

Static equations with positive cosmological constant

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Abstract. A proof of the uniqueness of the de Sitter space-time without unnecessary assumptions is still not available in the published literature. This space-time is the only solution in the static class in the absence of black holes satisfying Einstein equations ${}^4R_{\alpha\beta} = \Lambda g_{\alpha\beta}$, where the cosmological constant Λ is positive. The problem has important relevance in differential geometry. Boucher showed that the surface gravity κ on the cosmological horizon satisfies $\frac{3}{\Lambda}\kappa^2 \geq 1$ and when the equality holds the solution is the de Sitter space-time. Remaining part of the proof is to establish the opposite inequality. In this note we show that a numerical upper bound for $\frac{3}{\Lambda}\kappa^2$ naturally follows from several well-known inequalities. The bound itself is too crude and useless. However the connection with bounds on the Yamabe constant, isoperimetric inequality, a Penrose type inequality, and the Sobolev constant possibly tells something about the differential geometric importance of the equations obtained by the usual 3+1 decomposition of the static Einstein equations with positive Λ . It also tells about the present state of our understanding of constant scalar curvature metrics in 3-dimension.

Key words. de Sitter space-time, positive cosmological constant, constant scalar curvature 3-metric

1 Introduction

The de Sitter space-time belongs to a class of static space-times satisfying Einstein equations ${}^4R_{\alpha\beta} = \Lambda g_{\alpha\beta}$, where the cosmological constant Λ is positive. Representing the static metric 4g by $-V^2 dt^2 + g$, where g is the

induced Riemannian 3-metric on an open orientable constant time hypersurface Σ^+ , and V and g are independent of time t , the Einstein equations become equivalent to

$$R_{ij} = V^{-1}V_{;ij} + \Lambda g_{ij}, \quad \Delta V = -\Lambda V \quad (1)$$

Here $;$ denotes covariant derivative and Δ denotes the Laplacian relative to g . V and g are assumed to be regular on the compact manifold with boundary $\Sigma^+ \cup \partial\Sigma^+$. $V > 0$ in Σ^+ and $V = 0$ on the boundary $\partial\Sigma^+$. It is known that $\partial\Sigma^+$ is then totally geodesic. We shall attach another copy of Σ^+ along the totally geodesic boundary $\partial\Sigma^+$ replacing V on the new copy by its opposite so that in the interior of the new copy Σ^- , g and V satisfy the same equations Eq. (1). Then we assume that both V and g extend at least in $C^{1,1}$ fashion across $\partial\Sigma^+$. In short we assume V and g are C^3 in Σ^+ and globally $C^{1,1}$ in the compact manifold $\Sigma^+ \cup \partial\Sigma^+ \cup \Sigma^-$. The field equations will then improve the regularity but we shall not use those results, and leave the concern regarding required minimum regularity issues to other investigators. We also assume that the maximum value of V is 1.

Let $W = |\nabla V|^2$ where the norm is relative to g , and on $\partial\Sigma^+$, $W = \kappa^2$. It is known that κ^2 is a constant on each smooth component of $\partial\Sigma^+$, and $\kappa^2 > 0$ when $\partial\Sigma^+$ is a smooth connected surface. One hopes to prove the following result([1], [2], [3]).

Conjecture 1.1. *Suppose $\partial\Sigma^+$ is diffeomorphic to a 2-sphere. The only solution (V, g) of the field equations Eq. (1) on Σ^+ is the de Sitter solution. That is, the de Sitter space-time metric*

$$ds^2 = -(1 - \Lambda r^2/3)dt^2 + (1 - \Lambda r^2/3)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (2)$$

is the only static solution of $\overset{4}{R}_{\alpha\beta} = \Lambda \overset{4}{g}_{\alpha\beta}$, $\Lambda > 0$ such that the interior of the event horizon is diffeomorphic to the product of an open ball in \mathbb{R}^3 with the time line.

