The Nahm Pole Boundary Condition

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I want to begin by recalling a couple of things from yesterday. The Bogomolny equations $F = \star \mathrm{d}_A \phi$ for gauge group $U(1)$ have an exact solution on $\mathbb{R}^3$ with a point singularity:

$$\phi = \frac{1}{2|\vec{x} - \vec{x}_0|}, \quad F = \star \mathrm{d}\phi.$$

(This is the basic solution with $n = 1$.) For any gauge group G, we picked a homomorphism $\rho : \mathfrak{u}(1) \to \mathfrak{t} \subset \mathfrak{g}$ from the Lie algebra of $\mathfrak{u}(1)$ to that of G, and embed the $U(1)$ solution as a solution in G simply by

$$(A, \phi) \to (\rho(A), \rho(\phi)).$$

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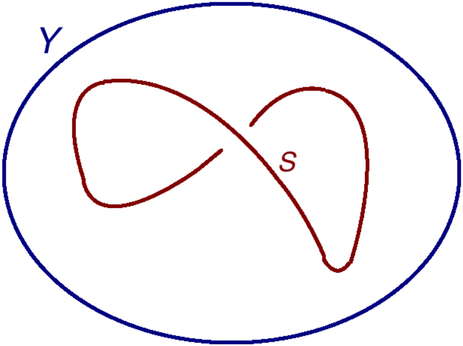
This gives a solution of the Bogomolny equations for G with a point singularity. I then introduced the KW equations

$$F - \lambda \wedge \lambda - \star \mathrm{d}_A \lambda, \quad \mathrm{d}_A \star \lambda = 0.$$

In general, any solution of the Bogomolny equations on a three-manifold W can be "lifted" to a solution of the KW equations on the four-manifold $W \times \mathbb{R}$, as follows. I parametrize $\mathbb{R}$ by y, and write $\pi : W \times \mathbb{R} \to W$ for the projection. If (A, ϕ) is a solution of the Bogomolny equations on W, then $(\pi^*(A), \pi^*(\phi)\mathrm{d}y)$ is a solution of the KW equations on $W \times \mathbb{R}$.

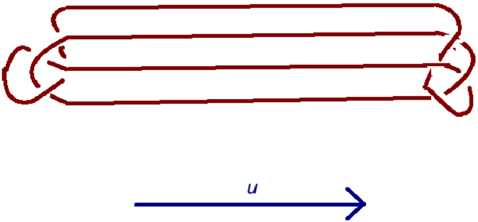
In particular, a solution of the Bogomolny equations on $\mathbb{R}^3$ with a point singularity can be lifted to a solution of the KW equations on $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$, but now the singularity is on a straight line in $\mathbb{R}^4$. We use this as a model solution that describes what kind of singularity we ask for along any embedded one-manifold in a four-manifold:



The KW equations in the presence of a prescribed singularity of this kind are a well-posed (elliptic) problem.

When we “categorify” by adding another dimension, the singularity is still in real codimension 3 but now we are in 5 dimensions and the singularity is on a 2-manifold.

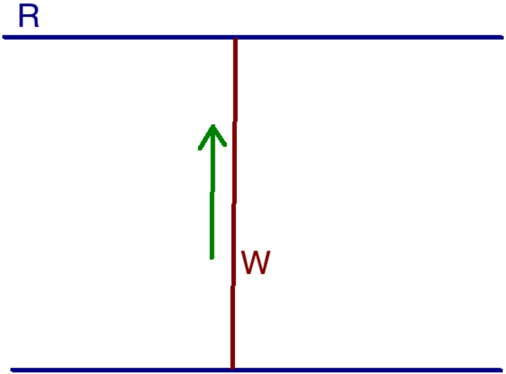
It seems there was some confusion yesterday as a result of my discussion of stretching a knot.



