MULTI-DIMENSIONAL DEGENERATE KELLER-SEGEL SYSTEM WITH CRITICAL DIFFUSION EXPONENT $2n/(n+2)$

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Abstract. This paper deals with a degenerate diffusion Patlak-Keller-Segel system in $n \geq 3$ dimension. The main difference between the current work and many recent works on the same model is that we study the diffusion exponent $m = 2n/(n+2)$ which is smaller than the exponent $m^* = 2 - 2/n$ used in those recent works. With the exponent $m = 2n/(n+2)$, the associated free energy is conformal invariant and there is a family of stationary solution $U_{\lambda,x_0}(|x|) = C(n)(\frac{\lambda}{\lambda^2+|x-x_0|^2})^{\frac{n+2}{2}}$, $\forall \lambda > 0$. For radially symmetric solutions, we prove that if the initial data is strictly below $U_{\lambda,0},(|x|)$ for some λ then the solution vanishes in L^1_{loc} as $t \to \infty$; if the initial data is strictly above $U_{\lambda,0}(|x|)$ for some λ then the solution concentrates at $r = 0$ as $t \to \infty$. We then prove that there is a global weak solution provided that the L^m norm of initial density is less than a universal constant, and the weak solution vanishes as time goes to infinity. We also prove a finite time blow up of the solution if the L^m norm for initial data is large then that of $U_\lambda(|x|)$ and the free energy of initial data smaller than that of $U_\lambda(|x|)$.

Keywords: Nonlinear diffusion, nonlocal aggregation, critical stationary solution, global existence, mass concentration, radially symmetric solution.

1. Introduction and preliminaries

In this paper, we study Patlak-Keller-Segel models in $n \geq 3$ dimension with homogeneous degenerate diffusion:

$$\begin{cases} \rho_t = \Delta\rho^m - \mathrm{div}(\rho\nabla c), & x \in \mathbb{R}^n,\ t \geq 0, \\ -\Delta c = \rho, & x \in \mathbb{R}^n,\ t \geq 0, \\ \rho(x,0) = \rho_0(x), & x \in \mathbb{R}^n \end{cases} \tag{1.1}$$

where diffusion exponent is taken to be $m = \frac{2n}{n+2} \in (1,2)$. This model is widely used to describe the collective motion of cells. Here $\rho(x,t)$ represents the bacteria density and $c(x,t)$ represents the chemical substance concentration. We assume the initial

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data $\rho_0(x) \in L^1_+(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)$. L^1_+ means nonnegative integrable functions. $c(x,t)$ in the second equation of (1.1) can be represented by the fundamental solution,

$$c(x,t) = \frac{1}{(n-2)n\alpha(n)} \int_{\mathbb{R}^n} \frac{\rho(y,t)}{|x-y|^{n-2}} dy, \tag{1.2}$$

where $\alpha(n) = \dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ is the volume of n-dimension unit ball.

The first equation of (1.1) can also be written in a form,

$$\rho_t = \Delta \rho^m + \rho^2 - \nabla c \cdot \nabla \rho.$$

The classical solution ρ of equation (1.1) preserves non-negativety if it is initially so. Hence one has

$$\rho(x,t) \geq 0, \qquad x \in \mathbb{R}^n,\ t \geq 0.$$

The associated free energy for (1.1) is given by

$$\mathcal{F}(\rho) = \frac{1}{m-1} \int_{\mathbb{R}^n} \rho^m(x,t) dx - \frac{1}{2} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \rho(x,t) c(y,t) dx dy. \tag{1.3}$$

Using (1.2), the free energy can be recast as

$$\mathcal{F}(\rho) = \frac{1}{m-1} \int_{\mathbb{R}^n} \rho^m(x,t) dx - \frac{1}{2(n-2)n\alpha(n)} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-2}} dx dy. \tag{1.4}$$

The different sign in above free energy represents the competition between diffusion and nonlocal aggregation. This is the key feature of this system.

There is a natural variational structure for (1.1). The first order variation of $\mathcal{F}$ gives the chemical potential:

$$\mu = \frac{\delta \mathcal{F}}{\delta \rho} = \frac{m}{m-1} \rho^{m-1} - c. \tag{1.5}$$

By defining the drift velocity $v = -\nabla \mu$, the first equation in (1.1) can be rewritten into a continuity equation:

$$\rho_t + \operatorname{div}(\rho v) = 0, \tag{1.6}$$

or

$$\rho_t = \operatorname{div}\left(\rho \nabla \left(\frac{m}{m-1}\rho^{m-1} - c\right)\right). \tag{1.7}$$

Take inner product of $\frac{\delta \mathcal{F}}{\delta u}$ with (1.6), one leads to the following energy-dissipation relation

$$\frac{d\mathcal{F}(\rho)}{dt} + \int_{\mathbb{R}^n} \rho |\nabla \mu|^2 dx = 0,$$

or

$$\frac{d\mathcal{F}(\rho)}{dt} + \int_{\mathbb{R}^n} \rho \left| \nabla \left(\frac{m}{m-1}\rho^{m-1} - c\right)\right|^2 dx = 0, \tag{1.8}$$

which leads to the fact that $\mathcal{F}(\rho(\cdot,t))$ is a monotone nonincreasing function of t.

The ith-moment of ρ, $i = 0, 1, 2$, is defined by

$$m_0(t) = \int_{\mathbb{R}^n} \rho(x,t)dx, \quad m_1(t) = \int_{\mathbb{R}^n} x\rho(x,t)dx, \quad m_2(t) = \int_{\mathbb{R}^n} |x|^2\rho(x,t)dx.$$

By a direct computation, we have the following conservation relations for these moments:

Proposition 1.1.

$$m_0'(t) = \frac{d}{dt}\int_{\mathbb{R}^n} \rho(x,t)dx = 0, \tag{1.9}$$

$$m_1'(t) = \frac{d}{dt}\int_{\mathbb{R}^n} x\rho(x,t)dx = 0, \tag{1.10}$$

$$m_2'(t) = \frac{d}{dt}\int_{\mathbb{R}^n} |x|^2\rho(x,t)dx = -4\int_{\mathbb{R}^n} \rho^m(x,t)dx + 2(n-2)\mathcal{F}(\rho(\cdot,t)). \tag{1.11}$$

The identity (1.11) will be used to show a finite time blow-up behavior when the L^m-norm of the initial data is larger than a critical value in Section 3.

The classical version of the Keller-Segel model has linear diffusion, which played the key role in the two dimensional case. In two dimension, the nonlocal aggregation comes from the logarithmic potential which is the fundamental solution of Laplacian. A logarithmic version of Haddy-Littlewood-Sobolev inequality determines the critical mass $m_0 = 8\pi$ which derives solution behavior of global existence and finite time blow up for two dimensional Keller-Segel equation, see Chapter 5 in a recent book by Perthame [9]. There is a in-depth analysis for the case of critical mass $m_0 = 8\pi$ by Blanchet, Carlen and Carrillo in [1]. By making full use of a family of stationary solution and relative entropy, they proved that there exist basins of attraction for each stationary solution.

In space dimension $n \geq 3$, there are several modifications of the Keller-Segel model. A simple and direct way is to use logarithmic interaction kernel instead of the $1/|x|^{n-2}$ kernel from Laplacian [6]. Another way is to use degenerate diffusion to balance the nonlocal aggregation. Sugiyama [10] argued that critical diffusion exponent for (1.1) is $m^* = 2 - 2/n$. He proved that if $m > m^*$, then diffusion dominates and there is a global solution in (1.1) (He refer to this case as subcritical); If $m < m^*$, i.e., the aggregation dominates the system and there is a finite time blow up in solution to (1.1) for some initial data (This case was called supercritical problem). For exponent $m^* = 2 - 2/n$, in the mass-invariant scaling $u_\lambda(x,t) = \lambda^n u(\lambda x, t)$ for the system (1.1), there is a balance between diffusion and potential drift. Blanchet, Carrillo and Laurencot in [2] showed that there is a critical mass M_c such that if $m_0 < M_c$ and in addition $\rho_0 \in L^\infty \cap H^1(\mathbb{R}^n)$, then a global weak solution exists and satisfies an energy-dissipation inequality. They also proved that if $m_0 > M_c$, $\rho_0 \in L^\infty \cap H^1(\mathbb{R}^n)$ and the free energy is negative initially, then there is a finite time blow up for the solution in $L^m(\mathbb{R}^n)$. They also discussed the large time behavior for the critical mass $m_0 = M_c$ case. We refer to [3] for a discussion on (1.1) with general interaction potentials.

There are many reasons for us to take diffusion exponent $m = \frac{2n}{n+2}$ as we will discuss below. We first show that there is a family of positive stationary solutions to the equation (1.1). In fact, by taking $\rho = (\frac{m-1}{m}c)^{1/(m-1)}$ in (1.7) and plugging it into (1.1), we obtain the following equation

$$-\Delta c = \left(\frac{m-1}{m}\right)^p c^p, \quad x \in \mathbb{R}^n. \tag{1.12}$$

where $p = \frac{1}{m-1} = \frac{n+2}{n-2}$. The solution to the above equation is a stationary solution of (1.1). Indeed, from the energy-dissipation relation (1.8), one knows that all positive stationary solutions are given by the above equation.

