Phase Transition in Spherical Spin Glass Model

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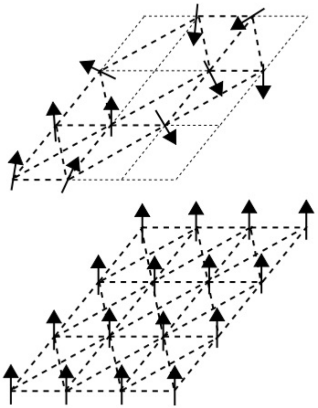
Spin glass

- First used to label the low temperature state of a class of substitutional magnetic alloys, such as CuMn or AuFe.
- Magnetic moments (spins) freeze in orientation without periodic ordering (glass).
- Exhibits a phase transition when external magnetic fields are kept very small.



Edwards-Anderson model

- Spin variable

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_N) \in \{-1, 1\}^N$$

- Hamiltonian

$$H_N(\boldsymbol{\sigma}) = -\sum_{i \sim j} J_{ij} \sigma_i \sigma_j$$

$J = (J_{ij})$ is a real (Gaussian) random symmetric matrix.

- Partition function

$$Z_N = \sum_{\boldsymbol{\sigma} \in \{-1,1\}^N} e^{\beta H_N(\boldsymbol{\sigma})}$$

- Gibbs measure

$$p_N(\boldsymbol{\sigma}) = \frac{e^{\beta H_N(\boldsymbol{\sigma})}}{Z_N}$$

- Free energy

$$F_N = \frac{1}{N} \log Z_N$$

Sherrington-Kirkpatrick model

- Spin variable
$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_N) \in \{-1, 1\}^N$$
- Hamiltonian
$$H_N(\boldsymbol{\sigma}) = -\sum_{i \neq j} J_{ij} \sigma_i \sigma_j$$
$J = (J_{ij})$ is a real (Gaussian) random symmetric matrix.
- Partition function
$$Z_N = \sum_{\boldsymbol{\sigma} \in \{-1,1\}^N} e^{\beta H_N(\boldsymbol{\sigma})}$$
- Gibbs measure
$$p_N(\boldsymbol{\sigma}) = \frac{e^{\beta H_N(\boldsymbol{\sigma})}}{Z_N}$$
- Free energy
$$F_N = \frac{1}{N} \log Z_N$$

Spherical SK model

- Spin variable

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_N) \in S_N = \{\mathbf{x} : x_1^2 + x_2^2 + \cdots + x_N^2 = N\}$$

- Hamiltonian

$$H_N(\boldsymbol{\sigma}) = -\sum_{i \neq j} J_{ij} \sigma_i \sigma_j$$

$J = (J_{ij})$ is a real (Gaussian) random symmetric matrix.

- Partition function

$$Z_N = \int_{\boldsymbol{\sigma} \in S_N} e^{\beta H_N(\boldsymbol{\sigma})} \mathrm{d}\Omega(S_N)$$

- Free energy

$$F_N = \frac{1}{N} \log Z_N$$

Extensions

- External magnetic field: $H_N(\sigma) = -\sum_{(i,j)\in S} J_{ij}\sigma_i\sigma_j + h\sum_i \sigma_i$
- p-spin model: $H_N(\sigma) = -\sum_{(i_1,\cdots,i_p)\in S} J_{i_1,\cdots,i_p}\sigma_{i_1}\cdots\sigma_{i_p}$
- Aging of spin glass
  See Ben Arous, Dembo, Guionnet (2001).
- Random energy model
  See Derrida (1981).

Replica method

- Consider the identity

$$\log Z = \lim_{n \to 0} \frac{Z^n - 1}{n}.$$

- Compute $(Z^n - 1)/n$ for positive integer n with steepest-descent method. Assume that the critical point is attained when all n copies are symmetric. (Replica symmetry)
- Take $n \to 0$.
- If the replica method does not work, apply replica symmetry breaking.