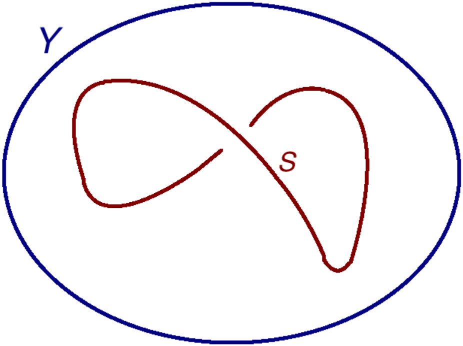
The basic definitions I proposed of the Jones polynomial and Khovanov homology had nothing to do with stretching. That is the whole point: the definitions just involve counting solutions of the KW equations and their five-dimensional analog (with some prescribed singularities) and that viewpoint potentially has manifest three- or four-dimensional symmetry. If stretching is part of the definition, we could not expect manifest three- or four-dimensional symmetry.

The only reason I discussed stretching was to motivate considering the KW equations. I thought an explanation based on the relation to symplectic Khovanov homology would be more compelling to some of you than an explanation of the physical setup. Anyway I wanted to try a different approach from what I've done in previous lectures.

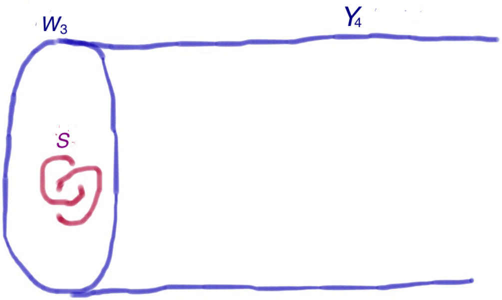
In yesterday's lecture, I explained that in order for categorification to be possible, the four-manifold on which we do the KW equations has to be $W \times \mathbb{R}$ with knots contained in $W \times p$, for some point $p \in \mathbb{R}$. If we "stretch" to facilitate computation, we do this along W, not along $\mathbb{R}$; in other words, "stretching" is in the vertical direction in the figure:



In yesterday's lecture, we discussed singly-graded Khovanov homology, with only the cohomological grading. As I remarked at the end, the physical picture makes clear where the additional "q"-grading of Khovanov homology would come from. It is supposed to come from the second Chern class, integrated over the 4-manifold Y_4. But for topological reasons, this q-grading cannot be defined in the construction as I have presented it so far. The second Chern class cannot be defined in the presence of the singularities that we've assumed:

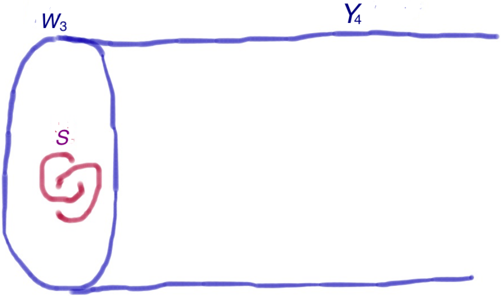


The physical picture tells us what we have to do to get the q-grading: Y_4 should be a manifold with boundary, with the knot placed in its boundary:



The boundary condition is subtle to describe, but has the property that the bundle is trivialized on the boundary, so the second Chern class can be defined. My main goal today is to describe this boundary condition.

We could actually get the q-grading for any Y_4 with boundary, but to also allow categorification, we want more specifically $Y_4 = W \times \mathbb{R}_+$, where $\mathbb{R}_+$ is a half-line, parametrized by y. For the Jones polynomial and Khovanov homology, we further take $W = \mathbb{R}^3$ (or S^3).



For $y \to \infty$, we ask for $A, \lambda \to 0$. For $y \to 0$, there is a subtle boundary condition which is one of the main points of the theory. Describing it is actually my main goal for today. This boundary condition depends on the knot K, and on the labeling of K (or of each component of a link L) by a representation $R^\vee$ of the Langlands or GNO dual group $G^\vee$. That is the only place that K enters the setup. It is an elliptic boundary condition but a subtle one. Elliptic means that although the definition is completely different, the resulting properties are similar to what one would get with more familiar elliptic boundary conditions such as Dirichlet or Neumann. For example, the linearization of the KW equation becomes a Fredholm operator and (on a compact four-manifold) has discrete spectrum.