It is a well known result [4, 5] that (1.12) has critical exponent $p_c = \frac{n+2}{n-2}$, or equivalently $m = \frac{2n}{n+2}$. Whenever $p < p_c$, or equivalently $m > \frac{2n}{n+2}$ (for example $m^* = 2 - 2/n > \frac{2n}{n+2}$), all nonnegative solution of (1.12) must be 0.

Proposition 1.2. *At $p = p_c$, all nonzero nonnegative solution of (1.12) must be of the form*

$$C_{\lambda,x_0}(x) = \frac{2^{\frac{n+2}{4}} n^{\frac{n}{2}}}{n-2} \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n-2}{2}}, \quad \text{for some } \lambda > 0, x_0 \in \mathbb{R}^n. \tag{1.13}$$

The corresponding stationary solution of (1.1), $\rho(x)$, is given by:

$$U_{\lambda,x_0}(x) = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} C_{\lambda,x_0}^{\frac{1}{m-1}}(x) = 2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n+2}{2}}. \tag{1.14}$$

Using the conservation laws (1.9)-(1.10), one can uniquely determinate the parameters λ and x_0 in the stationary solution and we state the result in the following proposition:

Proposition 1.3. *If $U_{\lambda,x_0}(x) = \lim_{t\to\infty} \rho(x,t)$, then the parameters $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ are uniquely determined by m^0 and m^1,*

$$x_0 = m_1/m_0, \quad \lambda^{\frac{n-2}{2}} \frac{2\pi}{n} \left(\frac{n-2}{2n}\right)^{\frac{n+2}{n-2}} [n(n-2)]^{\frac{n+2}{4}} = m_0.$$

Now we discuss connections among $U_{\lambda,x_0}(x)$, free energy and Hardy-Littlewood-Sobolev inequality. One has from (1.5) that $\frac{\delta \mathcal{F}}{\delta \rho}(U_{\lambda,x_0}(x)) = 0$. In other words, $U_{\lambda,x_0}(x)$ is also a family of critical points to $\mathcal{F}(\rho)$. Moreover, stationary solution $U_{\lambda,x_0}(x)$ reaches equality in Hardy-Littlewood-Sobolev inequality. A special version of Hardy-Littlewood-Sobolev inequality [8] is given by

Lemma 1.1. *Let $\rho \in L^m(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\rho(x)\rho(y)}{|x-y|^{n-2}} dxdy \le C(n) \|\rho\|_{L^m}^2, \tag{1.15}$$

where

$$C(n) = \pi^{(n-2)/2} \frac{1}{\Gamma(n/2+1)} \left\{\frac{\Gamma(n/2)}{\Gamma(n)}\right\}^{-2/n}. \tag{1.16}$$

Moreover, the equality holds if and only if $\rho(x) = AU_{\lambda,x_0}(x)$*, for some constant* A *and parameters* $\lambda > 0$*,* $x_0 \in \mathbb{R}^n$*.*

Consequently, we have the following decomposition of the free energy

$$\begin{aligned}\mathcal{F}(\rho) &= \frac{1}{m-1}\|\rho\|_{L^m}^m\left(1-\frac{(m-1)c_nC(n)}{2}\|\rho\|_{L^m}^{4/(n+2)}\right)\\ &\quad+\frac{c_n}{2}\left(C(n)\|\rho\|_{L^m}^2-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\rho(x)\rho(y)}{|x-y|^{n-2}}dxdy\right)\\ &:= \mathcal{F}_1(\rho)+\mathcal{F}_2(\rho). \end{aligned} \tag{1.17}$$

where $c_n = 1/(n(n-2)\alpha(n))$. Since $U_{\lambda,x_0}(x)$ is a critical point for both $\mathcal{F}(\rho)$ and $\mathcal{F}_2(\rho)$, it is also a critical point for $\mathcal{F}_1(\rho)$. Indeed we will show that it is a maximum point for $\mathcal{F}_1(\rho)$. This property will be used in the proof of a finite time blow up behavior in Section 3.

All the facts we listed above are reflected by the following proposition. i.e. with diffusion exponent $m = \frac{2n}{n+2}$, the free energy $\mathcal{F}(\rho)$ is invariant under translations, similarities, orthogonal transformations and inversions (Kelvin transformations).

Proposition 1.4. *The following facts hold*

(1) $\mathcal{F}(\rho_{\bar{x}}) = \mathcal{F}(\rho)$ *with* $\rho_{\bar{x}}(x) := \rho(x+\bar{x})$, $\forall \bar{x} \in \mathbb{R}^n$;
(2) $\mathcal{F}(\rho_\lambda) = \mathcal{F}(\rho)$ *with* $\rho_\lambda(x) := \lambda^{\frac{n+2}{2}}\rho(\lambda x)$, $\forall \lambda > 0$;
(3) $\mathcal{F}(\rho_{\mathcal{R}}) = \mathcal{F}(\rho)$ *with* $\rho_{\mathcal{R}}(x) := \rho(\mathcal{R}^{-1}x)$, $\forall\ \mathcal{R}^*\mathcal{R} = I$;
(4) $\mathcal{F}(\rho_{\bar{x},\lambda}) = \mathcal{F}(\rho)$ *with* $\rho_{\bar{x},\lambda}(x) := \Big(\frac{\lambda}{|x-\bar{x}|}\Big)^{n+2}\rho\Big(\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\Big)$, $\forall \bar{x} \in \mathbb{R}^n$, $\lambda > 0$.

We will give a proof of this proposition in the Appendix.

Remark 1.1. By Liouville's theorem [7], any smooth conformal mapping on a domain of $\mathbb{R}^n$, $n > 2$, can be expressed as a composition of translations, similarities, orthogonal transformations and Kelvin transformations. These transformations are all Möbius transformations.

Remark 1.2. The last transformation in Prop. 1.4 is enlightened by the Kelvin transformation of c, i.e.,

$$c_{\bar{x},\lambda}(x) = \Big(\frac{\lambda}{|x-\bar{x}|}\Big)^{n-2}c\Big(\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\Big).$$

Remark 1.3. Invariants of the translations and similarities in Prop. 1.4 allow that parameters x_0 and λ in $U_{\lambda,x_0}(x)$ are free. Invariants for free energy on the orthogonal and Kelvin transformations guarantee that the form of stationary solution is unique.

Moreover, using (1.3) with $C_{\lambda,x_0}(x) = \frac{m}{m-1}U_{\lambda,x_0}^{m-1}(x)$, we get

$$\mathcal{F}(U_{\lambda,x_0}(x)) = \frac{2}{n-2}\|U_{\lambda,x_0}(x)\|_{L^m}^m, \tag{1.18}$$

while we can calculate the right hand side explicitly,

$$\|U_{\lambda,x_0}(x)\|_{L^m}^m = n^n \pi^{\frac{n+2}{2}} 2^{1-\frac{n}{2}} \frac{1}{\Gamma(\frac{n+1}{2})}.$$

This paper is arranged as follows. In Section 2, we prove that, for radially symmetric solutions, if the initial data is strictly below $U_\lambda(|x|)$ for some λ then the solution vanishes in L^1_{loc} as $t \to \infty$; if the initial data is strictly above $U_\lambda(|x|)$ for some λ then the solution concentrates at $r = 0$ as $t \to \infty$.

In Section 3, we prove that there is a global weak solution provided that the L^m norm of initial density is less than a universal constant, and the weak solution vanishes as time goes to infinity. We also prove a finite time blow up of the solution if the L^m norm for initial data is larger than that of $U_\lambda(|x|)$ and the free energy of initial data smaller than that of $U_\lambda(|x|)$.

2. Decay and blow-up for radially symmetric solution

In the section, we will study large time behavior to the radially symmetric solution of (1.1). Radially symmetric solutions $(\rho(t,r),\ c(t,r))$ of the system (1.1) satisfy

$$\begin{cases} (r^{n-1}\rho)_t = \big(r^{n-1}(\rho^m)'\big)' - (r^{n-1}\rho c')', & r \in (0,\infty), t \geq 0, \\ -(r^{n-1}c')' = r^{n-1}\rho, & r \in (0,\infty), t \geq 0, \\ \rho'(t, r=0) = 0, & t \geq 0, \\ \rho(t=0, r) = \rho_0(r), & r \in (0,\infty). \end{cases} \tag{2.1}$$

where $'$ stands for the derivative with respect to r. We will show that the stationary solution $U_{\lambda,0}(x)$ ($x_0 = 0$ in (1.14)) is a critical profile in the following sense, if the initial data ρ_0 is strictly below a stationary solution for some λ, then all radially symmetric solutions are vanishing in $L^1_{loc}(\mathbb{R}^n)$, if the initial data ρ_0 is strictly above a stationary solution for some λ, then all radially symmetric solutions has a mass concentration at $x = 0$ point as $t \to \infty$. For simplicity, we will use notation $U_\lambda(|x|) = U_{\lambda,0}(x)$ in this section. The following is our main theorem in this section.

