosing a $\chi \in C_0^\infty([0, 1])$ which is 0 near the origin and 1 near 1. We then define

$$\tilde{\rho}_r(z) := Q(z) + \chi(\|z\|/r) o(\|z\|^2),$$

for $0 < r \ll 1$. Then $\tilde{\rho}_r$ converges to $\tilde{\rho}$ in \mathcal{C}^2 -norm as $r \rightarrow 0$, since the remainder vanishes to order three, $\tilde{\rho}_r$ does not have more critical points than

$\tilde{\rho}$, and $\tilde{\rho}(z) = Q(z)$ near the origin. Such critical points, where $\tilde{\rho}_r$ is equal to its second order expansion, will be called *nice* critical points. By our discussion, any q -complete manifold admits a smooth exhaustion function with only nice critical points.

Finally, by perturbing ρ near a critical point we may assume that the δ_j 's are equal to one for $j = 1, \dots, q-r$ and as small as we like for $j = q-r+1, \dots, q$. To see this, let χ be a cut off function with support in a neighborhood of 0. Define

$$\tilde{\rho} = \rho + \chi\left(\frac{z}{r}\right) \sum_{j \leq q} \alpha_j y_j^2.$$

Here $|\alpha_j|$ are small. It is easy to check that $\tilde{\rho}$ has no critical points except 0 if $|\alpha_s|$ is not too large. Near zero we have improved the δ_j 's. The constants α_j are controlled by how large the positive eigenvalues of ρ are where χ varies. We can continue the process, with possibly smaller r 's, in finitely many steps.

Lemma 3.3. *Let X be a Stein manifold and let ρ be a nice strongly plurisubharmonic exhaustion function. Let $p \in X$ be a critical point for ρ . Then there exists an open neighborhood U_p of p in X such that any holomorphic line bundle $L \rightarrow \{\rho > \rho(p)\}$ is trivial over $U_p \cap \{\rho > \rho(p)\}$.*

Proof. In local coordinates, say on the unit ball \mathbb{B}^n in \mathbb{C}^n , we may write

$$\rho(z) = \sum_{j=1}^{n_1} x_j^2 + y_j^2 + \sum_{j=n_1+1}^n x_j^2 - \delta_j y_j^2,$$

where we have normalized and assumed that $\rho(p) = \rho(0) = 0$. We know from Theorem 2.4 that we may solve $\bar{\partial}$ for $(0,1)$ -forms on $\{\rho > 0\} \cap U$ where U is some neighborhood of the origin in \mathbb{C}^n , and so the main point is to show that L is topologically trivial on $\{\rho > 0\} \cap U$. We see that $\{\rho > 0\} \cap \mathbb{B}^n$ retracts onto a $(2 \cdot n_1 + n - 1)$ -sphere, and so the only case we need to consider is if $n = 3$ and $n_1 = 0$.

Choose a suitable smooth function χ which vanishes near the origin, is increasing and equal to 1 for $t > \delta$ for some small δ . Then defining $\tilde{\rho}(z) := \sum_{j=1}^3 x_j^2 - \delta_j \cdot \chi(\|z\|^2) \cdot y_j^2$, we have that $\tilde{\rho}$ has no critical points and is strictly plurisubharmonic on $\chi > 0$ (recall that the δ_j 's can be assumed to be as small as we like). Note that $\tilde{\rho}$ extends to $X \setminus \mathbb{B}^3$ since $\tilde{\rho}$ is equal to ρ near $b\mathbb{B}^3$. (Here the reader could also consult Section 3.10 of [7])