With this boundary condition, which I will describe in some detail, the restriction to the boundary $W \times \{y = 0\}$ of the bundle E and connection A are specified. As a result, one can define a second Chern class

$$n = \frac{1}{8\pi^2} \int_{W \times \mathbb{R}_+} \mathrm{Tr}\, F \wedge F,$$

which really takes values in a $\mathbb{Z}$-torsor associated to framings of W and K. "Torsor" means that it is not true that n is an integer; rather the value of n mod $\mathbb{Z}$ depends only on the boundary conditions and not on the specific gauge field that satisfies them. One can "trivialize the torsor" and make n an integer by picking framings of W and K. The torsor comes in because the boundary condition specifies the restriction of E to the boundary (as a bundle with connection) but does not trivialize it.

To define a knot polynomial, one counts (with signs, in a standard way) the number a_n of solutions of the KW equations with second Chern class n, and then one defines

$$J(q; K, R^\vee) = \sum_{n \in \mathbb{Z}} a_n q^n.$$

Compactness (not yet proved with the appropriate boundary conditions) of the solutions of the KW equations will mean that there are only finitely many terms in the sum so that this is a Laurent polynomial.

As I explained yesterday, with this definition of the Jones polynomial, the “categorification” that leads to Khovanov homology is straightforward in principle. It arises because the KW equations can be “lifted,” in a certain sense, to certain elliptic differential equations in five dimensions, and these equations can be interpreted as gradient flow equations. But rather than say more about that today, what I want to do is to describe the boundary condition that is needed for the four (or five) dimensional equations. This boundary condition is of a possibly somewhat unfamiliar type, and understanding it is essential for making progress with this subject.

The boundary conditions have been studied in R. Mazzeo and EW, "The Nahm Pole Boundary Condition," arXiv:1311.3167, where we showed that the boundary condition is elliptic in the absence of a knot. A paper is in progress on the case with a knot and I will tell a little about that case later. I will carry out this discussion in the 4d language since going to five dimensions does not change much. (In our paper, we showed that the extension to 5d was fairly trivial.)

The boundary condition is not a simple Dirichlet or Neumann boundary condition – it is not defined by saying what fields or derivatives of fields vanish along the boundary. Rather, the boundary condition is defined by specifying a model solution of the KW equations that has a singularity along the boundary, and saying that one only wants to consider solutions of the KW equation that are asymptotic to this singular solution along the boundary.

The model solution is a solution on $\mathbb{R}^3 \times \mathbb{R}_+$, where I will parametrize $\mathbb{R}^3$ by x_1, x_2, x_3 and $\mathbb{R}_+$ by y. There is a simple exact solution with $A = 0$ and

$$\lambda = \sum_{i=1}^{3} \frac{\mathfrak{t}_i \cdot \mathrm{d}x_i}{y},$$

where $\mathfrak{t}_i$ are elements of the Lie algebra $\mathfrak{g}$ of G that obey the $\mathfrak{su}(2)$ commutation relations

$$[\mathfrak{t}_1, \mathfrak{t}_2] = \mathfrak{t}_3, \quad \text{and cyclic permutations.}$$

Thus the $\mathfrak{t}_i$ are images of a standard basis of $\mathfrak{su}(2)$ under some homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. Every ρ leads to an interesting theory, but to get the Jones polynomial and Khovanov homology, we take ρ to be a principal embedding in the sense of Kostant. (For $G = SU(2)$, this simply means that ρ is the identity map $\mathfrak{su}(2) \to \mathfrak{g}$. For $G = SU(N)$, it means that the N-dimensional representation of $\mathfrak{g}$ is irreducible with respect to $\rho(\mathfrak{su}(2))$.)

The solution I have just described is what I call the Nahm pole solution since it was introduced by W. Nahm around 1979 in his work on magnetic monopoles. That was in the context of "Nahm's equation,"

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}y} + [\lambda_2, \lambda_3] = 0, \quad \text{and cyclic permutations}$$

(which is an ODE to which our PDE can be specialized). On $\mathbb{R}^3 \times \mathbb{R}_+$, the Nahm pole boundary condition just says that a solution is supposed to be asymptotic to the Nahm pole solution for $y \to 0$.