Theorem 2.1. *Assume that the initial data $\rho_0 \geq 0$ is radially symmetric,*

(1) *If $\exists\ \lambda_0 > 0$ s.t.*
$$\rho_0(r) < U_{\lambda_0}(r), \quad r > 0,$$
then any radially symmetric solution $\rho(r,t)$ of (1.1) is vanishing in $L^1_{loc}(\mathbb{R}^n)$ as $t \to \infty$.

(2) *If $\exists\ \lambda_0 > 0$ s.t.*
$$\rho_0(r) > U_{\lambda_0}(r), \quad r > 0,$$
then any radially symmetric solution $\rho(r,t)$ of (1.1) must blow up (mass concentration) at $r = 0$ as time goes to infinity in the sense that there is $r(t) \to 0$ as $t \to \infty$ and a positive constant C such that
$$\int_{B(0,r(t))} \rho dx \geq C.$$

Inspired by a similar result in two dimensional case [9], we work on the following weighted primitive variable(integral of density ρ in the ball with radius r and center at origin)

$$M(t,r) := n\alpha(n)\int_0^r \sigma^{n-1}\rho(t,\sigma)d\sigma \tag{2.2}$$

by the second equation in (2.1), one has

$$M(t,r) = -n\alpha(n)r^{n-1}c'$$

Then (2.1) can be reduced to a single equation for $M(t,r)$. By integrating (2.1), we have

$$\begin{cases} M_t = n\alpha(n)r^{n-1}\left[\left(\dfrac{M'}{n\alpha(n)r^{n-1}}\right)^m\right]' + \dfrac{M'M}{n\alpha(n)r^{n-1}}, & r\in(0,\infty), t\geq 0,\\ M(t,0)=0, M(t,\infty)=m_0, & t\geq 0,\\ M(0,r) = n\alpha(n)\displaystyle\int_0^r \sigma^{n-1}\rho_0(\sigma)d\sigma, & r\in(0,\infty). \end{cases} \tag{2.3}$$

From the second equation of (2.1) with (2.2), we have $M' = n\alpha(n)r^{n-1}\rho$, for all $r\in(0,\infty), t\geq 0$. Thus $M'(t,r)\geq 0$, i.e., $M(t,r)$ is monotone increasing respect to r. The main advantage of using equation (2.3) instead of using (2.1) is that we can use comparison principle by constructing a super solution for decay estimates and constructing a sub solution for mass concentration estimates.

The stationary problem of (2.3) is reduced to

$$\begin{aligned} &n\alpha(n)r^{n-1}\left[\left(\frac{M'}{n\alpha(n)r^{n-1}}\right)^m\right]' + \frac{M'M}{n\alpha(n)r^{n-1}} = 0,\ r\in(0,\infty),\\ &M(0)=0, \quad M(\infty)=m_0 \end{aligned} \tag{2.4}$$

Recall from (1.13) and (1.14), that stationary solutions to (1.1) are given by

$$(U_\lambda(r), C_\lambda(r)) = \left(2^{\frac{n+2}{4}}n^{\frac{n+2}{2}}\left(\frac{\lambda}{\lambda^2+r^2}\right)^{\frac{n+2}{2}},\ 2^{\frac{n+2}{4}}n^{\frac{n}{2}}(n-2)^{-1}\left(\frac{\lambda}{\lambda^2+r^2}\right)^{\frac{n-2}{2}}\right)$$

where $\lambda>0$ is a free parameter. Hence, there is a family of explicit solutions to (2.4) as given by

$$\tilde{M}_\lambda(r) \ = n\alpha(n)\textstyle\int_0^r \sigma^{n-1}U_\lambda(\sigma)d\sigma = K_\lambda(n)\frac{1}{(1+\lambda^2r^{-2})^{\frac{n}{2}}}, \tag{2.5}$$

where $K_\lambda(n) = \alpha(n)2^{\frac{n+2}{4}}n^{\frac{n+2}{2}}\lambda^{\frac{n-2}{2}}$.

Proof of Theorem 2.1. In the following two subsections we will prove (1) and (2) of the theorem in the form of Lemma 2.1 and Lemma 2.2, respectively. □

2.1. **Super solution with subcritical initial data.** In this subsection, we will show that the solutions of the radially symmetric problem (2.3) vanish uniformly in time as $t\to\infty$ for any finite space interval, when initial data is controlled by a stationary solution $\tilde{M}_{\lambda_0}(r)$ in (2.5) for some $\lambda_0>0$. More precisely, we have

Lemma 2.1. *For $n \geq 3$, assume that*

$$m_0 = M(t,\infty) < K_{\lambda_0}(n), \ \ M(0,r) < \tilde{M}_{\lambda_0}(r), \forall r > 0.$$

for some $\lambda_0 > 0$. Then the solutions of (2.3) diminish in time in the following sense

$$M(t,r) \to 0 \text{ as } t \to \infty \text{ uniformly on any interval } 0 \leq r \leq R$$

and thus $\rho(t,x)$ in (1.1) vanishes in $L^1_{loc}(\mathbb{R}^n)$ as $t \to \infty$.

Proof. Since $M(0,r)$ and $\tilde{M}_{\lambda_0}(r)$ are bounded nondecreasing functions and $M(0,r) < \tilde{M}_{\lambda_0}(r)$, then there exist a $\mu \in (0,1)$ s.t. $M(0,r) \leq \mu \tilde{M}_{\lambda_0}(r)$.

Notice that $M(t,\infty) = m_0 < K_{\lambda_0}(n)$, without lose of generality, we can choose the same μ such that $M(t,\infty) = m_0 < \mu K_{\lambda_0}(n)$.

We construct a super-solution of (2.3) by modifying constant λ in the denominator of its stationary solution (2.5). We take $\lambda = \lambda(t) = (A_1 t + \lambda_0^n)^{1/n}$, for some $A_1 > 0$, and then we cut it off by a constant m_0 for $r \geq R(t)$, such that the super-solution $\bar{N}(t,r)$ is given by

$$\bar{N}(t,r) = \min\Big\{m_0, \frac{\mu K_{\lambda_0}(n)}{(1+\lambda^2(t)r^{-2})^{n/2}}\Big\},$$

where the cut off location $R(t)$ is given by

$$m_0 = \frac{\mu K_{\lambda_0}(n)}{\Big(1+\lambda^2(t)R^{-2}(t)\Big)^{n/2}}, \tag{2.6}$$

i.e.,

$$\Big(1+\lambda^2(t)R^{-2}(t)\Big)^{n/2} = \frac{\mu K_{\lambda_0}(n)}{m_0},$$

or

$$R(t) = \Big(\frac{\lambda^2(t)}{\big(\frac{\mu K_{\lambda_0}(n)}{m_0}\big)^{2/n} - 1}\Big)^{1/2}.$$

Hence $\bar{N}(t,r) = m_0$ for $r > R(t)$, and $\bar{N}(t,r) = \frac{\mu K_{\lambda_0}(n)}{(1+\lambda^2(t)r^{-2})^{n/2}}$ for $r \leq R(t)$ and $\bar{N}(t,r)$ is continuous.

Now we prove that $\bar{N}(t,r)$ is a supersolution to (2.3). Obviously, the constant state m_0 is also a super-solution. Next we only need to show that $\bar{N}(t,r) = \frac{\mu K_{\lambda_0}(n)}{(1+\lambda^2(t)r^{-2})^{n/2}}$ is a super-solution for $r \leq R(t)$, i.e. to prove

$$LHS = \bar{N}_t - n\alpha(n)r^{n-1}\Big[\big(\frac{\bar{N}'}{n\alpha(n)r^{n-1}}\big)^m\Big]' - \frac{\bar{N}'\bar{N}}{n\alpha(n)r^{n-1}} \geq 0. \tag{2.7}$$

A direct calculation of (2.7) term by term gives that

$$\begin{aligned}
\bar N_t &= \bar N \frac{-n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\lambda'(t),\\
n\alpha(n)r^{n-1}\Big[\big(\frac{\bar N'}{n\alpha(n)r^{n-1}}\big)^m\Big]' &= -2\bar N n^2\alpha(n)^{1-m}\lambda^{2m}(t)\frac{(\mu K_{\lambda_0}(n))^{m-1}r^{-n-2}}{(1+\lambda^2(t)r^{-2})^{\frac{n}{2}+1}},\\
\frac{\bar N'\bar N}{n\alpha(n)r^{n-1}} &= \mu K_{\lambda_0}(n)\bar N\frac{\lambda^2(t)r^{-n-2}}{\alpha(n)\big(1+\lambda^2(t)r^{-2}\big)^{\frac{n}{2}+1}}.
\end{aligned}$$