To prove that L is (locally near p) topologically trivial it is enough to prove that L is topologically trivial when restricted to

$$T := \{z \in \mathbb{B}^3 : |x|^2 > 1/2, |y| = 0\}.$$

(Since $\{\rho > 0\} \cap \mathbb{B}^3$ retracts to T .) We start with the line bundle $\tilde{L} := L|_{\{\tilde{\rho} > 1/2\}}$ (we look at the extension of $\tilde{\rho}$ to X). We first claim that \tilde{L} extends

holomorphically to $\{\tilde{\rho} > 0\}$. This follows from Lemma 3.2 by considering the set

$$\mathcal{S} := \inf_{s < 1/2} \{\tilde{L} \text{ extends uniquely to } \{\tilde{\rho} > s\}\}.$$

But the set $\tilde{\rho} = 0$ is a real analytic totally real manifold near the origin, and so by Proposition 2.5 we have that \tilde{L} extends to a neighborhood of the origin. This shows that \tilde{L} is topologically trivial on some tube $T_\delta := \{z \in \mathbb{B}^3 : |y^2| < \delta\}$, hence L is trivial on T . \square

We are now ready to prove Theorem 1.7 in the special case where X is Stein.

Theorem 3.4. *Let X be a Stein manifold of dimension greater than or equal to three, and let ρ be a strongly plurisubharmonic exhaustion of X . Then any holomorphic line bundle $L \rightarrow \{\rho > c\}$ extends uniquely to a holomorphic line bundle on X .*

Proof. We may assume that ρ is a nice strongly plurisubharmonic exhaustion. Assume for simplicity that ρ has a single minimum point x with $\rho(x) = 0$. We set

$$s := \inf\{c' < c : L \text{ extends uniquely to } \{\rho > c'\}\},$$

and claim that $s = 0$. Otherwise it follows by Lemma 3.2 that $\{\rho = s\}$ contains a critical point p , and in local coordinates we may write

$$\rho(z) = \rho(p) + \sum_{j=1}^{n_1} (x_j^2 + y_j^2) + \sum_{j=n_1+1}^n (x_j^2 - \delta_j \cdot y_j^2).$$

By Lemma 3.3 there exists an $r > 0$ such that L extends to a line bundle on $\{\rho > s\} \cap \mathbb{B}_r$. Let χ be a strictly positive function which is 1 near the origin and compactly supported on $[0, 1)$, and consider

$$\rho_\delta(z) = \rho(z) - \delta \cdot \chi(\|z\|^2/r^2).$$

If δ is chosen small enough we have that ρ_δ is strictly plurisubharmonic and that the origin is the only critical point of ρ_δ in \mathbb{B}_r . Now L lives on $\{\rho_\delta > 0\}$. By Lemma 3.2 we have that L extends to a smooth domain $\{\rho_\delta > c''\} \supset \{\rho > c'\}$ with $c' < s$. This contradicts the assumption that $s > 0$. Uniqueness follow from Lemma 3.5. \square

Lemma 3.5. *Let X be a complex manifold with a strongly q -convex exhaustion function ρ with $q \geq 2$, and let $L \rightarrow X$ be a holomorphic line bundle. If L is trivial over $\Omega_c = \{\rho > c\}$ then L is trivial. Consequently, if $\tilde{L} \rightarrow \{\rho > c\}$ is a holomorphic line bundle, then there is a unique extension of \tilde{L} to X , if such an extension exists, and if $L_k \rightarrow X$ satisfies $L_k^{\otimes k} = L$ on $\{\rho > c\}$ then $L_k^{\otimes k} = L$.*

Proof. We may assume that ρ is Morse with only nice critical points. We may assume that $b\Omega_c = \{\rho = c\}$ is smooth. Let $U_0 = \Omega_c$ and let U_1, \dots, U_m be a cover of $b\Omega_c$ similar to that of Lemma 3.1. Then for $c' < c$ close enough to c we have that $L \rightarrow \Omega_{c'}$ is represented by a cocycle (f_{ij}) with respect to this cover. The cocycle splits over Ω_c by assumption, and the splitting extends to $\Omega_{c'}$ for c' close to c .

Let c_0 be the largest $c_0 < c$ such that $b\Omega_{c_0}$ contains a critical point p . The argument above implies that L is trivial over Ω_{c_0} .

