CLASSIFICATION OF RECURRENT DOMAINS FOR HOLOMORPHIC MAPS ON $\mathbf{P}^3$

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Abstract. We generalize the classification of recurrent domains for holomorphic maps on $\mathbf{P}^2$ given in [FS2] to dimension three. In particular, we show that an invariant recurrent Fatou component is either an attracting basin or a Siegel domain or it retracts to a one-dimensional or two-dimensional Siegel manifold.

1. Introduction

Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map on the k-dimensional complex projective space. By definition, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{C}^{k+1}\backslash\{0\} & \xrightarrow{\tilde{f}} & \mathbf{C}^{k+1}\backslash\{0\} \\ \big\downarrow \pi & & \big\downarrow \pi \\ \mathbf{P}^k & \xrightarrow[f]{} & \mathbf{P}^k, \end{array}$$

where π is the standard projection from $\mathbf{C}^{k+1}\backslash\{0\}$ onto $\mathbf{P}^k$ and $\tilde{f}$ is the lifting of f, which is a homogeneous polynomial map of degree d. We always assume that $d \geq 2$.

The *Fatou set* $\mathcal{F}(f)$ is the maximum open set of $\mathbf{P}^k$ where the family $\{f^n\}_{n\in\mathbf{N}}$ is equicontinuous. A Fatou component is any connected component of $\mathcal{F}(f)$. In [U1], Ueda showed that any Fatou component is Kobayashi hyperbolic and Stein (cf. [FS1]).

A Fatou component Ω is *recurrent* if for some $p_0 \in \Omega$ the ω-limit set of p_0 intersects Ω, i.e. there exists $p_0 \in \Omega$ such that $f^{n_i}(p_0)$ is relatively compact in Ω for some subsequence n_i. In [FS2], the first author and Sibony gave a classification of recurrent domains for holomorphic maps on $\mathbf{P}^2$. The goal of this paper is to give a similar classification in dimension three.

A Fatou component Ω is a *Siegel domain* if there exists a subsequence $\{f^{n_i}\}$ converging uniformly on compact subsets of Ω to the identity. If $V \subset \Omega$ is a subvariety, then a *retraction* $\rho : \Omega \to V$ is a holomorphic map such that $\rho|_V = Id$. The following is our main result.

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Theorem 1.1. *Let f be a holomorphic map on $\mathbf{P}^3$ of degree $d \geq 2$. Let Ω be a recurrent Fatou component such that $f(\Omega) = \Omega$. Then one of the following happens:*
(i) *There is an attracting fixed point $p \in \Omega$, i.e. the eigenvalues $\{\lambda_i\}_{1\leq i\leq 3}$ of f' at p satisfy $|\lambda_i| < 1$, $1 \leq i \leq 3$.*
(ii) *There exist a closed m-dimensional complex (Kobayashi hyperbolic) submanifold M of Ω $(1 \leq m \leq 2)$, and a holomorphic retraction $\rho : \Omega \to M$ such that any limit h of a convergent subsequence of $\{f^n\}$ is of the form $h = \phi \circ \rho$, where $\phi \in$Aut(M).*
(iii) *The domain Ω is a Siegel domain. Any limit of a convergent subsequence of $\{f^n\}$ is an automorphism of Ω.*

We prove the theorem in the next section. First we give some remarks.

Remark 1.2. For $m = 1$ in the above theorem, we know from the one-dimensional theory ([M]) that M is biholomorphic to either a disc, a punctured disc or an annulus and $f|_M$ is an irrational rotation. Furthermore, by a result of Ueda ([U2]), the case of a punctured disc can not occur.

Remark 1.3. If Ω is a Siegel domain, let G denote the closure of $\{f^n\}_{n\geq 0}$ in the topology of uniform convergence on compact sets. Then G is a sub-Lie group of Aut(Ω) and is isomorphic to $\mathbf{T}^l \times F$ where F is a finite group and $1 \leq l \leq 3$. The proof is essentially the same as that of [FS2, Proposition 1.3]. Similarly in case (ii) of the above theorem, denoting by G_m the closure of $\{(f|_M)^n\}$, we have G_m is a sub-Lie group of Aut(M) and is isomorphic to $\mathbf{T}^l \times F$ where F is a finite group and $1 \leq l \leq m$.