So, we have

$$\begin{aligned}
LHS &= \bar N\frac{n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\Big(-\lambda'(t)+2n\alpha(n)^{1-m}\lambda^{2m-1}(t)\frac{(\mu K_{\lambda_0}(n))^{m-1}r^{-n}}{(1+\lambda^2(t)r^{-2})^{\frac{n}{2}}} \qquad (2.8)\\
&\qquad -\mu K_{\lambda_0}(n)\frac{\lambda(t)r^{-n}}{n\alpha(n)\big(1+\lambda^2(t)r^{-2}\big)^{\frac{n}{2}}}\Big)\\
&= \bar N\frac{n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\Big(-\lambda'(t)+A(t)\frac{(\mu K_{\lambda_0}(n))^{m-1}\lambda(t)r^{-n}}{n\alpha(n)(1+\lambda^2(t)r^{-2})^{\frac{n}{2}}}\Big), \qquad (2.9)
\end{aligned}$$

where

$$A(t) = 2n^2\alpha(n)^{2-m}\lambda^{2m-2}(t) - (\mu K_{\lambda_0}(n))^{2-m}.$$

By using the expression $m=\frac{2n}{n+2}$ and $K_{\lambda_0}(n)=\alpha(n)2^{\frac{n+2}{4}}n^{\frac{n+2}{2}}\lambda_0^{\frac{n-2}{2}}$, we have

$$A(t) \;\ge 2n^2\alpha(n)^{2-m}\lambda_0^{\frac{2(n-2)}{n+2}}(1-\mu^{2-m}) := A_0 > 0. \qquad (2.10)$$

(2.8) and (2.10) imply that

$$\begin{aligned}
LHS &\ge \bar N\frac{n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\left(-\lambda'(t)+A_0\frac{(\mu K_{\lambda_0}(n))^{m-1}\lambda(t)r^{-n}}{n\alpha(n)(1+\lambda^2(t)r^{-2})^{\frac{n}{2}}}\right)\\
&\ge \bar N\frac{n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\left(-\lambda'(t)+A_0\frac{(\mu K_{\lambda_0}(n))^{m-1}\lambda(t)R^{-n}(t)}{n\alpha(n)\big(1+\lambda^2(t)R^{-2}(t)\big)^{\frac{n}{2}}}\right)\\
&\ge \bar N\frac{n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\left(-\lambda'(t)+A_1n^{-1}\lambda^{-n+1}(t)\right)=0 \qquad (2.11)
\end{aligned}$$

provided that $\lambda'(t)=A_1n^{-1}\lambda^{-n+1}(t)$, i.e.,

$$\lambda(t) = (A_1t+\lambda_0^n)^{1/n},\; t\ge 0, \qquad (2.12)$$

where

$$A_1 = A_0(\mu K_{\lambda_0}(n))^{m-2}m_0\alpha(n)^{-1}\left[\left(\frac{\mu K_{\lambda_0}(n)}{m_0}\right)^{\frac{2}{n}}-1\right]^{n/2} > 0.$$

Furthermore $M(0,r) \le \tilde{M}_{\lambda_0}(r)$, and $M(t,0) \le \lim_{r\to 0} \bar{N}(t,r) = \lim_{r\to 0} \frac{\mu K_{\lambda_0}(n)}{\left(1+\lambda^2(t)r^{-2}\right)^{n/2}} = 0$, thus $\frac{\mu K_{\lambda_0}(n)}{\left(1+\lambda^2(t)r^{-2}\right)^{n/2}}$ is a super-solution of (2.3). The minimum of two super-solutions is also a super-solution, i.e., $\bar{N}(t,r)$ is a super-solution to (2.3).

By the comparison principle, we deduce that the solution of (2.3) satisfies $M(t,r) \le \bar{N}(t,r)$ in $[0,\infty)\times[0,\infty)$. By (2.6) and (2.12), we have $\lambda(t),\ R(t) \to \infty$ as $t \to \infty$. So, for a given interval $r \in (0,R_0)$, it holds that

$$M(t,r) \le \frac{\mu K_{\lambda_0}(n)}{(1+\lambda^2(t)R_0^{-2})^{n/2}} \to 0,\ \text{as } t\to\infty,$$

This completes the proof of Lemma 2.1. □

2.2. **Blow-up with super-critical initial data.** In this subsection, we will prove that if the initial data is above a stationary solution $\tilde{M}_{\lambda_0}(r)$ in (2.5) for some $\lambda_0 > 0$, then radially symmetric solutions must have mass concentration at $x = 0$ as time goes infinity.

Lemma 2.2. *For dimension $n \ge 3$. Assume that*

$$m_0 = M(t,\infty) > K_{\lambda_0}(n),\ M(0,r) > \tilde{M}_{\lambda_0}(r), r > 0,$$

for some $\lambda_0 > 0$, Then there is $r(t) \to 0$ as $t \to \infty$ and $C > 0$ such that all solutions $M(r,t)$ satisfy

$$M(r(t),t) \ge C.$$

Or equivalently radially symmetric solutions ρ to (1.1) *have mass concentration at $x = 0$, i.e.*

$$\int_{B(0,r(t))} \rho dx \ge C.$$

Proof. We will show that there exists a radius $r(t) > 0$ depends on t such that as $t \to \infty$, we have $r(t) \to 0$ and

$$M(t,r(t)) \ge \text{ Const.} > 0, \tag{2.13}$$

i.e.,

$$\int_{B(0,r(t))} \rho dx \ge C > 0.$$

Similar to the discussion in the beginning of the proof of Lemma 2.1, we can choose $\mu_0 > 1$ such that $\mu_0 K_{\lambda_0}(n) < m_0 = M(t,\infty)$ and $M(0,r) > \mu_0 \tilde{M}_{\lambda_0}(r)$. We construct a sub-solution $\frac{\mu_0 K_{\lambda_0}(n)}{(1+\lambda(t)^2 r^{-2})^{n/2}}$ of the equation (2.3) from the stationary solution $\tilde{M}_\lambda(r)$ in (2.5) by taking $\lambda = \lambda(t) = \lambda_0 e^{B_1 t}$ for some $B_1 < 0$. Similar to the construction of super-solution in previous subsection, we cut off it by $\frac{m_0}{(1+\lambda_0^2 r^{-2})^{n/2}}$ for $r \ge R(t)$ for some $R(t)$ which will be specified later. We take the sub-solution in the following form:

$$\underline{N}(t,r) = \max\left\{ \frac{m_0}{(1+\lambda_0^2 r^{-2})^{n/2}}, \frac{\mu_0 K_{\lambda_0}(n)}{(1+\lambda(t)^2 r^{-2})^{n/2}} \right\}, \tag{2.14}$$

We show both terms on the right hand side of (2.14) are sub-solutions. For the first term, one has

$$\begin{aligned} & \underline{N}_t - n\alpha(n) r^{n-1} \left[\left(\frac{\underline{N}'}{n\alpha(n) r^{n-1}} \right)^m \right]' - \frac{\underline{N}'\underline{N}}{n\alpha(n) r^{n-1}} \\ = & \left[2\alpha(n)^{2-m} n^2 \lambda_0^{2m} - (m_0)^{2-m} \lambda_0^2 \right] \frac{n(m_0)^m r^{-n-2}}{n\alpha(n)(1+\lambda_0^2 r^{-2})^{n+1}} \\ \le & \left[2\alpha(n)^{2-m} n^2 \lambda_0^{2m} - K_{\lambda_0}(n)^{2-m} \lambda_0^2 \right] \frac{n(m_0)^m r^{-n-2}}{n\alpha(n)(1+\lambda_0^2 r^{-2})^{n+1}} \\ = & \ 0. \end{aligned} \tag{2.15}$$

Together with the boundary conditions

$$\begin{aligned} \underline{N}(t,0) &= \lim_{r\to 0} \frac{m_0}{(1+\lambda_0^2 r^{-2})^{n/2}} = 0, \\ \underline{N}(t,\infty) &= \lim_{r\to\infty} \frac{m_0}{(1+\lambda_0^2 r^{-2})^{n/2}} \le m_0 = M(t,\infty), \end{aligned}$$

and initial condition $\underline{N}(0,r) \le M(0,r)$. Therefore, $\dfrac{m_0}{(1+\lambda_0^2 r^{-2})^{n/2}}$ is a sub-solution.