Let $U_1 = \{\rho > c_0\}$. By Theorem 13.8 in [10] there is a neighborhood U_2 of p such that any holomorphic function on $U_1 \cap U_2$ extends to U_2 . This shows that L is trivial over $\Omega_{c_0} \cup U_2$. Next modify ρ as in the proof of Theorem 3.4, and we may use the above argument to show that L is trivial until we reach the next critical value. \square

Theorem 3.6. *Let U be a complex manifold of dimension n . Assume ρ is a smooth exhaustion function which is strictly q -convex with $q \geq 3$. Let \mathcal{N} be a holomorphic line bundle defined in $\{\rho > c\}$. Then \mathcal{N} extends uniquely to a holomorphic line bundle on U .*

Proof. We can assume that ρ is a Morse function. A change with respect to the previous proof is that at a critical point p for ρ , there are *real* coordinates $z_j = x_j + iy_j$ such that

$$\rho = \sum_{s=1}^{n_1} (x_s^2 + y_s^2) + \sum_{n_1+1}^{n_2} (x_s^2 - \delta_s y_s^2) + \sum_{n_2+1}^n a_k x_k^2 + b_k y_k^2,$$

with $n_2 \geq q \geq 3$, $\delta_s < 1$, and $a_k, b_k = \pm 1$. The only difference with respect to the previous proof is that we need to prove that any line bundle is trivial near such a critical point. If the number of positive eigenvalues is greater than three, it follows that any bundle is topologically trivial, and since we can solve $\bar{\partial}$ we get the desired result.

Otherwise we have in local coordinates,

$$\tilde{\rho}(z) = \sum_{s=1}^3 x_s^2 - \delta_j y_s^2 - \sum_{j=4}^n (x_s^2 + y_s^2).$$

We want to show that L restricted to $\{\rho > 0\} \cap \{z_4 = \dots = z_n = 0\}$ is topologically trivial near the origin, because $\{\rho > 0\}$ retracts to this set near the origin.

Choose a function χ as in the proof of Theorem 3.4 and define

$$\rho(z) = \rho(z) = \sum_{s=1}^3 x_s^2 - \chi(\|z\|)\delta_j y_s^2 - \sum_{j=4}^n (x_s^2 + y_s^2),$$

such that $\tilde{\rho}$ has no critical points outside $\chi = 0$ and such that $\tilde{\rho}$ is strongly q -convex. Again $\tilde{\rho}$ extends to X . We may now extend L restricted to $\{\tilde{\rho} > c > 0\}$ to a bundle \tilde{L} on $\{\tilde{\rho} > 0\}$. But now \tilde{L} restricted to $\{\tilde{\rho} >$

$0\} \cap \{z_4 = \cdots = z_n = 0\}$ is a holomorphic line bundle in the complement of a totally real real analytic three dimensional manifold near the origin, hence the restriction of \tilde{L} extends across. Recall that the coordinate change ψ^{-1} from the discussion following Theorem 2.4 is holomorphic when restricted to $\{z_4 = \cdots = z_n = 0\}$. As in the proof of Theorem 3.4 this implies that the original L was trivial near the origin on $\{z_4 = \cdots = z_n = 0\}$. \square

We can then prove Theorem 1.8 if we rely on the smoothing of q -convex functions with corners obtained by the first author and K. Diederich [4]. We first recall their notation and statement.

Suppose we have a strictly q -convex function with corners in the meaning of the present paper. So if Q denotes the q in [4], then the functions have $n - Q + 1$ strictly positive eigenvalues. Hence $q = n - Q + 1$, $Q = n - q + 1$. After smoothing we are left with a \tilde{Q} -convex function with

$$\tilde{Q} = n - \left[\frac{n}{Q} \right] + 1$$

So we want to find \tilde{q} :

$$\tilde{q} = n - \tilde{Q} + 1.$$

So

$$\tilde{q} = n - \left(n - \left[\frac{n}{Q} \right] + 1 \right) + 1$$

$$\tilde{q} = \left[\frac{n}{Q} \right] = \left[\frac{n}{n - q + 1} \right]$$

Then in our notation a consequence of the smoothing theorem is.

