Gauge Theory And Khovanov Homology

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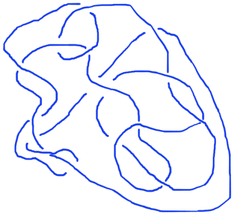
This will be a lecture on the Jones polynomial of a knot in three-space and its refinement that is known as Khovanov homology.

The first physics-based proposal concerning Khovanov homology of knots was made by Gukov, Vafa, and Schwarz (2004), who suggested that vector spaces associated to knots that had been introduced a few years earlier by Ooguri and Vafa were related to what appears in Khovanov homology.

A number of years later, I re-expressed this type of construction in terms of gauge theory and the counting of solutions of PDE's (see "Fivebranes and Knots," arXiv:1101.3216). That is the story I will describe today. Several previous lectures are available online (see arXiv:1401.6996 and arXiv:1108.3103) and I decided to take a different approach today.

In any event, the goal is to construct invariants of a knot



embedded in $\mathbb{R}^3$:

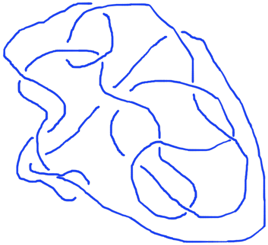
In the simplest version, the invariants will be obtained by simply counting, with signs, the solutions of an equation. The solutions will have an integer-valued topological invariant P and if a_n is the "number" (counted algebraically) of solutions with $P = n$, then the Jones polynomial of the knot will be

$$J(q) = \sum_n a_n q^n.$$

To get Khovanov homology, this situation is supposed to be "categorified," that is, we want for each n to define a complex of vector spaces whose Euler characteristic is a_n. We will come back to this later. For now I will just say that "categorification" will mean counting the solutions of some equations in one dimension more.

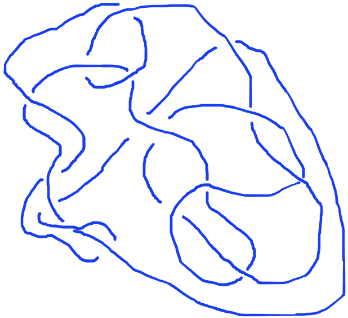
The equations whose solutions I claim should be counted to define the Jones polynomial and ultimately Khovanov homology might look ad hoc if written down without an explanation of where they come from. I could have started today's lecture by explaining the physical setup, but for today I decided instead to try a different approach of motivating the equations from what appears in an established mathematical approach to Khovanov homology, namely symplectic Khovanov homology (Seidel and Smith; Manolescu; Abouzeid and Smith).

Going all the way back to the original work of Vaughn Jones in 1983, most approaches to the Jones polynomial define an invariant in terms of some sort of presentation of a knot, for example a projection to a plane

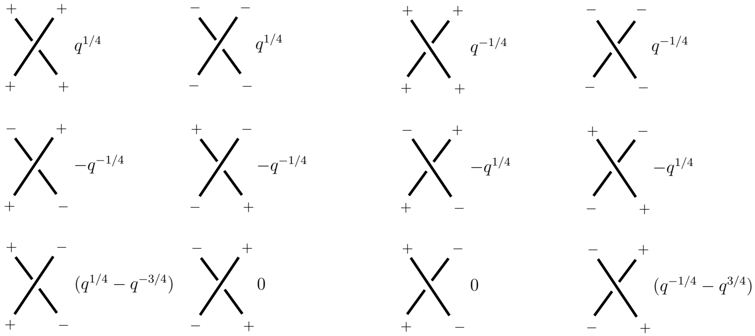


or some other "knot presentation." One defines something that is manifestly well-defined once such a presentation is given. What one defines is not obviously independent of the knot presentation, but turns out to be. That step is where the magic is. And there always is some magic.

To illustrate what I mean, and because I do not want to assume that everyone is familiar with the Jones polynomial, I will explain one standard definition of the Jones polynomial, the "vertex model," developed by Jones, L. Kauffman and others: Given a projection of a knot to a two-dimensional plane with only simple crossings and only simple maxima and minima of the height

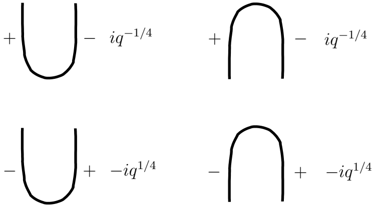


one labels the intervals between crossings, maxima, and minima by symbols $+$ or $-$. One sums over all such labelings with a suitable factor for each crossing



(Weights for other crossings are 0.)