Next we show that the second term $\dfrac{\mu_0 K_{\lambda_0}(n)}{(1+\lambda^2(t) r^{-2})^{n/2}}$ is also a sub-solution in the interval $0 \le r \le R(t)$ where $\underline{N}$ achieves its maximum by $\dfrac{\mu_0 K_{\lambda_0}(n)}{(1+\lambda^2(t) r^{-2})^{n/2}}$. The radius $R(t)$ is defined by the following equality

$$\frac{m_0}{(1+\lambda_0^2 R^{-2}(t))^{n/2}} = \frac{\mu_0 K_{\lambda_0}(n)}{(1+\lambda^2(t) R^{-2}(t))^{n/2}}.$$

Notice that there exists constant $R_0 :\ r \le R(t) \le R_0$ such that $\frac{m_0}{(1+\lambda_0^2 R_0^{-2})^{n/2}} = \mu_0 K_{\lambda_0}(n)$. Then a simple computation gives

$$\begin{aligned} & \underline{N}_t - n\alpha(n) r^{n-1} \Big[\Big(\frac{\underline{N}'}{n\alpha(n) r^{n-1}} \Big)^m \Big]' - \frac{\underline{N}'\underline{N}}{n\alpha(n) r^{n-1}} \\ = & \ \underline{N} \frac{n\lambda(t) r^{-2}}{1+\lambda^2(t) r^{-2}} \Big(-\lambda'(t) + B(t) \frac{(\mu_0 K_{\lambda_0}(n))^{m-1} \lambda(t) r^{-n}}{n\alpha(n)(1+\lambda^2(t) r^{-2})^{\frac{n}{2}}} \Big), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} & B(t) = 2n^2 \alpha(n)^{2-m} \lambda^{2m-2}(t) - (\mu_0 K_{\lambda_0}(n))^{2-m} \\ \le & \ 2n^2 \alpha(n)^{2-m} \lambda_0^{\frac{2(n-2)}{n+2}} (1-\mu_0^{2-m}) := B_0 < 0. \end{aligned} \tag{2.17}$$

By (2.16) and (2.17), we have

$$\begin{aligned}
&\underline{N}_t - n\alpha(n)r^{n-1}\Big[\big(\frac{\underline{N}'}{n\alpha(n)r^{n-1}}\big)^m\Big]' - \frac{\underline{N}'\underline{N}}{n\alpha(n)r^{n-1}} \\
\le\ & \underline{N}\frac{n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\Big(-\lambda'(t) + B_0\frac{(\mu_0 K_{\lambda_0}(n))^{m-1}\lambda(t)r^{-n}}{n\alpha(n)(1+\lambda^2(t)r^{-2})^{\frac{n}{2}}}\Big) \\
\le\ & \underline{N}\frac{n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\Big(-\lambda'(t) + B_0\frac{(\mu_0 K_{\lambda_0}(n))^{m-1}\lambda(t)R_0^{-n}}{n\alpha(n)\big(1+\lambda^2(t)R_0^{-2}\big)^{\frac{n}{2}}}\Big) \\
\le\ & \underline{N}\frac{n\lambda(t)r^{-2}}{1+\lambda^2(t)r^{-2}}\Big(-\lambda'(t) + B_1\lambda(t)\Big) = 0, \qquad (2.18)
\end{aligned}$$

where we have used the fact that $\lambda'(t) = B_1\lambda(t)$, i.e., $\lambda(t) = \lambda_0 e^{B_1 t}$ with

$$B_1 = B_0(\mu_0 K_{\lambda_0}(n))^m (m_0 n\alpha(n))^{-1} R_0^{-n} < 0. \qquad (2.19)$$

Now $\forall t > 0$, we have

$$M(t,r) \ge \underline{N}(t,r) \;=\; \frac{\mu_0 K_{\lambda_0}(n)}{(1+\lambda^2(t)r^{-2})^{n/2}} = \frac{K_{\lambda_0}(n)}{[1+(\lambda_0 e^{B_1 t})^2 r^{-2}]^{n/2}}.$$

Furthermore we can take $r(t) = \lambda_0 e^{B_1 t}$

$$M(t,r(t)) \ge \underline{N}(t,r(t)) = \frac{K_{\lambda_0}(n)}{2^{n/2}} > 0.$$

This completes the proof of the lemma. □

3. Existence and blow-up with general initial data

We will discuss the existence and blow up of the solution with more general initial data, not limited to the radially symmetric case. The main tools in this part are the entropy inequality and second moment estimate. For the sake of simplicity, we will use notations L^p to represent $L^p(\mathbb{R}^n)$, L^rL^p to denote $L^r((0,+\infty), L^p(\mathbb{R}^n)$ and $L^r_TL^p$ for $L^r((0,T), L^p(\mathbb{R}^n)$.

3.1. **Global existence.** In this subsection, we will prove the following theorem on global existence of weak solution of (1.1) if the initial data satisfies

$$\|\rho_0\|_{L^m} < C_s = \Big(\frac{4m^2}{(2m-1)^2 C_{GNS}}\Big)^{\frac{1}{2-m}},$$

where C_{GNS} is the universal constant appeared in Gagliardo-Nirenberg-Sobolev inequality.

Theorem 3.1. *For initial date $\rho_0 \in L^1_+ \cap L^m$ and $\|\rho_0\|_{L^m} < C_s$, there is a global weak solution to* (1.1)*. Moreover $\|\rho(\cdot,t)\|_{L^m}$ decays algebraically,*

$$\|\rho(\cdot,t)\|_{L^m} \le Ct^{-\frac{1}{m(\beta-1)}}, \qquad \textit{for large } t, \qquad (3.1)$$

where $\beta = \frac{2m^2-3m+2}{m(m-1)} > 1$.

Proof. We split the proof into five steps.

Step 1. We start with the regularized problem, for $\varepsilon > 0$,

$$\begin{cases} \partial_t \rho_\varepsilon = \Delta \rho_\varepsilon^m + \varepsilon \Delta \rho_\varepsilon - \mathrm{div}(\rho_\varepsilon \nabla c_\varepsilon), & x \in \mathbb{R}^n, t \geq 0, \\ -\Delta c_\varepsilon = J_\varepsilon * \rho_\varepsilon & x \in \mathbb{R}^n, t \geq 0, \\ \rho(x,0) = \rho_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.2}$$

where J_ε is a mollifier with radius ε. We know from parabolic theory that the above regularized problem has a global smooth positive solution u_ε for $t > 0$ if the initial data is nonnegative.

Step 2. We will show that

$$\rho_\varepsilon \in L^\infty L^m \cap L^{m+1} L^{m+1}, \qquad \nabla \rho_\varepsilon^{m-\frac{1}{2}} \in L^2 L^2 \tag{3.3}$$

$$\nabla c_\varepsilon \in L^\infty L^2, \qquad \partial_t \rho_\varepsilon \in L^2_T W^{-1,p}. \tag{3.4}$$

Taking $m\rho_\varepsilon^{m-1}$ as a test function in (1.1), we have

$$\begin{aligned} & \frac{d}{dt}\int \rho_\varepsilon^m dx + \frac{4m^2(m-1)}{(2m-1)^2}\int \left|\nabla \rho_\varepsilon^{m-\frac{1}{2}}\right|^2 dx + \varepsilon \frac{4(m-1)}{m}\int \left|\nabla \rho_\varepsilon^{\frac{m}{2}}\right|^2 dx \\ = \ & -(m-1)\int \nabla \rho_\varepsilon^m \nabla c_\varepsilon dx \end{aligned} \tag{3.5}$$

The right hand side can be estimated by Gagliardo-Nirenberg-Sobolev inequality

$$\begin{aligned} & (m-1)\int \rho_\varepsilon^m J_\varepsilon * \rho_\varepsilon dx \leq (m-1)\int \rho_\varepsilon^{m+1} dx \\ = \ & (m-1)\left\|\rho_\varepsilon^{m-\frac{1}{2}}\right\|_{L^{\frac{m+1}{m-\frac{1}{2}}}}^{\frac{m+1}{m-\frac{1}{2}}} \\ \leq \ & (m-1)C_{GNS}\left\|\nabla \rho_\varepsilon^{m-\frac{1}{2}}\right\|_{L^2}^2 \left\|\rho_\varepsilon^{m-\frac{1}{2}}\right\|_{L^{\frac{m}{m-\frac{1}{2}}}}^{\frac{2(2-m)}{2m-1}} \\ \leq \ & (m-1)C_{GNS}\left\|\nabla \rho_\varepsilon^{m-\frac{1}{2}}\right\|_{L^2}^2 \|\rho_\varepsilon\|_{L^m}^{2-m} \end{aligned} \tag{3.6}$$

If the last term of (3.6) can be strictly dominated by the second term of (3.5) which can be realized by taking initial data

$$\|\rho_0\|_{L^m} < C_s = \left(\frac{4m^2}{(2m-1)^2 C_{GNS}}\right)^{\frac{1}{2-m}},$$

then we can close the estimate. In other words, if we choose ρ_0 such that

$$(m-1)\left(-C_{GNS}\|\rho_0\|_{L^m}^{2-m} + \frac{4m^2}{(2m-1)^2}\right) := \delta > 0,$$

then we can obtain the estimate,

$$\frac{d}{dt}\int \rho_\varepsilon^m dx + \delta \int \left|\nabla \rho_\varepsilon^{m-\frac{1}{2}}\right|^2 dx \leq 0, \tag{3.7}$$

i.e.,

$$\|\rho_\varepsilon\|_{L^m} < C_s. \tag{3.8}$$

Moreover, we also have from (3.7) that

$$\int_0^\infty \int \left|\nabla \rho_\varepsilon^{m-\frac{1}{2}}\right|^2 dx \le C_s/\delta. \tag{3.9}$$

The last inequality of (3.6) with (3.8) and (3.9), we obtain that

$$\|\rho_\varepsilon\|_{L^{m+1}L^{m+1}} < C_s. \tag{3.10}$$

Apply Young's inequality to

$$\nabla c_\varepsilon = -\frac{1}{n\alpha(n)} \int \frac{x-y}{|x-y|^n} J_\varepsilon * \rho_\varepsilon(y) dy,$$

one has

$$\|\nabla c_\varepsilon\|_{L^\infty L^2} \le C\|J_\varepsilon * \rho_\varepsilon\|_{L^\infty L^m} \le C\|\rho_\varepsilon\|_{L^\infty L^m} \le C. \tag{3.11}$$

With the help of the estimates we obtained above, we can get that

$$\nabla \rho_\varepsilon^m \in L^2 L^{\frac{2m}{m+1}},$$

$$\rho_\varepsilon \nabla c_\varepsilon \in L^{m+1} L^{\frac{2(m+1)}{m+3}},$$

and

$$\varepsilon^{1/2} \nabla \rho_\varepsilon \in L^2 L^m.$$

the estimate on $\partial_t \rho_\varepsilon$ can be easily obtained by using the equation in the sense of distribution, i.e.,

$$\partial_t \rho_\varepsilon \in L^2_T W^{-1,p}, \tag{3.12}$$

where $p = \min\{\frac{2m}{m+1}, \frac{2(m+1)}{m+3}, m\} = \frac{2(m+1)}{m+3} > 1$.