SK Model

- Assume that $\mathbb{E}[J_{ij}] = 0$, $\mathbb{E}[J_{ij}^2] = 1/N$.
- The limit $F = \lim_{N\to\infty} F_N$ exists:
  Proposed by Parisi (1980) and proved by Talagrand (2006)
- Phase transition at $\beta = \beta_c := 1/2$.
- Gaussian fluctuation at high temperature ($\beta < 1/2$):

$$N\big(F_N - (\log 2 + \beta^2)\big) - \widetilde{f}(\beta) \to \mathcal{N}(0, -\frac{1}{2}\log(1-4\beta^2) - 2\beta^2)$$

  Proved by Aizenman, Lebowitz, Ruelle (1987)
- Fluctuation at low temperature is not known.

SSK Model

- Assume that $\mathbb{E}[J_{ij}] = 0$, $\mathbb{E}[J_{ij}^2] = 1/N$.
- The limit $F = \lim_{N\to\infty} F_N$ exists:
  Proposed by Kosterlitz, Thouless, and Jones (1976), generalized by Chrisanti and Sommers (1992), and proved by Talagrand (2006)

Theorem

The limiting free energy

$$F(\beta) = \begin{cases} \beta^2 & \text{if } \beta < 1/2 \\ 2\beta - \frac{1}{2}\log(2\beta) - \frac{3}{4} & \text{if } \beta > 1/2 \end{cases}.$$

In particular, $F(\beta)$ is C^2 but not C^3 at $\beta = 1/2$.

(See Panchenko and Talagrand (2007).)

High temperature: Gaussian fluctuation

Theorem

Assume that $\mathbb{E}[J_{ij}] = 0$, $\mathbb{E}[J_{ij}^2] = 1/N$ $(i \neq j)$, $\mathbb{E}[J_{ii}^2] = w_2/N$, $\mathbb{E}[J_{ij}^4] = W_4/N^2$, *and* $\mathbb{E}[J_{ij}^p] \leq C_p N^{-p/2}$. *Then, for* $\beta < 1/2$,

$$N(F_N - F) \to \mathcal{N}(c, f)$$

with

$$c = \frac{1}{4}\left(\log(1 - 4\beta^2) + 4\beta^2(w_2 - 2) + 8\beta^4(W_4 - 3)\right)$$

and

$$f = \frac{1}{2}\left(-\log(1 - 4\beta^2) + 2\beta^2(w_2 - 2) + 4\beta^4(W_4 - 3)\right).$$

Note: The variance coincides with the result by Aizenman, Lebowitz, and Ruelle.

Low temperature: Tracy-Widom fluctuation

Theorem

Assume that $\mathbb{E}[J_{ij}] = 0$, $\mathbb{E}[J_{ij}^2] = 1/N$ $(i \neq j)$, *and* $\mathbb{E}[J_{ij}^p] \leq C_p N^{-p/2}$. *Then, for* $\beta > 1/2$,

$$\left(\beta - \frac{1}{2}\right)^{-1} N^{2/3}(F_N - F) \to TW_{GOE}.$$

Note: TW_{GOE} is the limit law of the fluctuation of the largest eigenvalue of the Gaussian orthogonal ensemble.

Universality

- The results are universal except that the mean and the variance of the fluctuation in the high temperature regime depend on w_2 and W_4.
- Analogous results hold for other models with, e.g.,
  1. invariant ensembles,
  2. sample covariance matrices, and
  3. Hermitian random matrices (with complex spins).

Heuristics

$$Z_N = \int_{\boldsymbol{\sigma} \in S_N} e^{\beta \langle \boldsymbol{\sigma}, (-J) \boldsymbol{\sigma} \rangle} \mathrm{d}\Omega(S_N)$$

- In the zero-temperature case ($\beta \to \infty$),

$$\frac{1}{\beta N} \log Z_N \to \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{x}, (-J) \boldsymbol{x} \rangle = \lambda_1 .$$

- In the low temperature regime, λ_1 dominates.
- In the high temperature regime, all eigenvalues contribute.