To state the Nahm pole boundary condition on $M_4 = W \times \mathbb{R}_+$, for a more general 3-manifold W, one needs to specify some terms of $\mathcal{O}(1)$ in the solution (for $y \to 0$) as well as the singular terms of order $1/y$. For $G = SU(2)$, one takes the G-bundle E on which we are solving the equations to be, when restricted to $W \times \{y = 0\}$, the frame bundle of W, so that $\mathrm{ad}(E) = TW$. Then one takes A, restricted to the boundary, to be the Levi-Civita connection on TW. With this choice of E, the formula

$$\lambda = \sum_{i=1}^{3} \frac{\mathfrak{t}_i \cdot \mathrm{d}x_i}{y}$$

makes sense (one can think of the numerator $\sum_i \mathfrak{t}_i \cdot \mathrm{d}x_i$ as stating the identification $\mathrm{ad}(E) \cong TW$). One can show that this choice of (A, λ) obeys the Nahm pole boundary condition up to an error of $\mathcal{O}(y)$, and the Nahm pole boundary condition simply says that the solution should agree with what I have described up to $\mathcal{O}(y)$. (One can generalize this to the case that the metric of M_4 is not a product near the boundary.)

Showing that the Nahm pole boundary condition is elliptic is mostly an exercise in "uniformly degenerate elliptic operators" but one needs to know some specific facts about the KW equations. The main thing that one needs to know is that if $\mathcal{L}$ is the linearization of the KW equations around the Nahm pole solution, then $\mathcal{L}$ as an operator between appropriate Hilbert spaces of functions on $\mathbb{R}^4_+$ has no kernel or cokernel. Actually one can show in an elementary way that $\mathcal{L}^\dagger = -N\mathcal{L}N^{-1}$ with an explicit matrix N, so it suffices to show that there is no kernel.

Much the same argument that proves this actually proves the following statement: The only solution of the KW equations on $\mathbb{R}^4_+$, approaching the Nahm pole solution for $y \to 0$ and also for $\sqrt{\vec{x}^2 + y^2} \to \infty$, is the Nahm pole solution. In terms of Khovanov homology, this means that the Khovanov homology of the empty link is of rank 1.

Before trying to prove these vanishing results, I will explain a simpler vanishing result for the KW equations on a four-manifold $M = M_4$ without boundary. This will help us know what to aim for.

The KW equations actually have many different useful Weitzenbock formulas. I will first state some formulas that are useful if we are on a manifold without boundary. Let $\mathcal{V} = F - \lambda \wedge \lambda - \star \mathrm{d}_A \lambda$, $\mathcal{W} = \mathrm{d}_A \star \lambda$, so the KW equations are $\mathcal{V} = \mathcal{W} = 0$. Clearly then the KW equations are equivalent to the vanishing of

$$I = -\int_M \mathrm{Tr}\left(\mathcal{V} \wedge \star\mathcal{V} + \mathcal{W} \wedge \star\mathcal{W}\right).$$

A short calculation gives

$$I = -\int_M \mathrm{d}^4 x \sqrt{g}\, \mathrm{Tr}\left(\frac{1}{2} F_{ij} F^{ij} + D_i \lambda_j D^i \lambda^j + R_{ij} \lambda^i \lambda^j + \frac{1}{2}[\lambda_i, \lambda_j][\lambda^i, \lambda^j]\right)$$

with R_{ij} the Ricci tensor. If R_{ij} is non-negative, then this is a sum of non-negative terms, so the condition $I = 0$ leads to a strong constraint.

But if is possible to say something useful even if R_{ij} is not non-negative, because it is possible to find a family of Weitzenbock formulas. Define the selfdual and anti-selfdual two-forms $\mathcal{V}^+(t) = (F - \lambda\wedge\lambda + t\mathrm{d}_A\lambda)^+$, $\mathcal{V}^-(t) = (F - \lambda\wedge\lambda - t^{-1}\mathrm{d}_A\lambda)^-$. The equations $\mathcal{V}^+(t) = \mathcal{V}^-(t) = \mathcal{W} = 0$ are a 1-parameter family of elliptic equations, parametrized by $t \in \mathbb{RP}^1$. One finds that

$$-\int_M \mathrm{d}^4x\sqrt{g}\,\mathrm{Tr}\left(\frac{t^{-1}}{t+t^{-1}}\mathcal{V}^+_{ij}(t)\mathcal{V}^{+\,ij}(t) + \frac{t}{t+t^{-1}}\mathcal{V}^-_{ij}(t)\mathcal{V}^{-\,ij}(t) + \mathcal{W}^2\right)$$
$$= I + \frac{t-t^{-1}}{4(t+t^{-1})}\int_M \mathrm{d}^4x\,\epsilon^{ijkl}\mathrm{Tr}F_{ij}F_{kl}.$$