Step 3. In this step we show that

$$\nabla \rho_\varepsilon \in L^r_T L^r, \qquad \text{for all} \quad n \ge 3.$$

where $r = \min\{2, \frac{2(m+1)}{4-m}\}$. The estimate will be divided into two cases: $n < 6$ and $n \ge 6$.

For the case $n < 6$: We recast $\nabla \rho_\varepsilon$ as

$$\nabla \rho_\varepsilon = \frac{1}{m-1/2} \rho_\varepsilon^{3/2-m} \nabla \rho_\varepsilon^{m-1/2}.$$

From the above estimates $\rho_\varepsilon \in L^{m+1}L^{m+1}$ and $\nabla \rho_\varepsilon^{m-1/2} \in L^2 L^2$, by Hölder's inequality, we have

$$\begin{aligned} \int |\nabla \rho_\varepsilon|^{\frac{2(m+1)}{4-m}} dx \qquad &= C \int |\rho_\varepsilon^{3/2-m}|^{\frac{2(m+1)}{4-m}} |\nabla \rho_\varepsilon^{m-1/2}|^{\frac{2(m+1)}{4-m}} dx \\ &\le C\Big(\int |v|^{\frac{2(m+1)}{4-m}p} dx\Big)^{1/p} \Big(\int |\nabla \rho_\varepsilon^{m-1/2}|^{\frac{2(m+1)}{4-m}q} dx\Big)^{1/q}, \end{aligned}$$

where $v := \rho_\varepsilon^{3/2-m} \in L^{\frac{m+1}{3/2-m}}L^{\frac{m+1}{3/2-m}}$, with $\frac{m+1}{3/2-m} > 2$. We choose $p = \frac{4-m}{3-2m} > 1$ and $q = \frac{4-m}{m+1} > 1$ such that $\frac{2(m+1)}{4-m}q = 2$. Hence we have

$$\|\nabla\rho_\varepsilon\|_{L^{\frac{2(m+1)}{4-m}}} \le C\|v\|_{L^{\frac{m+1}{3/2-m}}}\|\nabla\rho_\varepsilon^{m-1/2}\|_{L^2}.$$

Furthermore, by Hölder's inequality for time integration, it follows that

$$\begin{aligned}\int_0^\infty \|\nabla\rho_\varepsilon\|^{\frac{2(m+1)}{4-m}}_{L^{\frac{2(m+1)}{4-m}}}dt \quad &\le C(\int_0^\infty \|v\|^{\frac{2(m+1)}{4-m}p}_{L^{\frac{m+1}{3/2-m}}}dt)^{1/p}(\int_0^\infty \|\nabla\rho_\varepsilon^{m-1/2}\|^{\frac{2(m+1)}{4-m}q}_{L^2}dt)^{1/q}\\ &= C(\int_0^\infty \|v\|^{\frac{m+1}{3/2-m}}_{L^{\frac{m+1}{3/2-m}}}dt)^{1/p}(\int_0^\infty \|\nabla\rho_\varepsilon^{m-1/2}\|^2_{L^2}dt)^{1/q}\\ &\le C\end{aligned}$$

where p and q are the same as before. Thus,

$$\nabla\rho_\varepsilon \in L^{\frac{2(m+1)}{4-m}}L^{\frac{2(m+1)}{4-m}}.$$

Combining with the fact that $\rho_\varepsilon \in L^\infty(L^1 \cap L^m) \cap L^{m+1}L^{m+1}$, we deduce that

$$\rho_\varepsilon \in L^{\frac{2(m+1)}{4-m}}W^{1,\frac{2(m+1)}{4-m}}.$$

In the case $n \ge 6$, Taking inner product of (1.1) with ρ_ε^{2-m}, one has

$$\begin{aligned}&\frac{1}{3-m}\frac{d}{dt}\int \rho_\varepsilon^{3-m}dx + m(2-m)\int |\nabla\rho_\varepsilon|^2\,dx + \varepsilon(2-m)\int \rho_\varepsilon^{1-m}|\nabla\rho_\varepsilon|^2\,dx\\ = &\int \rho_\varepsilon\nabla c_\varepsilon\cdot\nabla\rho_\varepsilon^{2-m}dx = \frac{2-m}{3-m}\int \nabla c_\varepsilon\cdot\nabla\rho_\varepsilon^{3-m}dx \le C\int \rho_\varepsilon^{4-m}dx.\end{aligned}$$

Now we only need to estimate $\int \rho_\varepsilon^{4-m}dx$. Let $u := \rho_\varepsilon^{m-1/2}$, we use $u \in L^\infty L^{\frac{m}{m-1/2}}$ which is exactly $\rho_\varepsilon \in L^\infty L^m$. From (3.9), we have $\nabla u \in L^2L^2$. By Gagliardo Nirenberg Sobolev inequality,

$$\int \rho_\varepsilon^{4-m}dx = \int u^{\frac{4-m}{m-1/2}}dx \le C\|\nabla u\|^{\theta\frac{4-m}{m-1/2}}_{L^2}\|u\|^{(1-\theta)\frac{4-m}{m-1/2}}_{L^{\frac{m}{m-1/2}}} = C\|\nabla u\|^{\frac{16}{n+2}}_{L^2}\|u\|^{(1-\theta)\frac{4-m}{m-1/2}}_{L^{\frac{m}{m-1/2}}},$$

where $0 < \theta = \frac{4(3n-2)}{(n+2)(n+4)} < 1$. Hence for any fix $T > 0$, we have

$$\int_0^T\int \rho_\varepsilon^{4-m}dxdt \le C\int_0^T \|\nabla u\|^{\frac{16}{n+2}}_{L^2}\|u\|^{(1-\theta)\frac{4-m}{m-1/2}}_{L^{\frac{m}{m-1/2}}}dt \le C(\|u\|_{L^\infty L^{\frac{m}{m-1/2}}}, \|\nabla u\|_{L^2L^2}, T),$$

where we have used $\frac{16}{n+2} \le 2$ for $n \ge 6$. Consequently, for any fixed $T > 0$,

$$\begin{aligned}\frac{1}{(3-m)}\int \rho_\varepsilon^{3-m}dx \quad &+ \quad m(2-m)\int_0^T\int |\nabla\rho_\varepsilon|^2\,dxdt + \varepsilon(2-m)\int_0^T\int \rho_\varepsilon^{1-m}|\nabla\rho_\varepsilon|^2\,dxdt\\ &\le \quad \frac{1}{(3-m)}\|\rho_0\|^{3-m}_{L^{3-m}} + C \le C(\|\rho_0\|_{L^m}, \|\rho_0\|_{L^1}) + C.\end{aligned}$$

In above we have used the fact that $3 - m \le m$ in the case of $n \ge 6$.

Thus we have

$$\int_0^T \int |\nabla \rho_\varepsilon|^2 \, dxdt \leq C, \text{ for any fixed } T > 0.$$

Hence for all $n \geq 3$, it holds that

$$\rho_\varepsilon \in L_T^r W_{loc}^{1,r}. \tag{3.13}$$

where $r = \min\{2, \frac{2(m+1)}{4-m}\}$.

Step 4. From (3.12) and (3.13) and Lions-Aubin's lemma that there exists a sequence still labeled as ρ_ε such that

$$\rho_\varepsilon \to \rho \qquad in\ L_T^r L_{loc}^{\bar{p}},$$

where $\bar{p} = \min\{\frac{(3n+2)n}{n^2+n-2}, \frac{2n}{n-2}\} > 2$. This leads to existence of a global weak solution.

Step 5. In this step, we prove that the global weak solution obtained in step 4 decays to zero as $t \to \infty$.