Remark 1.4. In case (ii) of the above theorem, we have that M is Kobayashi hyperbolic and Stein as a closed submanifold of Ω. If M admits a $\mathbf{T}^m$-action, for $m = \dim M$, then M is biholomorphic to a Reinhardt domain by [BBD]. Then by [Z], we have that M is Kobayashi complete. Arguing as in [PZ], we can then conclude that Ω is Kobayashi complete in this case.

2. Proof of Theorem 1.1

Since Ω is recurrent, there exist $p_0 \in \Omega$ and $n_i \to \infty$ such that $f^{n_i}(p_0)$ is relatively compact in Ω.

Taking a subsequence of $\{n_i\}$, we assume that $f^{n_i}(p_0) \to p$, $n_{i+1} - n_i \to \infty$. Taking a subsequence of $n_{i+1} - n_i$, we can assume that $\{f^{n_{i+1}-n_i}\}$ converges uniformly on compact subsets of Ω to a holomorphic map $h : \Omega \to \bar{\Omega}$. It is easy to see that we have $h(p) = p$. Consider all maps $h : \Omega \to \bar{\Omega}$, with $h(p) = p$ for some $p \in \Omega$ and h being the limit for some subsequence $\{f^{n_i}\}$. Note that we always have $f \circ h = h \circ f$ in Ω.

If the rank of some h is zero, then $h(\Omega) = p$ and necessarily $f(p) = p$. Since some iterates of f converges to the constant map, all the eigenvalues of f' at p must have modulus strictly less than one. Hence this leads to case (i).

If the rank of some h is three, then some subsequence $\{f^{n_{i+1}-n_i}\}$ converges uniformly on compact subsets of Ω to the identity and hence Ω is a Siegel domain. The restriction of f to Ω is clearly an automorphism of Ω. Therefore we are in case (iii), and we need to show that if some subsequence of $\{f^n\}$ converges to a holomorphic map g then $g \in$ Aut(Ω).

As mentioned in the introduction, Ueda showed that Ω is Kobayashi hyperbolic, hence Aut(Ω) has the structure of a Lie group ([K]). Let G be the closed subgroup of

$\mathrm{Aut}(\Omega)$ generated by $f|_\Omega$, which is a Lie group. Let G^0 be the connected component of Id in G, which is also a Lie group. Since some subsequence of $\{f^n\}$ converges to Id, G^0 is not reduced to Id. Since G^0 is commutative, it is isomorphic to $\mathbf{T}^l \times \mathbf{R}^m$. Let Φ denote the isomorphism from $\mathbf{T}^l \times \mathbf{R}^m$ to G^0. We then have $f^N = \Phi(a,b)$ for some $N \geq 1$ and some $(a,b) \in \mathbf{T}^l \times \mathbf{R}^m$. If $b \neq 0$ we cannot have a subsequence of $\{f^n\}$ converging to Id. Hence $b = 0$ and G^0 is isomorphic to $\mathbf{T}^l$. (Indeed, as in [FS2, Proposition 1.3], we have $1 \leq l \leq 3$.) In particular, G^0 is compact and any convergent subsequence of $\{f^n\}$ tends to an element of $\mathrm{Aut}(\Omega)$.

Assume now that for all h, the maximal rank of h is m, $m = 1$ or 2. Fix an h with the maximal rank and let $\Sigma := h(\Omega) \subset \bar{\Omega}$.

If $z \in \Sigma$ then $z = h(w)$ for some $w \in \Omega$ and $f(z) = f(h(w)) = h(f(w)) \in \Sigma$. Hence $f(\Sigma) \subset \Sigma$. Choose $w' \in \Omega$ such that $f(w') = w$ and set $z' = h(w')$. Then $z' \in \Sigma$ and $f(z') = f(h(w')) = h(f(w')) = h(z') = z$. Therefore $f(\Sigma) = \Sigma$.

Set $\Sigma^0 := \Sigma \cap \Omega$ and $\Sigma^1 := \Sigma \cap \partial\Omega$. Then $\Sigma = \Sigma^0 \cup \Sigma^1$. Since $f(\Omega) = \Omega$ and f is an open map, we have $f(\Sigma^0) = \Sigma^0$ and $f(\Sigma^1) = \Sigma^1$.