Theorem 3.7. *Let U be a complex manifold of dimension n . Let ρ be an exhaustion function which is strictly q -convex with corners. Then there is a smooth exhaustion function $\tilde{\rho}$ which is \tilde{q} -convex,*

$$\tilde{q} = \left[\frac{n}{n - q + 1} \right].$$

Hence in order to prove Theorem 1.8 we just use the smoothing. We need

$$\left[\frac{n}{n - q + 1} \right] \geq 3.$$

This is equivalent to $\frac{n}{n - q + 1} \geq 3$ so $n \geq 3n - 3q + 3$ so $2n + 3 \leq 3q$ so $q \geq \frac{2n+3}{3}$. Then Theorem 1.8 is a consequence of Theorem 1.7.

Observe that for $n \geq 6$ we get values of q strictly smaller than n .

4. PROOF OF THEOREM 1.6

Lemma 4.1. *Let L be a smooth complex manifold of dimension q in an open set $W \subset \mathbb{P}^n$. Suppose that $p \in L$. Then there is a neighborhood of p , $W' \subset\subset W$, such that $-\log \sin^2 d(z, L)$ is C^∞ strictly $q+1$ -convex in W' . Moreover if K is a compact subset of $W' \setminus L$, then if L' is close enough to L then the positive eigenvalues of $-\log \sin^2 d(z, L')$ are uniformly bounded below by a positive constant on K .*

Proof. The proof is classical, but we include it for the benefit of the reader and because we need the estimates to be uniform in continuous variations of L . Recall, Barth [2], that if $x, y \in \mathbb{C}^{n+1} \setminus \{0\}$ and the corresponding points in \mathbb{P}^n are x^*, y^* , then the distance $d(x^*, y^*)$ in the Fubini-Study metric is given by the formula:

$$\sin^2(d(x^*, y^*)) = 1 - \frac{|\langle x, \bar{y} \rangle|^2}{|x|^2|y|^2}$$

where $\langle x, \bar{y} \rangle = \sum x_i \bar{y}_i$. So we are going to estimate the distance to a local piece of a smooth q -dimensional complex manifold L . We choose a smooth collection of rotations A_p of \mathbb{C}^{n+1} for $p^* \in L$ so that $p^* = [0 : \dots : 0 : 1]$ and L is given by $[z_1 : \dots : z_q : f_p^{q+1}(z_1, \dots, z_q) : \dots : f_p^n(z_1, \dots, z_q) : 1]$ and the functions f_p^j vanish to second order at the origin. Let N_L^p denote the complex normal plane $\{[0 : \dots : 0 : z_{q+1} : \dots : z_n : 1]\}$. We write $z' = (z_1, \dots, z_q)$, $z'' = (z_{q+1}, \dots, z_n)$. Also write $f_p(z') = (f_p^{q+1}(z_1, \dots, z_q), \dots, f_p^n(z_1, \dots, z_q))$. Set $x = (z', f_p(z'), 1)$ and $y = (0, z'', 1)$. According to the above formula,

$$\begin{aligned} \sin^2 d(x^*, y^*) &= 1 - \frac{|1 + \sum_{j=q+1}^n \bar{z}_j f_p^j(z')|^2}{(1 + |z'|^2 + |f_p(z')|^2)(1 + |z''|^2)} \\ &\geq 1 - \frac{1 + C|z''||z'|^2}{(1 + |z'|^2 + |f_p(z')|^2)(1 + |z''|^2)} \\ &\geq 1 - \frac{1}{1 + |z''|^2} \\ &= \sin^2 d(0, y^*). \end{aligned}$$