By Gagliardo Nirenberg Sobolev inequality,

$$\begin{aligned} \int \rho^{m+1} dx &= \left\| \rho^{m-\frac{1}{2}} \right\|_{L^{\frac{m+1}{m-\frac{1}{2}}}}^{\frac{m+1}{m-\frac{1}{2}}} \\ \leq\ & C_{GNS} \left\| \nabla \rho^{m-\frac{1}{2}} \right\|_{L^2}^2 \left\| \rho^{m-\frac{1}{2}} \right\|_{L^{\frac{m}{m-\frac{1}{2}}}}^{\frac{2(2-m)}{2m-1}} = C_{CNS} \left\| \nabla \rho^{m-\frac{1}{2}} \right\|_{L^2}^2 \|\rho\|_{L^m}^{2-m}. \end{aligned} \tag{3.14}$$

Or equivalently,

$$\left\| \nabla \rho^{m-\frac{1}{2}} \right\|_{L^2}^2 \geq \frac{1}{C_{GNS} \|\rho\|_{L^m}^{2-m}} \int \rho^{m+1} dx.$$

We have the following inequality for weak solution,

$$\frac{d}{dt} \int \rho^m dx \leq -\delta \int |\nabla \rho^{m-\frac{1}{2}}|^2 dx \leq -\frac{\delta}{C_{GNS} \|\rho\|_{L^m}^{2-m}} \int \rho^{m+1} dx.$$

On the other hand, we have

$$\|\rho\|_{L^m} \leq \|\rho\|_{L^{m+1}}^\theta \|\rho\|_{L^1}^{1-\theta}, \qquad \theta = \frac{m^2-1}{m^2},$$

Combining with the previous inequality, we have an inequality for $\|\rho\|_{L^m}$,

$$\frac{d}{dt} \int \rho^m dx \leq -\frac{\delta}{C_{GNS}} \|\rho\|_{L^m}^{m-2} \|\rho\|_{L^m}^{\frac{m^2}{m-1}} \|\rho\|_{L^1}^{\frac{1}{1-m}} = -C_d \left(\int \rho^m dx \right)^\beta,$$

where $C_d = \dfrac{\delta}{C_{GNS}} \|\rho\|_{L^1}^{\frac{1}{1-m}}$, $\beta = \frac{2m^2-3m+2}{m(m-1)} > 1$.

Then by solving this ordinary differential inequality, we have

$$\|\rho(\cdot, t)\|_{L^m} \leq \left(\frac{1}{(\beta-1) C_d t + \|\rho_0\|_{L^m}^{m(1-\beta)}} \right)^{\frac{1}{m(\beta-1)}}.$$

which implies that the solution decays to zero in L^m norm as $t \to \infty$.

$$\|\rho(\cdot,t)\|_{L^m} \le Ct^{-\frac{1}{m(\beta-1)}}, \qquad \text{for large } t.$$

□

3.2. **blow-up of general solution.** In this subsection, we will discuss the blow-up of the solution when $\|\rho_0\|_{L^m} > \|U_{\lambda,x_0}\|_{L^m}$ and $\mathcal{F}(\rho_0) < \mathcal{F}(U_{\lambda,x_0})$.

Recall that (1.17) gives a decomposition of the free energy

$$\begin{aligned} \mathcal{F}(\rho) &= \frac{1}{m-1}\|\rho\|_{L^m}^m \left(1 - \frac{(m-1)c_n C(n)}{2}\|\rho\|_{L^m}^{4/(n+2)}\right) \\ &\quad + \frac{c_n}{2}\left(C(n)\|\rho\|_{L^m}^2 - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\rho(x)\rho(y)}{|x-y|^{n-2}}dxdy\right) \\ &:= \mathcal{F}_1(\rho) + \mathcal{F}_2(\rho). \end{aligned}$$

where $c_n = 1/(n(n-2)\alpha(n))$. where $\mathcal{F}_2(\rho) \ge 0$ from Hardy-Littlewood-Sobolev's inequality and

$$\mathcal{F}_1(\rho) = f(\|\rho\|_{L^m}^m), \qquad f(s) = \frac{1}{m-1}s - \frac{c_n}{2}C(n)s^{\frac{2}{m}}$$

As we already mentioned in the introduction that $U_{\lambda,x_0}(x)$ is a critical point for both $\mathcal{F}(\rho)$ and $\mathcal{F}_2(\rho)$. Hence it is also a critical point for $\mathcal{F}_1(\rho)$. In the following lemma, we show that $\|U_{\lambda,x_0}\|_{L^m}^m$ is indeed a maximum point for $f(s)$.

Lemma 3.1. *$f(s)$ is a strict concave function and it reaches the unique maximum point at*

$$s^* = \|U_{\lambda,x_0}\|_{L^m}^m = \Big(\frac{2n^2\alpha(n)}{C(n)}\Big)^{\frac{n}{2}}.$$

where U_{λ,x_0} is any stationary solutions of the equation (1.1).

Proof. Take first and second order derivatives for $f(s)$, one has $f'(s) = \dfrac{1}{m-1} - \dfrac{c_n}{m}C(n)s^{\frac{2-m}{m}}$ and $f''(s) = -\dfrac{c_n(2-m)}{m^2}C(n)s^{\frac{2-m}{m}-1} < 0$ for all $s > 0$. Thus $f(s)$ attains its maximum at

$$s^* = \Big(\frac{2n^2\alpha(n)}{C(n)}\Big)^{\frac{n}{2}}.$$

Now we show that $s^* = \|U_{\lambda,x_0}\|_{L^m}^m$. By the formula of free energy with $\rho = U_{\lambda,x_0}$

$$\mathcal{F}(U_{\lambda,x_0}) = \frac{1}{m-1}\|U_{\lambda,x_0}\|_{L^m}^m - \frac{1}{2}\int_{\mathbb{R}^n} U_{\lambda,x_0}C_{\lambda,x_0}dx,$$

and the critical case of Hardy-Littlewood-Sobolev's inequality (1.15)

$$\begin{aligned}\mathcal{F}(U_{\lambda,x_0}) &= \frac{1}{m-1}\|U_{\lambda,x_0}\|_{L^m}^m - \frac{c_n}{2}\int\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{U_{\lambda,x_0}(x)U_{\lambda,x_0}(y)}{|x-y|^{n-2}}dxdy,\\ &= \frac{1}{m-1}\|U_{\lambda,x_0}\|_{L^m}^m - \frac{1}{2(n-2)n\alpha(n)}C(n)\|U_{\lambda,x_0}\|_{L^m}^2,\end{aligned}$$

we have

$$\int_{\mathbb{R}^n} U_{\lambda,x_0}C_{\lambda,x_0}dx = \frac{1}{(n-2)n\alpha(n)}C(n)\|U_{\lambda,x_0}\|_{L^m}^2.$$

Notice that $C_{\lambda,x_0} = \dfrac{2n}{n-2}U_{\lambda,x_0}^{m-1}$ as in the introduction. Thus we have

$$2n^2\alpha(n)\|U_{\lambda,x_0}\|_{L^m}^m = C(n)\|U_{\lambda,x_0}\|_{L^m}^2,$$

from which we have

$$\|U_{\lambda,x_0}\|_{L^m}^m = \Big(\frac{2n^2\alpha(n)}{C(n)}\Big)^{\frac{n}{2}} = s^*.$$

□

Before state and prove our main theorem in this section, let us first prove the following technical lemma.

Lemma 3.2. *Assume $\mathcal{F}(\rho_0) < \mathcal{F}(U_{\lambda,x_0})$, $\|\rho_0\|_{L^m} > \|U_{\lambda,x_0}\|_{L^m}$ and ρ is solution of (1.1), then there is $\mu > 1$ such that ρ satisfy*

$$\|\rho\|_{L^m}^m > \mu\|U_{\lambda,x_0}\|_{L^m}^m.$$

Proof. Since $\mathcal{F}(\rho_0) < \mathcal{F}(U_{\lambda,x_0})$, we can choose $\delta : 0 < \delta < 1$ such that $\mathcal{F}(\rho_0) < \delta\mathcal{F}(U_{\lambda,x_0})$. by (1.17) with Hardy-Littlewood-Sobolev's inequality (1.15) and the monotone non-increasing of $\mathcal{F}(\rho(\cdot,t))$ for t, we have

$$f(\|\rho\|_{L^m}^m) = \mathcal{F}_1(\rho) \le \mathcal{F}(\rho) \le \mathcal{F}(\rho_0) < \delta\mathcal{F}(U_{\lambda,x_0}) = \delta f(s^*).$$

Then for any $s > \|U_{\lambda,x_0}\|_{L^m}^m$, $f(s)$ is a strictly decreasing function. Then it has a strictly decreasing inverse function f^{-1}. Hence if $\|\rho_0\|_{L^m} > \|U_{\lambda,x_0}\|_{L^m}$, we have for some $\mu > 1$,

$$\|\rho\|_{L^m}^m > \mu\|U_{\lambda,x_0}\|_{L^m}^m.$$

□

Theorem 3.2. *Assume $m_2(0) = \int_{\mathbb{R}^n}|x|^2\rho_0(x)dx < \infty$, $\mathcal{F}(\rho_0) < \mathcal{F}(U_{\lambda,x_0})$ and $\|\rho_0\|_{L^m} > \|U_{\lambda,x_0}\|_{L^m(\mathbb{R}^n)}$, then the solutions of* (1.1) *develop singularities, i.e., the solutions of* (1.1) *is blowing up in a finite time.*

Proof. Here we use the formula

$$\nabla c = -\frac{1}{n\alpha(n)}\frac{x}{|x|^n} * \rho(x).$$

By Lemma 3.2, (1.11) and the monotonicity of free energy, we deduce that

$$\begin{aligned} \frac{d}{dt}m_2(t) &\le -4\mu\|U_{\lambda,x_0}\|_{L^m}^m + 2(n-2)\mathcal{F}(\rho_0) \\ &\le -4\mu\|U_{\lambda,x_0}\|_{L^m}^m + 2(n-2)\mathcal{F}(U_{\lambda,x_0}) \\ &= -4(\mu-1)\|U_{\lambda,x_0}\|_{L^m}^m < 0, \end{aligned}$$

where we have used $\mathcal{F}(U_{\lambda,x_0}) = \frac{2}{n-2}\|U_{\lambda,x_0}\|_{L^m}^m$ in the third equality, see (1.18). This means that there is a $T>0$ such that $\lim_{t\to T} m_2(t) = 0$.