In other words, the same quantity I can be written as a sum of squares in many different ways, modulo the topological invariant

$$J(t) = \frac{t - t^{-1}}{4(t + t^{-1})} P, \quad P = \int_M \mathrm{d}^4 x \, \epsilon^{ijkl} \mathrm{Tr} F_{ij} F_{kl}.$$

Now we can deduce the following: (1) The KW equations cannot have any solutions for $t \neq 0, \infty$ except with $P = 0$ (if $P \neq 0$, then by looking at the Weitzenbock formula at some value of t' with $J(t') < J(t)$, we reach a contradiction). And (2): If the KW equations are obeyed at one value of t other than $0, \infty$, then they are obeyed at all t. This is an immediate consequence of the Weitzenbock formula, once we know that $P = 0$. The equations then reduce to $\mathcal{F} = 0$, where $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$, with $\mathcal{A}$ the complex connection $\mathcal{A} = A + i\lambda$, along with $\mathrm{d}_A \star \lambda = 0$. According to a theorem of K. Corlette, the solutions correspond to homomorphisms $\pi_1(M) \to G_{\mathbb{C}}$ that are in a certain sense semi-stable.

The moral of the story is that the KW equations participate in many different Weitzenbock formulas, not just one, and it is important to know all of them. However, none of the formulas that I have written down so far are useful for understanding the Nahm pole boundary condition. The reason is that if $\partial M \neq \varnothing$, then the preceding formulas will have boundary contributions that are divergent in the case of a solution with Nahm pole behavior along the boundary. This is inevitable because the expression that I called I in writing the Weitzenbock formula is divergent in the case of a solution with Nahm pole on the boundary. A formula like

$$-\int \mathrm{Tr}\left(\mathcal{V}\wedge\star\mathcal{V}+\mathcal{W}\wedge\star\mathcal{W}\right) = I + \text{boundary correction}$$

must have a boundary correction $-\infty$ in the case of a Nahm pole, since the left hand side is 0 and $I = +\infty$.

To get around this, the best we could hope for would be a Weitenbock formula on $\mathbb{R}^4_+$ in which I is replaced by a sum of squares of quantities whose vanishing characterizes the Nahm pole solution $A = 0$, $\lambda = \vec{\mathfrak{t}} \cdot \mathrm{d}\vec{x}/y$. The quantities that vanish in the Nahm pole solution are the curvature F, covariant derivatives and commutators that involve λ_y, namely $D_i \lambda_y$ and $[\lambda_i, \lambda_y]$, covariant derivatives of λ along $\mathbb{R}^3$ such as $D_i \lambda_j$, and finally $W_i = D_y \lambda_i + \varepsilon_{ijk}[\lambda_j, \lambda_k]$. (Nahm's equation is $W_i = 0$.) What we need is true. Define the following sum of squares of the objects whose vanishing characterizes the Nahm pole solution:

$$I' = -\int_{\mathbb{R}^3 \times \mathbb{R}_+} \mathrm{d}^4 x \, \mathrm{Tr} \left(\frac{1}{2} \sum_{i,j} F_{ij}^2 + \sum_{a,b} (D_a \lambda_b)^2 + \sum_i (D_i \lambda_y)^2 + \sum_a [\lambda_y, \lambda_a]^2 + \sum_a W_a^2 \right).$$

Then there is an identity along the lines that we need:

$$-\int_M \mathrm{Tr}\left(\mathcal{V}^2 + \mathcal{W}^2\right) = I' + \Delta$$

where Δ is the boundary term

$$\Delta = -\int_{y=0} \mathrm{d}^3x \mathrm{Tr}\left(\lambda \wedge F + \dots\right).$$

(I've omitted some further terms in Δ.) Δ is the sum of a contribution at the boundary $y = 0$ and on a large hemisphere $\sqrt{\vec{x}^2 + y^2} >> 1$.