Integral representation

Lemma

Let

$$G(z) = 2\beta z - \frac{1}{N} \sum_i \log(z - \lambda_i)$$

where log *denotes the principal branch of the logarithmic function. Then, the partition function* Z_N *satisfies*

$$Z_N = \frac{-\mathrm{i}}{|S^{N-1}|} \left(\frac{\pi}{N\beta} \right)^{(N/2)-1} \lim_{T \to \infty} \int_{\gamma - \mathrm{i}T}^{\gamma + \mathrm{i}T} e^{\frac{N}{2} G(z)} \mathrm{d}z$$

for any $\gamma > \lambda_1$*. Here,* S^{N-1} *denotes the unit sphere in* $\mathbb{R}^N$*.*

cf. Kosterlitz, Thouless, and Jones (1976).

Integral representation (Idea of the proof)

- Diagonalize $(-J)$ and rewrite the partition function as
$$Z_N = \int_{S^{N-1}} e^{\beta N \sum \lambda_i x_i^2} \mathrm{d}\Omega.$$
- Consider the integral
$$Q(z) = \int_{\mathbb{R}^N} e^{\beta N \sum \lambda_i y_i^2} e^{-\beta N z \sum y_i^2} \mathrm{d}\boldsymbol{y}.$$
- Evaluate $Q(z)$ in two different ways: (i) by Gaussian integral and (ii) by using polar coordinates:
$$Q(z) = \int_0^\infty e^{-\beta N z r^2} r^{N-1} I(r) \mathrm{d}r$$
with
$$I(r) = \int_{S^{N-1}} e^{\beta N r^2 \sum \lambda_i x_i^2} \mathrm{d}\Omega.$$
- Take inverse Laplace transform to find $Z_N = I(1)$.

Rigidity

For a positive integer $k \in [1, N]$, let $\hat{k} := \min\{k, N+1-k\}$. Let γ_k be the classical location defined by

$$\int_{\gamma_k}^{\infty} \mathrm{d}\rho_{sc} = \frac{1}{N}\left(k - \frac{1}{2}\right).$$

Then, for any $\epsilon > 0$,

$$|\lambda_k - \gamma_k| \leq \hat{k}^{-1/3} N^{-2/3+\epsilon}.$$

(See Erdős, Yau, and Yin (2012).)

Linear statistics

For every function $\varphi : \mathbb{R} \to \mathbb{R}$ that is analytic in an open neighborhood of $[-2, 2]$, the random variable

$$\sum_i \varphi(\lambda_i) - N \int_{-2}^{2} \varphi(\lambda) \mathrm{d}\rho_{sc}(\lambda)$$

converges in distribution to a Gaussian random variable.

(See Bai and Yao (2005).)

Tracy-Widom distribution

The rescaled largest eigenvalue $N^{2/3}(\lambda_1 - 2)$ converges in distribution to the Tracy-Widom distribution.

(See Erdős, Yau, and Yin (2012).)

Critical point

- Heuristically,

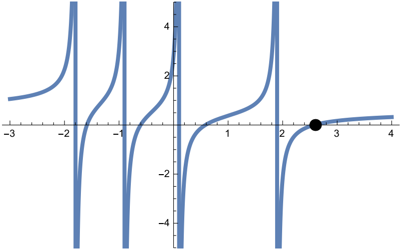
$$G'(z) = 2\beta - \frac{1}{N}\sum_i \frac{1}{z-\lambda_i}$$
$$\simeq 2\beta - \int_{-2}^{2} \frac{1}{z-\lambda}\mathrm{d}\rho_{sc}(\lambda) = 2\beta - \frac{z-\sqrt{z^2-4}}{2}.$$

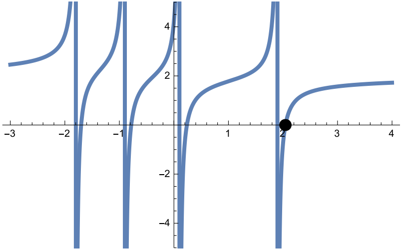
- The equation

$$\frac{z-\sqrt{z^2-4}}{2} = 2\beta$$

has a unique solution $z = 2\beta + \frac{1}{2\beta}$ if $2\beta < 1$.

Critical point





- Let z_c be the critical point of G on (λ_1, ∞).
- If $\beta < 1/2$, $z_c \simeq z_c^\infty$ for some non-random z_c^∞ with $|z_c^\infty - 2| \sim 1$.
- If $\beta > 1/2$, $|z_c - \lambda_1| \sim N^{-1}$.