On the other hand, $\forall R>0$, by using Hölder inequality, we have

$$\int_{\mathbb{R}^n} \rho(x)dx \le \int_{B_R} \rho(x)dx + \int_{B_R^c} \rho(x)dx \le CR^{(n-2)/2}\|\rho\|_{L^m} + \frac{1}{R^2}m_2(t).$$

Now by choosing $R = (\frac{m_2(t)}{C\|\rho\|_{L^m}})^{2/(n+2)}$, we have

$$\|\rho\|_{L^1} \le C\|\rho\|_{L^m}^{\frac{4}{n+2}} m_2(t)^{\frac{n-2}{n+2}}.$$

So, $\lim_{t\to T}\|\rho\|_{L^m}^{\frac{4}{n+2}} \ge \lim_{t\to T} \dfrac{\|\rho\|_{L^1}}{\bar{C}(n)m_2(t)^{\frac{n-2}{n+2}}} = \infty.$ □

Remark 3.1. There is a gap between C_s and $\|U_{\lambda,x_0}\|_{L^m}$. So there is still a space, in the case that $C_s \le \|u_0\|_{L^m} \le \|U_{\lambda,x_0}\|_{L^m}$ that we don't know the solution exists or blows up. See appendix for the comparison between C_s and $\|U_{\lambda,x_0}\|_{L^m}$.

4. Appendix

4.1. **Proof of Prop. 1.4.** The invariance of the free energy $F(\rho)$ is obvious in the translation transformation $\rho_{\bar{x}}(x) = \rho(x+\bar{x})$ and the orthogonal transformation $\mathcal{R}\rho(x) = \rho(\mathcal{R}^{-1}x)$. Since the determinant to Jacobian matric is 1 for the two kinds of transformation.

By the scaling transformation $\rho_\lambda(x) = \lambda^{\frac{n+2}{2}}\rho(\lambda x)$ and direct computation, we have

$$\begin{aligned} \mathcal{F}(\rho_\lambda) &= \int_{\mathbb{R}^n} \frac{\lambda^{\frac{(n+2)m}{2}}\rho^m(\lambda x)}{m-1}dx - \frac{c_n}{2}\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\lambda^{n+2}\rho(\lambda x)\rho(\lambda y)}{|x-y|^{n-2}}dxdy \\ &= \int_{\mathbb{R}^n} \frac{\lambda^n\rho^m(\lambda x)}{m-1}dx - \frac{c_n}{2}\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho(\lambda x)\rho(\lambda y)}{|x-y|^{n-2}\lambda^{n-2}}\lambda^{n-2}\lambda^{n+2}dxdy \\ &= \mathcal{F}(\rho). \end{aligned}$$

Notice that the Kalvin transformation of c is

$$c_{\bar{x},\lambda}(x) = \Big(\frac{\lambda}{|x-\bar{x}|}\Big)^{n-2} c\Big(\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\Big),$$

we have the related transformation for ρ in the following

$$\rho_{\bar{x},\lambda}(x) = \Big(\frac{\lambda}{|x-\bar{x}|}\Big)^{n+2} \rho\Big(\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\Big).$$

Thus the free energy for $\rho_{\bar{x},\lambda}$ is

$$\mathcal{F}(\rho_{\bar{x},\lambda}) = \int \frac{1}{m-1}\Big(\frac{\lambda}{|x-\bar{x}|}\Big)^{(n+2)m}\Big[\rho\Big(\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\Big)\Big]^m dx$$
$$-\frac{c_n}{2}\int\int \frac{1}{|x-y|^{n-2}}\rho\Big(\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\Big)\rho\Big(\bar{x}+\frac{\lambda^2(y-\bar{x})}{|y-\bar{x}|^2}\Big)\Big(\frac{\lambda^2}{|x-\bar{x}||y-\bar{x}|}\Big)^{n+2}dxdy,$$
$$= I_1 - \frac{c_n}{2}I_2.$$

Here

$$\begin{aligned} I_1 &= \int \frac{1}{m-1}\Big(\frac{\lambda}{|x-\bar{x}|}\Big)^{2n}\Big[\rho\Big(\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\Big)\Big]^m dx \\ &= \int \frac{1}{m-1}\Big[\rho\Big(\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\Big)\Big]^m d\Big(\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\Big) \\ I_2 &= \int\int \Big[\frac{\lambda^2|x-y|}{|x-\bar{x}|\cdot|y-\bar{x}|}\Big]^{2-n}\rho(z)\rho(w)dzdw \\ &= \int\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho(z)\rho(w)}{|z-w|^{n-2}}dzdw. \end{aligned}$$

where $z = \bar{x}+\dfrac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}$, $w = \bar{x}+\dfrac{\lambda^2(y-\bar{x})}{|y-\bar{x}|^2}$, and we have used

$$|z-w| = \Big|\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2} - \frac{\lambda^2(y-\bar{x})}{|y-\bar{x}|^2}\Big| = \frac{\lambda^2|x-y|}{|x-\bar{x}|\cdot|y-\bar{x}|}.$$

Hence we have

$$\mathcal{F}(\rho_{\bar{x},\lambda}) = I_1 - \frac{c_n}{2}I_2 = \int \frac{\rho^m(z)}{m-1}dz - \frac{c_n}{2}\int\int \frac{\rho(z)\rho(w)}{|z-w|^{n-2}}dzdw = \mathcal{F}(\rho).$$

4.2. **gap between C_s and $\|U_{\lambda,x_0}\|_{L^m}$.** Since L^m norm of U_{λ,x_0} doesn't depend on λ and x_0, we will use $\|U\|_{L^m} = \|U_{\lambda,x_0}\|_{L^m}$ in the following. Among the estimates in the proof of Theorem 3.1, we used only an important Gagliardo-Nirenberg-Sobolev inequality which is

$$\|v\|_{L^r} \le \|v\|_{L^{2^*}}^{\theta}\|v\|_{L^q}^{1-\theta} \le \bar{C}_{GNS}\|\nabla v\|_{L^2}^{\theta}\|v\|_{L^q}^{1-\theta},$$

where in our case, $r = \frac{2(m+1)}{2m-1}$, $\theta = \frac{2m-1}{m+1}$ and $C_{GNS} = (\bar{C}_{GNS})^{\frac{m+1}{m-\frac{1}{2}}}$.

From Lieb-[8] P202, we know the best constant for Sobolev embedding

$$\|\nabla f\|_{L^2}^2 \ge S_n\|f\|_{L^{2^*}}^2,$$
$$S_n = \frac{n(n-2)}{4}2^{\frac{2}{n}}\pi^{1+\frac{1}{n}}\Gamma(\frac{n+1}{2})^{-\frac{2}{n}}$$

hence, $C_{GNS} = \left(S_n^{-\frac{\theta}{2}}\right)^{\frac{m+1}{m-\frac{1}{2}}}$. We can calculate that $C_s < \|U\|_m$. In fact,

$$\begin{aligned}
& C_s - \|U\|_{L^m} \\
= \;& \Big(\frac{4m^2}{(2m-1)^2 C_{GNS}}\Big)^{\frac{1}{2-m}} - \Big(n^n \pi^{\frac{n+1}{2}} 2^{1-\frac{n}{2}} \Gamma^{-1}(\frac{n+1}{2})\Big)^{\frac{1}{m}} \\
= \;& \Big(\frac{4m^2}{(2m-1)^2 \frac{4}{n(n-2)} 2^{-\frac{2}{n}} \pi^{-(1+\frac{1}{n})} \Gamma^{\frac{2}{n}}(\frac{n+1}{2})}\Big)^{\frac{n+2}{4}} - \Big(n^n \pi^{\frac{n+1}{2}} 2^{1-\frac{n}{2}} \Gamma^{-1}(\frac{n+1}{2})\Big)^{\frac{n+2}{2n}} \\
= \;& \Big[\Big(\frac{m^2(n-2)}{(2m-1)^2}\Big)^{\frac{n+2}{4}} - n^{\frac{n+2}{4}} 2^{-\frac{n+2}{4}}\Big] 2^{\frac{n+2}{2n}} \pi^{\frac{(n+1)(n+2)}{4n}} \Gamma^{-\frac{n+2}{2n}}(\frac{n+1}{2}) n^{\frac{n+2}{4}} < 0
\end{aligned}$$

due to $\dfrac{m^2(n-2)}{(2m-1)^2} < n/2$ for all $n \geq 3$.

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