To estimate the Levi form of a function ρ we are going to use that if $\rho \geq \psi$ and $\rho(p) = \psi(p)$ then the Levi form of ρ at p is larger than the Levi form of ψ at p . So we have to find appropriate functions ψ . The above inequality shows that the points on N_L^p are closest points to $p \in L$ and hence that the distance function to L is smooth. Next we estimate the Levi form of $\sin^2 d(L, y^*)$. Let $y = (z', z'', 1)$, $y_0 = (0, z''_0, 1)$ and pick the point $x = (z', f_p(z'), 1)$ on L . We want to estimate the behaviour of $\sin^2 d(L, y^*)$ on the plane given by

$(z', tz_0'', 1)$, $z' \in \mathbb{C}^q$, $t \in \mathbb{C}$ with z' close to 0 and t close to 1. So this is a $q + 1$ -dimensional plane through y_0^* . We then have

$$\begin{aligned}
A &:= \sin^2 d(L, (z', tz_0'', 1)) - \sin^2 d((0, 0, 1), (0, z_0'', 1)) \\
&\leq \sin^2 d((z', f_p(z'), 1), (z', tz_0'', 1)) - \sin^2 d((0, 0, 1), (0, z_0'', 1)) \\
&= \left[1 - \frac{||z'|^2 + t \sum_{j=q+1}^n \bar{F}_p^j(z')(z_0'')_j + 1|^2}{(|z'|^2 + |f_p|^2 + 1)(|z'|^2 + |t|^2|z_0''|^2 + 1)} \right] \\
&\quad - \left[1 - \frac{1}{(1 + |z_0''|^2)} \right] \\
&= \frac{(|z'|^2 + |f_p|^2 + 1)(|z'|^2 + |t|^2|z_0''|^2 + 1)}{(|z'|^2 + |f_p|^2 + 1)(|z'|^2 + |t|^2|z_0''|^2 + 1)(1 + |z_0''|^2)} \\
&\quad - \frac{||z'|^2 + t \sum_{j=q+1}^n \bar{F}_p^j(z')(z_0'')_j + 1|^2(1 + |z_0''|^2)}{(|z'|^2 + |f_p|^2 + 1)(|z'|^2 + |t|^2|z_0''|^2 + 1)(1 + |z_0''|^2)} \\
&= \mathcal{O}((z')^4, (z')^2(t - 1)) + \frac{|z_0''|^2(|t|^2 - 1)}{(|t|^2|z_0''|^2 + 1)(1 + |z_0''|^2)} \\
&\quad - \frac{2\Re \sum_{j=q+1}^n \bar{F}_p^j(z')(z_0'')_j}{(1 + |z_0''|^2)} \\
&\quad - \frac{|z_0''|^2|z'|^2}{(1 + |z_0''|^2)^2}
\end{aligned}$$

Write $\delta = \frac{(|t|^2 - 1)|z_0''|^2}{1 + |z_0''|^2}$, then

$$\begin{aligned}
\frac{1}{(|t|^2|z_0''|^2 + 1)} &= \frac{1}{(1 + |z_0''|^2) + (|t|^2 - 1)|z_0''|^2} \\
&= \frac{1}{(1 + |z_0''|^2)(1 + \delta)} \\
&= \frac{1}{1 + |z_0''|^2} (1 - \delta + \delta^2 + \dots)
\end{aligned}$$

We get then, if \mathcal{PO} refers to pluriharmonic error terms.

$$\begin{aligned}
A &\leq \mathcal{O}((z')^4, (z')^2(t - 1), |t - 1|^3) + \frac{|z_0''|^2(|t|^2 - 1)}{(1 + |z_0''|^2)^2} \left[1 - \frac{(|t|^2 - 1)|z_0''|^2}{1 + |z_0''|^2} \right] \\
&\quad + \mathcal{PO}((z')^2) - \frac{|z_0''|^2|z'|^2}{(1 + |z_0''|^2)^2}
\end{aligned}$$