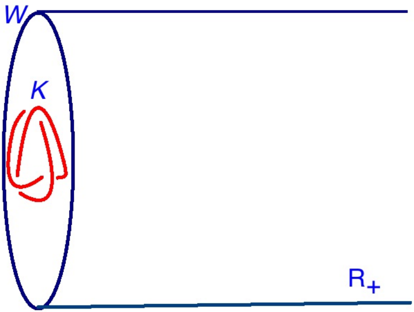
Now to get a vanishing theorem that will say that the global Nahm pole solution is the only solution on the half-space that obeys Nahm pole boundary conditions, we need to do the following: We have to prove that if A, λ approach the Nahm pole solution for $y \to 0$ and $\sqrt{\vec{x}^2 + y^2} \to \infty$, then they approach it fast enough so that $\Delta = 0$. Once this is established, the Weitzenbock formula will say that a KW solution that satisfies the boundary condition must have $I' = 0$. But I' was constructed so that it vanishes for and only for the Nahm pole solution.

To find the expected behavior of a solution for $y \to 0$ is a matter of looking at an ODE in which one ignores the $\vec{x}$ dependence, since that is nonsingular. In effect, then, we just have to look at the eigenvalues of the linearization of Nahm's equation (or more exactly a doubled version of Nahm's equation with A as well as λ). Half of the linearized eigenvalues are negative and half are positive. The Nahm pole boundary condition amounts to setting to 0 the coefficients of perturbations with negative eigenvalues, and allowing the positive ones. The positive eigenvalues are large enough to ensure that $\Delta = 0$ when the Nahm pole boundary condition is obeyed.

This shows that there is no contribution to Δ from the boundary at $y = 0$. To show that there is no contribution to Δ at $\sqrt{\vec{x}^2 + y^2} \to \infty$, one needs to look at the eigenvalues of the "angular" part of the operator $\mathcal{L}$, which is an operator on a hemisphere S^3_+ with Nahm pole boundary conditions along the boundary. Those eigenvalues determine how fast a solution will vanish at infinity, assuming that it does vanish at infinity. Again the spectrum is such that there is no contribution to Δ.

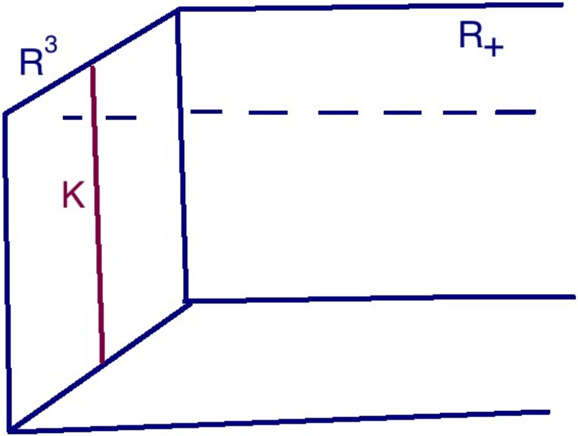
This leads to the nonlinear vanishing theorem – the global Nahm pole solution is the only solution on the half-space that satisfies the boundary conditions. Much the same argument proves a linearization of the same statement: the operator $\mathcal{L}$ obtained by linearizing around the Nahm pole solution has trivial kernel (and hence also trivial cokernel, since $\mathcal{L}$ is conjugate to $-\mathcal{L}^\dagger$).

Together with the machinery of uniformly degenerate elliptic operators, this leads to the ellipticity of the Nahm pole boundary condition in the absence of knots. But what are we supposed to say in the presence of knots? As I said at the outset, the knot will be in the boundary:



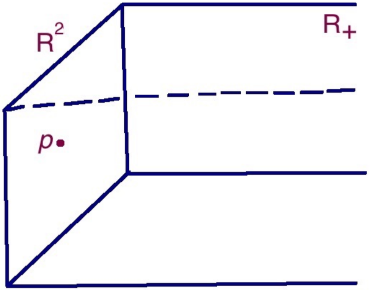
We introduce a refinement of the Nahm pole boundary condition, such that (A, λ) obeys the Nahm pole boundary condition at a generic boundary point away from a knot, has some more subtle behavior near the knot.