High temperature regime

- Method of steepest-descent: (choose $\gamma = z_c$)

$$\log \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{\frac{N}{2} G(z)} \mathrm{d}z \simeq \frac{N}{2} G(z_c)$$

- For $\beta < 1/2$, approximate $G(z_c)$ by $G_\infty(z_c^\infty)$, where

$$G_\infty(z) = 2\beta z - \int_{-2}^{2} \log(z - \lambda) \mathrm{d}\rho_{sc}(\lambda)$$

and

$$z_c^\infty = 2\beta + \frac{1}{2\beta}.$$

- Use the rigidity in the approximation $NG(z_c) \simeq NG(z_c^\infty)$.

High temperature regime

- Use the linear statistics to prove the Gaussian fluctuation:

$$N[G(z_c^\infty) - G_\infty(z_c^\infty)]$$
$$= \sum_i \log(z_c^\infty - \lambda_i) - N \int_{-2}^{2} \log(z_c^\infty - \lambda) \mathrm{d}\rho_{sc}(\lambda)$$

Low temperature regime

- Method of steepest-descent: (choose $\gamma = z_c$)

$$\log \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{\frac{N}{2} G(z)} \mathrm{d}z \simeq \frac{N}{2} G(z_c)$$

- Use the rigidity to prove that

$$|z_c - \lambda_1| \sim N^{-1}$$

- The fluctuation of $G(z_c)$ can be approximated by the (rescaled) fluctuation of λ_1.

Low temperature regime

- Problem: the critical point is very near the singularity.
- Technically, $G^{(\ell)}(z_c) \sim N^{\ell-1}$, hence $G(z)$ can fluctuate heavily even in a very small neighborhood of z_c.
- We prove the following lemma to justify the use of the steepest-descent method:

Lemma

Let $\beta > \beta_c$. Then, there exists $K \equiv K(N)$ satisfying $N^{-C} < K < C$ for some constant $C > 0$ such that

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}G(z)} \mathrm{d}z = \mathrm{i} e^{\frac{N}{2}G(z_c)} K$$

with high probability.

Low temperature regime

Idea of the proof.

- The upper bound can be easily obtained. In order to prove the lowever bound, we introduce Γ, the curve of steepest-descent that passes through z_c, and compute the integral on Γ.
- Find a (rough) behavior of Γ on $B_{N^{-2}}(z_c)$.
- Find a lower bound by estimating the integral on $\Gamma \cap B_{N^{-2}}(z_c)$.

□

Low temperature regime

$$
\begin{aligned}
G(z_c) &= 2\beta z_c - \frac{1}{N}\sum_{i=1}^{N}\log(z_c-\lambda_i) \\
&\simeq 2\beta\lambda_1 - \frac{1}{N}\sum_{i=1}^{N}\log(\lambda_1-\lambda_i) \\
&\simeq 2\beta\lambda_1 - \frac{1}{N}\sum_{i=1}^{N}\left[\log(2-\lambda_i) + \frac{1}{2-\lambda_i}(\lambda_1-2)\right] \\
&\simeq 2\beta\lambda_1 - \int_{-2}^{2}\log(2-s)\mathrm{d}\rho_{sc}(\lambda) - \int_{-2}^{2}\frac{1}{2-s}(\lambda_1-2)\mathrm{d}\rho_{sc}(\lambda) \\
&= (2\beta-1)\lambda_1 + \frac{3}{2}
\end{aligned}
$$

Summary

1. $\beta < 1/2$:

$$F_N \simeq F(\beta) + \frac{1}{N}\mathcal{N}(f, c)$$

2. $\beta > 1/2$:

$$F_N \simeq F(\beta) + \frac{1}{N^{2/3}}\left(\beta - \frac{1}{2}\right) TW_{GOE}$$

where

$$F(\beta) = \begin{cases} \beta^2 & \text{if } \beta < 1/2 \\ 2\beta - \frac{1}{2}\log(2\beta) - \frac{3}{4} & \text{if } \beta > 1/2 \end{cases}.$$