A short history of the uniqueness problem for the de Sitter space-time is as follows. Boucher and Gibbons [1] gave an elliptic formulation for

the uniqueness problem of the de Sitter space-time. Boucher, Gibbons and Horowitz [2] proved the uniqueness of the anti-de Sitter space-time ($\Lambda < 0$). In the same paper an inequality was proved for the case $\Lambda > 0$ where equality implies the uniqueness. Since then the uniqueness of the de Sitter space-time has been proved by many authors under various conditions and in different dimensions. Boucher [3] and Friedrich [4] proved uniqueness assuming Penrose compactification of the space-time and certain condition on the conformal infinity. Galloway [5] proved the uniqueness when the space-time contains a null line. Shiromizu [6], Shiromizu, Ida and Torii [7] studied mass and energy for asymptotically de Sitter space-times in relation to uniqueness and cosmic no-hair conjectures. Chrusciel [8] proved some results valid in a general dimension $n \geq 3$ and with some conditions on the boundary. However the elliptic problem originally elaborated by Boucher and Gibbons [1] is still unsolved. This problem has important relevance in the study of constant scalar curvature Riemannian metrics and the critical points of the scalar curvature map (see Kobayashi [9], Shen [10],[11], Moncrief [12], Lafontaine [13] and Hwang [14]). Gibbons, Hartnoll and Pope [15] provided counterexamples showing non-uniqueness in some higher dimensions. Wang [16] proved the uniqueness of the anti-de Sitter space-time in higher dimensions. Kobayashi [9] and Shen [11] are interested in the general situation when the set $V = 0$ has more than one component. If g is conformally flat all complete solutions have been found by Kobayashi [9] and Lafontaine [17].

From the field equations Eq. (1) we find that the scalar curvature of g is $R = 2\Lambda$. Following lemma is possibly a restatement of results already known. Compare with Boucher [3] who used the identity in Eq. (3) below for the case of $\Lambda < 0$. Here $W_1 = \frac{\Lambda}{3}(1 - V^2)$.

Lemma 1.1. *Either g is spherically symmetric and $\kappa^2 = \Lambda/3$, or $\kappa^2 > \Lambda/3$. In any case $W - W_1 \leq \kappa^2 - \Lambda/3$.*

Proof: From Eq. (1) we get the following identity due to Lindblom [18]. Here ∇_i denotes covariant derivative for g and indices are raised by g .

$$\nabla^i \left(\frac{1}{V} \nabla_i (W - W_1) \right) = \frac{V^3}{4W} R_{abc} R^{abc} + \frac{3}{4VW} |\nabla(W - W_1)|^2 \quad (3)$$

where $R_{abc} = \nabla_c R_{ab} - \nabla_b R_{ac} + \frac{1}{4}(g_{ac}\nabla_b R - g_{ab}\nabla_c R)$ vanishes if and only if g is conformally flat. Now at the maximum of V , that is, when $V = 1$, $W - W_1 = 0$. So positive maximum of $W - W_1$ must be reached near the boundary $\partial\Sigma^+$ where $V = 0$ unless $W = W_1$ identically. Thus $(W - W_1)|_{\partial\Sigma^+} \geq 0$ or $W = W_1$ identically. When $W = W_1$ identically, g is conformally flat and ${}^{(4)}g$ can be easily shown to be the de Sitter solution. In any case $W - W_1 \leq (W - W_1)|_{\partial\Sigma^+} = \kappa^2 - \Lambda/3 \geq 0$. \square

Let

$$\mu^2 = (3/\Lambda)\kappa^2.$$

Lemma 1.1 says that $\mu \geq 1$ and the equality gives the spherical symmetry and uniqueness.

2 Upper Bound on μ

Let $\Sigma = \Sigma^+ \cup \Sigma^- \cup \partial\Sigma^+$ where Σ^- is another copy of Σ^+ attached along the totally geodesic boundary $\partial\Sigma^+$. Since $V = 0$ on $\partial\Sigma^+$, V extends to a negative function in Σ^- . If the points $p^\pm \in \Sigma$ are images of each other under doubling then $V(p^+) = -V(p^-)$. W is defined in Σ^- in terms of V and g by the same formulas as in Σ^+ . W has the same value at p^\pm .