Passing to a subsequence, we can assume that $\{f^{n_{i+1}-n_i}\}$ converges uniformly on compact subsets of Ω to a holomorphic map $\rho : \Omega \to \bar{\Omega}$. We have $h(w) = \rho(h(w))$ for $w \in \Omega$ if $h(w) \in \Sigma^0$. So we have $\Sigma^0 \subset V := \{p \in \Omega;\ \rho(p) = p\}$. From the rank of $Id - \rho'$ at each point of V, we get that V is smooth (cf. [B, Theorem 1.1]).

For any $z \in V$, we have $\rho(f(z)) = f(\rho(z)) = f(z)$, thus $f(V) \subset V$. The equicontinuity of f in Ω implies the equicontinuity of f on V. And by the definition of V, we have that every point of V is in the ω-limit set of V. By [D], we know that $(f|_V)^n$ and all its limits are automorphisms of V. Thus for any limit map g of $\{f^n\}$, we have $V \subset g(\Omega) \cap \Omega$. In particular, we get that $\Sigma_0 = V$. Let G^0 be the connected component of Id in G, the closed subgroup of $\mathrm{Aut}(\Sigma^0)$ generated by $\{(f|_{\Sigma^0})^n\}$. Since Σ^0 is Kobayashi hyperbolic, a similar argument as in case (iii) above shows that G^0 is isomorphic to $\mathbf{T}^l$ with $1 \leq l \leq m$. If we set $\phi := h|_{\Sigma^0}$, then $\phi \in \mathrm{Aut}(\Sigma^0)$ and $h = \phi \circ \rho$. Hence to show that we are in case (ii), we only need to show that $\Sigma^1 = \emptyset$.

Let $U \subset \Omega$ be a small relatively compact set. Assume that $\{f^{m_i}\}$ is a convergent subsequence. Then for any subsequence $\{p_j\}$ of $\{n_i\}$ we have that $\{f^{m_i+p_j}\}$ converges uniformly on relatively compact sets of Ω to a holomorphic map g. For any m_i, $f^{m_i}(U)$ is relatively compact in Ω. Therefore, for any $\epsilon > 0$ we can choose p_j such that for any $x \in U$ we have $|f^{p_j+m_i}(x) - h(f^{m_i}(x))| < \epsilon/2$. For any $\epsilon > 0$ and any $x \in U$ we have for $m_i + p_j$ big enough that $|f^{p_j+m_i}(x) - g(x)| < \epsilon/2$. So for any $x \in U$ we then have $|f^{m_i}(h(x)) - g(x)| < \epsilon$. This shows that $\{(f|_\Sigma)^{m_i}\}$ converges on $h(U)$ to $g(U)$.

Let $\hat{\Sigma}$ be the minimal analytic continuation of V that contains all $g(\Omega)$, where g is any limit of $\{f^n\}$. Let $\hat{f} : \hat{\Sigma} \to \hat{\Sigma}$ be the natural map induced by f. Then, passing to a subsequence, we have that $\{\hat{f}^{m_{i+1}-m_i}\}$ converges to Id on $\hat{\Sigma}$. Therefore we have $\hat{f} \in \mathrm{Aut}(\hat{\Sigma})$. Thus, by a similar argument as above, we have that the connected component of Id in the closed subgroup of $\mathrm{Aut}(\hat{\Sigma})$ generated by $\{\hat{f}^n\}$ is isomorphic to $\mathbf{T}^l$ with $1 \leq l \leq m$. Since the $\mathbf{T}^l$ action on $\hat{\Sigma}$ is ergodic and $\hat{f}$ maps singular points to singular points, we have that $\hat{\Sigma}$ is actually smooth.