and for each creation or annihilation event

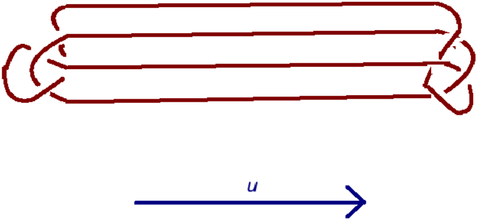


The sum is a sort of finite version of the sums of statistical mechanics, and in this case it is clear that the sum is a Laurent polynomial in $q^{1/2}$, known as the Jones polynomial. The output of the finite sum does not depend on the choice of how the knot was projected to the plane and so the Jones polynomial is a knot-invariant.

In short, like other traditional approaches to the Jones polynomial, the definition that I have explained is manifestly computable, but it is completely not obvious why it gives a knot invariant.

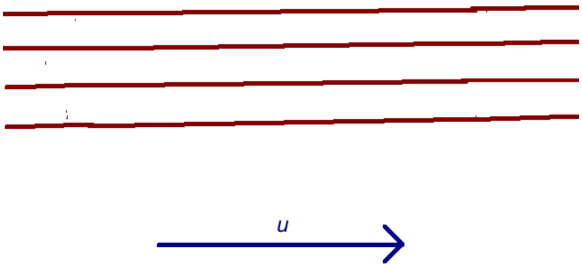
An approach based on counting solutions of PDE's has the opposite advantages and drawbacks: Topological invariance is potentially manifest (given certain generalities about elliptic PDE"s and assuming compactness is under control), but it may not be clear how to calculate. The ideal is to have manifest three- or (in the categorified case) four-dimensional symmetry together with a method of calculation. How might this be achieved?

Adapting what has been done mathematically in many problems involving counting of solutions of PDE's (going back to the Atiyah-Jones conjecture in Donaldson/Floer theory), a natural try, which I followed in work with D. Gaiotto (arXiv:1106.4789) is to stretch a knot in one direction:

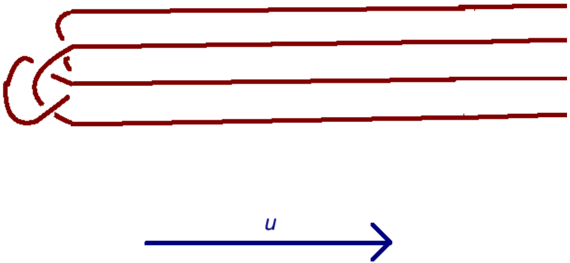


Then one wants it to be the case that except near the ends, the solutions are independent of u. (This is not automatically the case and we had to make a perturbation to get to a situation in which this would be true.)

Then we define a space $\mathcal{M}$ of u-independent solutions. We can think of these as the solutions in the presence of infinite long strands that extend in the u direction:

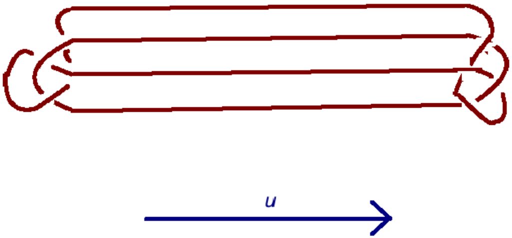


In $\mathcal{M}$, we define two "subspaces" $\mathcal{L}_\ell$ and $\mathcal{L}_r$ consisting of solutions that extend over the left or over the right. (For simplicity in my terminology, I will assume a given solution extends in at most one way, but this assumption is not necessary.) For example, a point in $\mathcal{L}_\ell$ represents a solution in a semiinfinite situation in which the strands terminate on the left:



Likewise $\mathcal{L}_r$ parametrizes solutions that extend over the right end.

For a global knot with the strands ending on both ends



the solutions are solutions in the middle that extend over both ends. So the global solutions are intersection points of $\mathcal{L}_\ell$ and $\mathcal{L}_r$. The integer a_n that appears as a coefficient in the Jones polynomial is supposed to be the algebraic intersection number of $\mathcal{L}_\ell$ and $\mathcal{L}_r$:

$$a_n = \mathcal{L}_\ell \cap \mathcal{L}_r.$$

(To be more exact, a_n is this intersection number computed by counting only intersections with $P = n$.)