We obtain a numerical upper bound on μ by combining following five relations:

- a bound on the volume of Σ due to a theorem of O Murchadha [19] which uses bounds on the Yamabe's constant Y proved by Trudinger [20], Aubin [21] and Schoen [22]. Since the scalar curvature is constant bounds on Y are $3(\pi^2/4)^{2/3} \geq Y \geq \frac{1}{8}R(\text{vol}(\Sigma))^{2/3}$.
- isoperimetric inequality.
- a Penrose inequality type bound on the area of $\partial\Sigma^+$ proved by Boucher, Gibbons and Horowitz [2].

- Cheng and Li estimate (Theorem 2 in [23]) on the Sobolev constant c_1 for Dirichlet boundary condition, that is, for functions vanishing on $\partial\Sigma^+$ by the first eigenvalue Λ . Λ is the first eigenvalue because being positive in Σ^+ , the corresponding eigenfunction V cannot be orthogonal to the positive first eigenfunction.
- a crude estimate $\int_{\Sigma^+} V < \text{vol}(\Sigma^+)$ which follows because $V \leq 1$. This is used to estimate the volume integral of ΔV . We denote the constant $(\text{vol}(\Sigma^+))^{-1} \int_{\Sigma^+} V$ by c_0 . The crude estimate then says $c_0 < 1$.

Equations describing the five relations are respectively as follows. $|\dots|$ denotes surface area.

$$\text{vol}(\Sigma) \leq \frac{\pi^2}{4} \left[\frac{12}{\Lambda} \right]^{3/2} \quad (4)$$

$$(\text{vol}(\Sigma^+))^{2/3} \leq c_1 |\partial\Sigma^+| \quad (5)$$

$$|\partial\Sigma^+| \leq 12\pi/\Lambda \quad (6)$$

$$\Lambda^{3/2} \geq 4 \left(\frac{c_1^2}{6} \right)^{3/2} e^{-1} (\text{vol}(\Sigma^+))^{-1} \quad (7)$$

$$\kappa |\partial\Sigma^+| = c_0 \Lambda \text{vol}(\Sigma^+) \quad (8)$$

Since $\kappa = \mu \sqrt{\frac{\Lambda}{3}}$ we get from (8) and (5), $\mu \sqrt{\frac{\Lambda}{3}} |\partial\Sigma^+| \leq c_0 \Lambda (c_1 |\partial\Sigma^+|)^{3/2}$ which gives

$$\mu \leq c_0 \sqrt{3\Lambda} (c_1)^{3/2} |\partial\Sigma^+|^{1/2} \quad (9)$$

So using (6) we get $\mu \leq c_0 \sqrt{36\pi} (c_1)^{3/2} = 6\sqrt{\pi} c_0 (c_1)^{3/2}$. Now $\text{vol}(\Sigma) = 2\text{vol}(\Sigma^+)$, and so (7) and (4) give $c_1^3 \leq 27\pi^2 e/\sqrt{2}$. Thus using the crude estimate $c_0 < 1$ we have $\mu < 18\pi \sqrt{3\pi e/\sqrt{2}} \approx 241$ (too big!).

3 Conclusion

Of the five relations first three give sharp estimates for our problem in the sense that equality holds for the de Sitter solution. The fourth one, which is

possibly mostly responsible for the big number bound, can be improved by using Cheng and Li estimates more efficiently. However the main problem is the fifth one which, unlike others, is not a general result. Although it is clear that one can reduce the bound on c_0 somewhat using estimates on W , optimal result in this direction is expected to be connected to conformal flatness and the uniqueness proof itself. At present this approach being hopeless we are looking for a general technique involving positive mass theorem and Green's function of the Yamabe operator to establish that $\mu \leq 1$.

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