So,

$$\begin{aligned} \sin^2 d(L, (z', tz_0'', 1)) &\leq \left[1 - \frac{1}{(1 + |z_0''|^2)}\right] + \mathcal{O}((z')^4, (z')^2(t-1), |t-1|^3) \\ &+ \frac{|z_0''|^2(|t|^2 - 1)}{(1 + |z_0''|^2)^2} \left[1 - \frac{(|t|^2 - 1)|z_0''|^2}{1 + |z_0''|^2}\right] \\ &+ \mathcal{PO}((z')^2) - \frac{|z_0''|^2|z'|^2}{(1 + |z_0''|^2)^2}. \end{aligned}$$

Next we need to compose with $-\log(x)$. Using the comparison trick, we first consider the z' direction: Let $A - B|z'|^2$ be the right hand side of the previous inequality. Define

$$\begin{aligned} r &:= -\log[A - B|z'|^2]. \text{ Then} \\ r_z &= \frac{B\bar{z}}{1 - B|z'|^2} \text{ and} \\ r_{z\bar{z}}(0) &= B > 0 \end{aligned}$$

For the t direction at $t = 1$, we decompose r differently,

$$\begin{aligned} r &:= -\log(A + B(|t|^2 - 1) - C(|t|^2 - 1)^2) \\ r_t &= -\frac{B\bar{t} - 2C(|t|^2 - 1)\bar{t}}{(A + B(|t|^2 - 1) - C(|t|^2 - 1)^2)} \\ r_{t\bar{t}}(1) &= \frac{[B][B] - [B - 2C][A]}{A^2} \\ &= \frac{B^2 - AB + 2AC}{A^2} \end{aligned}$$

$$\begin{aligned} B^2 - AB + 2AC &= \left[\frac{|z_0''|^2}{(1 + |z_0''|^2)^2}\right]^2 - \left[1 - \frac{1}{(1 + |z_0''|^2)}\right] \frac{|z_0''|^2}{(1 + |z_0''|^2)^2} \\ &+ 2\left[1 - \frac{1}{(1 + |z_0''|^2)}\right] \frac{|z_0''|^4}{(1 + |z_0''|^2)^3} \\ &= \frac{|z_0''|^4}{(1 + |z_0''|^2)^4} - \frac{|z_0''|^4}{(1 + |z_0''|^2)^3} + 2\frac{|z_0''|^6}{(1 + |z_0''|^2)^4} \\ &= \frac{|z_0''|^4}{(1 + |z_0''|^2)^4} [1 - (1 + |z_0''|^2) + 2|z_0''|^2] \\ &= \frac{|z_0''|^6}{(1 + |z_0''|^2)^4} > 0 \end{aligned}$$

These calculations show that $-\log \sin^2 d(L, y)$ is strictly plurisubhamonic on a $q + 1$ -dimensional plane in a neighborhood of L . \square

Observe that on compact subsets of $W' \setminus L$ any small \mathcal{C}^2 perturbation of $-\log \sin^2 d(z, L) := \rho(z, L)$ is still $q + 1$ -convex. We have a lower bound on the eigenvalues of the Levi form if L' is close to L . In a tubular neighborhood T of X we have

$$-\log \sin^2 d(z, X) = \sup_{i \in I} -\log \sin^2 d(z, L_i)$$

with L_i a countable family of plaques.

We are ready to finish the proof of Theorem 1.6.

Proof. Fix an affine chart U of \mathbb{P}^n . Let T be a tubular neighborhood of X in \mathbb{P}^n . Fix $p \in X$. It is possible to find real hyperplanes P_1, P_2 in the Euclidean sense and distant of R large enough, so that in a neighborhood of p in $T \cap U$, we have $d(z, X) = \min_i d(z, L_i)$ with L_i a countable family of the intersection of leaves with the region between P_1 and P_2 . Here d denotes the distance in the Fubini Study metric. Let $\rho(z, L_i) = -\log \sin^2 d(z, L_i)$, the functions constructed in Lemma 3.1, associated to the complex manifold L_i . We can write $T = \cup T_j$, $T_j = \{z; \frac{1}{2^{j+1}} \leq d(z, X) < \frac{1}{2^j}\}$.