To describe what this more subtle behavior should be, we consider the case that the knot is locally a straight line $\mathbb{R} \subset \mathbb{R}^3$, so we work on $\mathbb{R}^4_+$ with a knot that lives on a straight line K in the boundary.



The idea is going to be to find a singular model solution in the presence of the knot. This solution will coincide with the Nahm pole solution near a boundary point away from K, but it will look different near a point of K. The model solution will depend on the choice of an irreducible representation $R^\vee$ of the dual group $G^\vee$. Then a boundary condition is defined by saying that one only allows solutions of the KW equations that look like the model solution near a knot.

For this to make sense, the model solution must look the same near any point of K, so we assume that the model solution is invariant under translations along K. So we reduce to equations on $\mathbb{R}^2 \times \mathbb{R}_+$ with the knot now represented by a point $p \in \mathbb{R}^2$.



Once we reduce to 3 dimensions (and assume vanishing of A_1 and λ_y in a way that can be motivated by the Weitzenbock formula) the KW equations become tractable. Pick coordinates so that x_1 runs along the knot K; x_2, x_3 parametrize the normal plane to K in the boundary; and y measures the distance from the boundary. Define the three operators

$$\begin{aligned}
\mathcal{D}_1 &= D_2 + iD_3 = \frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3} + [A_2 + iA_3, \cdot\,] \\
\mathcal{D}_2 &= D_y - i[\lambda_1, \cdot\,] = \frac{\partial}{\partial y} + [A_y - i\lambda_1, \cdot\,] \\
\mathcal{D}_3 &= [\lambda_2 - i\lambda_3, \cdot\,],
\end{aligned}$$

and also the "moment map"

$$\mu = F_{23} - [\lambda_2, \lambda_3] - D_y\lambda_1.$$

The KW equations in this situation become

$$0 = [\mathcal{D}_i, \mathcal{D}_j], \quad i, j = 1, \ldots, 3$$

along with a "moment map" condition

$$\mu = 0.$$

These equations are a sort of hybrid of three much-studied equations in the mathematics of gauge theory. If we drop $\mathcal{D}_1$ (by assuming that the fields are independent of x_2 and x_3 and that $A_2 = A_3 = 0$), we get Nahm's equation; if we drop $\mathcal{D}_2$ (by assuming that the fields are independ ot y and that $A_y = \lambda_1 = 0$) we get Hitchin's equation; and if we drop $\mathcal{D}_3$ (by setting $\lambda_2 = \lambda_3 = 0$), we get the Bogomolny equation.

The full system of equations is tractable for the same reason each of those three specializations is: (a) the equations $[\mathcal{D}_i, \mathcal{D}_j] = 0$ are invariant under $G_{\mathbb{C}}$-valued gauge transformations ($G_{\mathbb{C}}$ is the complexification of G); (b) the combination of setting $\mu = 0$ and dividing by G-valued gauge transformations is the same as forgetting the condition $\mu = 0$ and dividing by $G_{\mathbb{C}}$-valued gauge transformations. This means that the solutions can be understood in terms of complex geometry.

It is reasonable to expect that the model solution possesses the symmetries of the knot. So we assume that the model solution is invariant under a rotation of the boundary $\mathbb{R}^2$ around the point $p \in \mathbb{R}^2$ at which the knot lives, and also invariant under a scaling of $\mathbb{R}^2 \times \mathbb{R}_+$ keeping p fixed. With these assumptions, the equations $[\mathcal{D}_i, \mathcal{D}_j] = \mu = 0$ reduce to affine Toda equations which are integrable. One can find all the solutions in closed form, and the solutions that satisfy the Nahm pole boundary condition away from the knot are classified by an irreducible representation $R^\vee$ of the dual group $G^\vee$. These solutions were found for $G = SO(3)$, $G^\vee = SU(2)$ in my paper arXiv:1101.3216, and for general G by V. Mikhaylov in arXiv:1202.4848.