If $m = 1$ then $f|_\Sigma$ is a circle action. For any $p \in \Sigma$, we can find a holomorphic coordinate system in a neighborhood T of p in Σ with the form $\{a < |z| < b\}$ such that $f|_T(z) = e^{i\theta} z$. Consider a neighborhood N of T, where over each point $q \in T$ we consider the complex space perpendicular to T. Since the complex space has real dimension four and T has real dimension two, it is easy to find a non-vanishing global real coordinate function x_1 in N. Let S be the real space spanned by the real coordinate x_1 and $S_{\mathbf{C}}$ be the complexification of S in N. Let $S'_{\mathbf{C}}$ be the orthogonal compliment of $S_{\mathbf{C}}$ in N and denote by x_2 the radial coordinate in $S'_{\mathbf{C}}$. Then the coordinates (x_1, x_2) measure the distance of a point in the complex space N_q to the point $q \in T$.

Write $f|_N = (f_1, f_2, f_3)$ and $(f|_N)^n = (f_1^n, f_2^n, f_3^n)$. Then we have

$$(f_2(z, x_1, x_2), f_3(z, x_1, x_2)) = (x_1 a_1(z) + x_2 b_1(z) + O(2), x_1 c_1(z) + x_2 d_1(z) + O(2)),$$

and

$$(f_2^n(z, x_1, x_2), f_3^n(z, x_1, x_2)) = (x_1 a_n(z) + x_2 b_n(z) + O(2), x_1 c_n(z) + x_2 d_n(z) + O(2)),$$

where $O(2) = O(x_1^2, x_2^2, x_1 x_2)$. If we set

$$A_1(z) = \begin{pmatrix} a_1(z) & b_1(z) \\ c_1(z) & d_1(z) \end{pmatrix} \quad \text{and} \quad A_n(z) = \begin{pmatrix} a_n(z) & b_n(z) \\ c_n(z) & d_n(z) \end{pmatrix},$$

then we have $A_n(z) = \prod_{j=0}^{n-1} A_1(e^{ij\theta} z)$.

For any matrix A and any matrix norm $\|\cdot\|$, it is well known that $\|A^m\|^{1/m}$ tends to $\rho(A)$, the spectral radius of A, as m goes to infinity. Fix n, consider the function $u_m(z) = \frac{1}{m} \log \|A_n^m(z)\|$, where we set $\|A\| = \sum_{i,j} |a_{ij}|$. Then $u_m(z)$ is continuous a.e. and uniformly bounded above. Therefore $\{u_m(z)\}$ converges in L^1 to $\log \rho(A_n(z)) =: v_n(z)$. Since the sequence $\{\frac{1}{n} v_n(z)\}$ is uniformly bounded above, it has a convergent subsequence that converges on T to a real function $\varphi(z)$, which is constant a.e. on $\{|z| = r\}$ by the ergodicity of f. Therefore the sequence $\{\frac{1}{n} v_n(z)\}$ converges on T to $\varphi(z)$ which is a function of $|z|$ a.e..

For a generic choice of $p \in T$ we have $a_1(z) \neq 0$. In a neighborhood W of p we can choose local coordinate (z, w, t) such that $W \cap T = \{w = t = 0\}$. Choose a subsequence n_j with $n_j \to \infty$ as $j \to \infty$, such that $f^{n_j}(q) \in W$ for $q \in W$. This is possible because f is ergodic on T and f is attracting in the normal direction in W. Then we can write f^{n_j} as

$$f^{n_j}(z, w, t) = (e^{i n_j \theta} z + O(w, t), H_{n_j}(z) Z + O(w^2, t^2, wt)),$$

where $Z = (w, t)^T$ and $H_{n_j}(z)$ are holomorphic 2×2 matrices. Thus in $W \cap T$ we have $\|A_{n_j}(z)\| = \|H_{n_j}(z)\|$, and therefore $\{\frac{1}{m} \log \|A_{n_j}^m(z)\|\}$ is a sequence of plurisubharmonic functions for any n_j. This shows that the limit function $\varphi(z)$ is plurisubharmonic a.e.. Since $\varphi(z) \leq 0$ a.e. on $\overline{\Sigma^0}$, we must have $\varphi(z) > 0$ on a subset of $\Sigma \backslash \overline{\Sigma^0}$ with positive measure if $\Sigma^1 \neq \emptyset$. But then f can not have a stable manifold of dimension two over that subset, a contradiction. Thus we conclude that $\Sigma^1 = \emptyset$.

If $m = 2$ then we have two cases depending on l (l as in $\mathbf{T}^l$).