Let χ_j be a smooth function supported on $T_{j+1} \cup T_j \cup T_{j-1}$ with value 1 on T_j . Let $\alpha(t) := -\log \sin^2 t$ for $t > 0$.

Using the uniformity of Lemma 4.1 it is possible to find finitely many $L'_i, i \in I_j$ so that on a neighborhood of \bar{T}_j , the function $\max_{i \in I_j} (\rho(z, L_i) + c_j \chi_j)$ is larger than $\alpha(d(z, X))$ and is strictly $q + 1$ -convex with corners. This permits to construct a function ρ which is strictly $q + 1$ -convex with corners and which is a small perturbation of the function $\alpha(d(z, X))$. Observe that on $T_{j+1} \cup T_j \cup T_{j-1}$, ρ is obtained as a perturbation of the above function and we do the construction inductively on j . Once we have obtained this strictly $q + 1$ -convex function with corners which goes to infinity near X , we apply Peternell's Theorem 1.6 [12].

□

5. EXAMPLES IN DIMENSION 2

In dimension 2, it is well known that we do not have Hartogs' extendability for line bundles. Consider the line bundle in $\mathbb{C}^2 \setminus \{(0, 0)\}$ given in $z \neq 0, w \neq 0$ by the transition function $f(z, w) = \exp(\frac{1}{zw})$. It does not extend through $(0, 0)$. If it did, we would have $f = \frac{f_1}{f_2}$ in a neighborhood of $(0, 0)$ with f_i holomorphic in $z \neq 0$ and f_2 holomorphic in $w \neq 0$. Using Laurent series expansion, one checks easily that this is impossible.

We can construct also an example of a line bundle in the complement of the unit ball, with the unit sphere as natural boundary. Let $\{(z_i, w_i)\}$ denote a dense set of points on the boundary of the unit ball. Let $f = \sum_i \epsilon_i \frac{1}{(z-z_i)(w-w_i)}$. If the ϵ_i go to zero fast enough, then this is a well defined holomorphic function on the set $\{1 < |z| < 2, 1 < |w| < 2\}$. Hence we can use $g = e^f$ as transition function for a holomorphic line bundle on

$\{|z|, |w| < 2\} \setminus \{|z| \leq 1, |w| \leq 1\}$. We show that this line bundle extends to $\{|z|, |w| < 2\} \setminus \{|z|^2 + |w|^2 \leq 1\}$, but that it does not have an extension across any point in the boundary of the unit ball.

We can divide the (z_i, w_i) into finitely many groups of points, lying in the same small polydisc. Then for each such small polydisc the corresponding line bundle is holomorphic in the complement of this polydisc. The line bundle is the product of these. Hence the line bundle extends to the complement of any $B(0, 1 + \delta)$. We need to show it does not extend through any point (z_i, w_i) . We choose any small polydisc D around (z_i, w_i) , containing only points (z_j, w_j) with ϵ_j very small compared to ϵ_i . Cover the rest with polydiscs also. Then the line bundle for each of the other polydiscs extends across (z_i, w_i) so the obstruction to extension comes only from D . Assume for simplicity that $z_i = w_i = 0$. We note that the corresponding additive Cousin problem is a holomorphic function which can be Laurent series expanded and has terms with negative powers in both z and w at the same time. These can not be written as a difference of two holomorphic functions as needed because these can have only poles in one variable at a time. So this shows non-extension.

6. LEVI FLAT MANIFOLDS IN OPEN SETS IN \mathbb{P}^n .

Lins Neto [10] has shown the nonexistence of real analytic Levi flat hypersurfaces Σ in \mathbb{P}^n for $n \geq 3$. He uses the extension of the foliation of Σ as a holomorphic foliation of codimension 1 in \mathbb{P}^n . Siu [13] has shown the nonexistence of a smooth Levi flat hypersurface in \mathbb{P}^n , $n \geq 3$, foliated by leaves of codimension 1. M. Brunella (personal communication) has observed that there are no exceptional minimal sets for foliations of dimension $q \geq \frac{n+1}{2}$ in \mathbb{P}^n as a consequence of the Baum-Bott formula.