How would one go about proving that the KW equations with a boundary condition defined by one of these model equations is a well-posed (elliptic) problem? Basically, modulo generalities about uniformly degenerate elliptic operators, we need to show that the operator $\mathcal{L}$ obtained by linearizing around one of these solutions has no kernel or cokernel. It is again sufficient to show that the kernel vanishes, since $\mathcal{L}^\dagger$ is conjugate to $-\mathcal{L}$.

Just as in the absence of a knot, we will actually find a nonlinear analog of the vanishing of the kernel of $\mathcal{L}$: any solution of the KW equations on $\mathbb{R}^3 \times \mathbb{R}_+$ (with the knot as an infinite straight line in the boundary, as before) that is asymptotic to the model solution both along the boundary and at infinity actually coincides with it.

The vanishing results we want are the sort that often follow from a Weitzenbock formula. But none of the Weitzenbock formulas that we considered before are well-adapted to the presence of a knot. Even the more subtle Weitzenbock formula that includes the Nahm pole singularity away from a knot

$$-\int_M \mathrm{Tr}\left(\mathcal{V}^2+\mathcal{W}^2\right) = I' + \Delta$$

does not give any useful information, because I' (which is the sum of squares of quantities that vanish in the Nahm pole solution without a knot) is divergent in the presence of a knot so I' will be $+\infty$ and hence Δ will be $-\infty$ with a knot present.

So we need a new Weitzenbock formula. We imitate what we did before. We find a collection of quantities X_i whose vanishing characterizes the model solution. (Some obvious X_i are real and imaginary parts of $[\mathcal{D}_i, \mathcal{D}_j]$, and also μ; the others are quantities like $[\lambda_i, \lambda_y]$ that vanish because the model solution has $A_1 = \lambda_y = 0$.) Then if

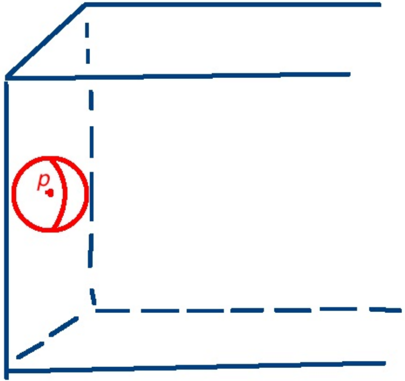
$$I'' = -\sum_i \int_{\mathbb{R}^3 \times \mathbb{R}_+} \mathrm{Tr}\, X_i^2,$$

we have to hope that there is an identity

$$-\int_{\mathbb{R}^3 \times \mathbb{R}_+} \mathrm{Tr}\, (\mathcal{V} \wedge \star\mathcal{V} + \mathcal{W} \wedge \star\mathcal{W}) = I'' + \widehat{\Delta},$$

where $\widehat{\Delta}$ is a new boundary term. It turns out that there is indeed an identity like this.

We still need to show that $\widehat{\Delta} = 0$ in the case of a solution that obeys the KW equations and is asymptotic near the knot to the model solution. For this, one needs to know what is the asymptotic behavior near the boundary and at infinity of a solution of the KW equations. We already know the behavior at a generic boundary point, which was used in our proof of the well-posedness of the Nahm pole boundary condition without a knot. to find the behavior near the knot and also at infinity, we now need to solve the angular part of the equation on a 2d hemisphere S^2_+.



Again it turns out that the eigenvalues of the angular operator are favorable, so there is no contribution to $\widehat{\Delta}$ either near p or at infinity. This fact together with the relevant Weitzenbock formula imply that a solution of the KW equations on $\mathbb{R}^3 \times \mathbb{R}_+$ that is asymptotic on the boundary and at infinity to the model solution with the knot actually coincides with that model solution. A linearized version of the same argument shows that the kernel of $\mathcal{L}$ vanishes, which is what we actually needed to know for ellipticity.

This is the main step in showing that $\mathcal{L}$ is a Fredholm operator in the presence of an arbitrary knot K embedded in any three-manifold W. Some details are still needed to show that this gives an elliptic boundary condition for the nonlinear KW equations in the general case of a curved knot. This is why the second paper has not appeared yet.