If $l = 2$, then by [BBD], there exist a Reinhardt domain U in $\mathbf{C}^2$ and a biholomorphic map $\Phi : \Sigma \to U$ such that for some j we have $\Phi \circ f^j = R \circ \Phi$ where $R(z, w) = (e^{i\theta} z, e^{i\varphi} w)$. For any $p \in \Sigma$, we can find a holomorphic coordinate system in a neighborhood T of p in Σ with the form $\{a < |z| < b,\ c < |w| < d\}$ such that $f|_T(z, w) = (e^{i\theta} z, e^{i\varphi} w)$. Consider a neighborhood L of T, where over each point $q \in T$ we consider the complex space perpendicular to T with local coordinate t. Then the real local coordinate $x = |t|$ is non-vanishing since it measures the distance of a point in the complex space L_q to the point $q \in T$. Write $f|_L = (f_1, f_2, f_3)$

and $(f|_L)^n = (f_1^n, f_2^n, f_3^n)$. Then we have $f_3(z,w,x) = xa_1(z,w) + O(x^2)$ and $f_3^n(z,w,x) = xa_n(z,w) + O(x^2)$ with $a_n(z,w) = \prod_{j=0}^{n-1} a_1(e^{ij\theta}z, e^{ij\varphi}w)$.

Since the sequence $\{\frac{1}{n}\log a_n(z,w)\}$ is in L^1, it has a convergent subsequence that converges on T to a real function $\varphi(z,w)$, which is constant a.e. on $\{|z| = r,\ |w| = s\}$ by the ergodicity of f. Therefore the sequence $\{\frac{1}{n}\log a_n(z,w)\}$ converges on T to $\varphi(z,w)$ which is a function of $(|z|,|w|)$ a.e..

For a generic choice of $p \in T$ we have $a_1(z,w) \neq 0$. In a neighborhood W of p we can choose local coordinate (z,w,t) such that $W \cap T = \{t = 0\}$. Choose a subsequence n_j with $n_j \to \infty$ as $j \to \infty$, such that $f^{n_j}(q) \in W$ for $q \in W$. This is possible because f is ergodic on T and f is attracting in the normal direction in W. Then we can write f^{n_j} as $f^{n_j}(z,w,t) = (e^{in_j\theta}z + O(t), e^{in_j\varphi}w + O(t), th_{n_j}(z,w) + O(t^2))$. Thus in $W \cap T$ we have $a_{n_j}(z,w) = |h_{n_j}(z,w)|$, and therefore $\{\frac{1}{n_j}\log a_{n_j}(z,w)\}$ is a sequence of plurisubharmonic functions. This shows that the limit function $\varphi(z,w)$ is plurisubharmonic a.e.. Since $\varphi(z,w) \le 0$ a.e. on $\overline{\Sigma^0}$, we must have $\varphi(z,w) > 0$ on a subset of $\Sigma\backslash\overline{\Sigma^0}$ with positive measure if $\Sigma^1 \neq \emptyset$. But then f is not attracting in the normal direction over that subset, a contradiction. Thus we conclude that $\Sigma^1 = \emptyset$.

If $l = 1$, then $\mathrm{Aut}(\Sigma)$ is an analytic Lie group (see [BM]), and the complexification of the real orbits are also orbits. For a generic choice of $p \in \Sigma$, let S be the invariant hyperbolic Riemann surface that contains p on which f acts as an irrational rotation. Since $\mathrm{Aut}(\Sigma)$ is analytic, Σ is foliated with invariant surfaces in a neighborhood of S. Let T, L and $A_n(z)$ be as in the case $m = 1$. Then since in a neighborhood of S in Σ it is foliated with invariant surfaces, we have that one of the eigenvalues of $A_n(z)$ always has modulus one, thus a constant λ. Denote by $v(z)$ the eigenvector associated with λ and write $\tilde{A}_n(z) = A_n(z) - \lambda v(z)\cdot v(z)^T$. We can then apply the same argument for $m = 1$ case to $\tilde{A}_n(z)$ and conclude that $S \cap \partial\Omega = \emptyset$. Since the choice of p is generic, this is enough to conclude that $\Sigma^1 = \emptyset$.

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