We have the following result regarding Levi-flat manifolds in open subsets of \mathbb{P}^n :

Theorem 6.1. *Let $\Omega \subset \mathbb{P}^n$ be an open set, and assume that \mathcal{F} is a q -dimensional holomorphic foliation on Ω with $q \geq \frac{2n}{3}$. Then there does not exist a \mathcal{C}^1 -smooth \mathcal{F} -invariant submanifold $\Sigma \subset \mathbb{P}^n$, $\Sigma \subset \Omega$, such that $\Sigma = \bigcap_{j=1}^{n-q} \Sigma_j$, where Σ_j is a hypersurface in \mathbb{P}^n , and the normals to the Σ_j 's are everywhere complex linearly independent on Σ .*

Proof. We assume to get a contradiction that there exists such a manifold Σ . Note that by Theorem 1.6 and Theorem 3.7 we have that $\mathbb{P}^n \setminus \Sigma$ admits a smooth 3-convex exhaustion function ρ .

We cover Σ by a finite number of balls B_j such that \mathcal{F} is defined by holomorphic $(n-q)$ -forms $\omega_j = df_j^1 \wedge \cdots \wedge df_j^{n-q}$ on B_j , and these give rise to a line bundle $\mathcal{N} := (f_{ij})$ with $f_{ij} = \omega_i / \omega_j$. Note that \mathcal{N} is holomorphically nontrivial: a splitting $f_{ij} = f_j / f_i$ would give rise to a holomorphic form $\omega := f_j \cdot \omega_j$ on Ω . By Hartogs extension, we have that ω would extend to

all of \mathbb{P}^n , but it is known that \mathbb{P}^n does not support a nontrivial holomorphic m -form for any $m \geq 1$.

On the other hand there exist $n - q$ non-vanishing continuous vector fields ξ_j on Σ , each pointing in the normal direction to Σ_j , and such that they span an $(n - q)$ -dimensional complex space over \mathbb{C} . This means that $\psi_j := \omega_j(\xi_1, \dots, \xi_{n-q})$ defines a continuous splitting of (f_{ij}) over Σ , hence $\mathcal{N} \rightarrow \Sigma$ is topologically trivial, and consequently has roots of all orders. This holds also in a neighborhood retract to Σ .

By Theorem 3.6 we have that \mathcal{N} along with all its roots \mathcal{N}_k , *i.e.*, $\mathcal{N}_k^{\otimes k} = \mathcal{N}$, extends uniquely to \mathbb{P}^n . It follows that the Chern class of (the extended) \mathcal{N} is zero, and this contradicts the non-triviality of \mathcal{N} on Ω . \square

Remark 6.2. An argument as above also gives the non-existence of real analytic Levi-flat hypersurfaces in \mathbb{P}^n for $n \geq 3$: the Levi foliation of such a hypersurface Σ extends to a neighborhood of Σ , and by the solution to the Levi-problem we have that $\mathbb{P}^n \setminus \Sigma$ is Stein.

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J. E. FORNÆSS, NTNU, 7491 TRONDHEIM, NORWAY. JOHN.FORNASS@MATH.NTNU.NO
AND MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY

N. SIBONY, UNIVERSITÉ PARIS-SUD, MATHÉMATIQUE, F-91405 ORSAY CEDEX, FRANCE,
NESSIM.SIBONY@MATH.U-PSUD.FR

E. F. WOLD, MATEMATISK INSTITUTT, UNIVERSITETET I OSLO, POSTBOKS 1053
BLINDERN, 0316 OSLO, NORWAY, ERLENDFW@MATH.UIO.NO