In this language of intersections, categorification can happen if $\mathcal{M}$ is in a natural way a symplectic manifold and $\mathcal{L}_\ell$ and $\mathcal{L}_r$ are Lagrangian submanifolds. Then Floer cohomology – i.e. the A-model or the Fukaya category – of $\mathcal{M}$ gives a framework for categorification. From the point of view of today's lecture, the reason that all this will happen is that, even before we stretched the knot to reduce to intersections in $\mathcal{M}$, the equations whose solutions we were counting are equations for critical points of some Morse function(al) h.

In "symplectic Khovanov homology," a version of such a story is developed for Khovanov homology (at least in a singly-graded version) with a very specific $\mathcal{M}$. A description of this $\mathcal{M}$ that was proposed by Kamnitzer (and exploited in a mirror version by Cautis and Kamnitzer) and which provided an important clue in my work is as follows: $\mathcal{M}$ can be understood as a space of Hecke modifications. Let me explain this concept. Let C be a Riemann surface and $E \to C$ a holomorphic $G_{\mathbb{C}}$ bundle over C, where $G_{\mathbb{C}}$ is some complex Lie group. A Hecke modification of E at a point $p \in C$ is a holomorphic $G_{\mathbb{C}}$ bundle $E' \to C$ with an isomorphism to E away from p:

$$\varphi : E'|_{C \backslash p} \cong E|_{C \backslash p}.$$

For example, if $G_{\mathbb{C}} = \mathbb{C}^*$, the we can think of E as a holomorphic line bundle $\mathcal{L} \to C$. A holomorphic bundle $\mathcal{L}'$ that is isomorphic to $\mathcal{L}$ away from p is

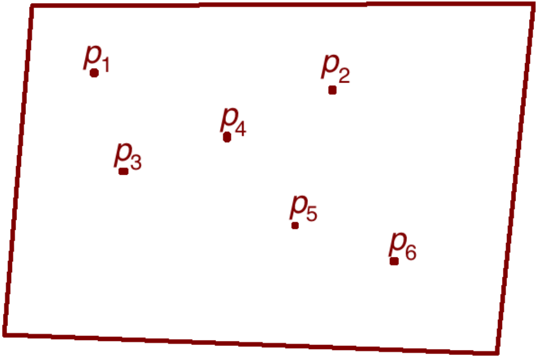
$$\mathcal{L}' = \mathcal{L}(np) = \mathcal{L} \otimes \mathcal{O}(p)^n$$

for some integer n. Here the integer n can be thought of as a weight of the Langlands-GNO dual group of $\mathbb{C}^*$, which is another copy of $\mathbb{C}^*$.

The reason that I write $G_{\mathbb{C}}$, making explicit that this is the complex form of the group, is that when we do gauge theory, the gauge group will be the compact real form and I will call this simply G. In general, for any G, there is a corresponding Langlands-GNO dual group $G^\vee$, with complexification $G^\vee_{\mathbb{C}}$, such that Hecke modifications of a holomorphic $G_{\mathbb{C}}$-bundle at a point $p \in C$ occur in families classified by dominant weights (or equivalently finite-dimensional representations) of $G^\vee_{\mathbb{C}}$ (or equivalently $G^\vee$).

For example, if $G_{\mathbb{C}} = GL(2, \mathbb{C})$, we can think of a $G_{\mathbb{C}}$-bundle $E \to C$ as a rank 2 complex vector bundle $E \to C$. The Langlanda-GNO dual group $G_{\mathbb{C}}^{\vee}$ is again $GL(2, \mathbb{C})$, and a Hecke modification dual to the 2-dimensional representation of $G_{\mathbb{C}}^{\vee}$ is as follows. For some local decomposition $E \cong \mathcal{O} \oplus \mathcal{O}$ in a neighborhood of $p \in C$, one has $E' \cong \mathcal{O}(p) \oplus \mathcal{O}$. The difference from the abelian case is that there is not just 1 Hecke modification of this type at p but a whole *family* of them, arising from the choice of decomposition of E as $\mathcal{O} \oplus \mathcal{O}$.

Because of the dependence on the decomposition of E, or more accurately on the choice of a subbundle $\mathcal{O} \subset E$ that is going to be replaced by $\mathcal{O}(p)$, the Hecke modifications of this type at p form a family, parametrized by $\mathbb{CP}^1$. Suppose we are given $2n$ points on $\mathbb{C} \cong \mathbb{R}^2$ at which we are going to make Hecke modifications of this type of a trivial bundle rank 2 complex vector bundle $E \to \mathbb{C}$:



The space of all such Hecke modifications would be a copy of $(\mathbb{CP}^1)^{2n}$, with one copy of $\mathbb{CP}^1$ at each point. However, there is a natural subvariety $\mathcal{M} \subset (\mathbb{CP}^1)^{2n}$ defined as follows. One adds a point ∞ at infinity to compactify $\mathbb{C}$ to $\mathbb{CP}^1$, so we are now making Hecke modifications of a trivial bundle $E = \mathcal{O} \oplus \mathcal{O} \to \mathbb{CP}^1$. A point in $(\mathbb{CP}^1)^{2n}$ determines a way to perform Hecke modifications at the points $p_1, p_2, \ldots, p_{2n}$ to make a new bundle E'. $\mathcal{M}$ is defined by requiring that $E' \otimes \mathcal{O}(-n\infty)$ is trivial. (If we were working in $PGL(2, \mathbb{C})$ rather than $GL(2, \mathbb{C})$, we would just say that E' should be trivial.)

Symplectic Khovanov homology is constructed by considering intersections of Lagrangian submanifolds of the space $\mathcal{M}$ of multiple Hecke modifications from a trivial bundle to itself.

We want to reinterpret this in terms of gauge theory PDE's.

In my work with Kapustin on gauge theory and geometric Langlands, an important fact was that $\mathcal{M}$ can be realized as a moduli space of solutions of a certain system of PDE's. However, although $\mathcal{M}$ is defined in terms of bundles on a 2-manifold $\mathbb{R}^2 \cong \mathbb{C}$, the PDE's are in 3 dimensions – on $\mathbb{R}^3$. As a result of this, everything in the rest of the lecture will be in a dimension 1 more than one might expect. To describe the Jones polynomial – an invariant of knots in 3-space – we will count solutions of certain PDE's in 4 dimensions, and the categorified version – Khovanov homology – will involve PDE's in 5 dimensions.

The 3-dimensional PDE's that we need are known as the Bogomolny equations. They are equations for a pair A, ϕ, where A is a connection on a G-bundle $E \to W_3$, with W_3 an oriented 3-dimensional Riemannian manifold, and ϕ is a section of $\mathrm{ad}(E) \to W_3$ (i.e. an adjoint-valued 0-form). If $F = \mathrm{d}A + A \wedge A$ is the curvature of A, then the Bogomolny equations are

$$F = \star \mathrm{d}_A \phi.$$

($\star$ is the Hodge star and d_A is the gauge-covariant extension of the exterior derivative.)

The Bogomolny equations have many remarkable properties and we will focus on just one aspect. We consider the Bogomolny equations on $W_3 = \mathbb{R} \times C$ with C a Riemann surface. Any connection A on a G-bundle $E \to C$ determines a holomorphic structure on E (or more exactly on its complexification): one simply writes $\mathrm{d}_A = \overline{\partial}_A + \partial_A$ and uses $\overline{\partial}_A$ to define the complex structure. (In complex dimension 1, there is no integrability condition that must be obeyed by a $\overline{\partial}$ operator.) So for any $y \in \mathbb{R}$, by restricting $E \to \mathbb{R} \times C$ to $E \to \{y\} \times C$, we get a holomorphic bundle $E_y \to C$. However, if the Bogomolny equations are satisfied, E_y is canonically independent of y. Indeed, a consequence of the Bogomolny equations is that $\overline{\partial}_A$ is independent of r up to conjugation. If we parametrize $\mathbb{R}$ by y then

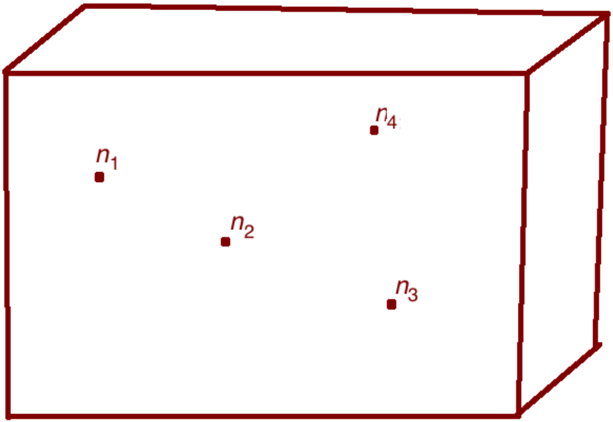
$$\left[\frac{D}{Dy} - \mathrm{i}\phi, \overline{\partial}_A\right] = 0.$$

The Bogomolny equations admit solutions with a singularity at isolated points in $\mathbb{R}^3$ (or in a more general 3-manifold W_3). Let me first describe the picture for $U(1)$. One fixes an integer n and one observes that the Bogomolny equation has an exact solution for any $x_0 \in \mathbb{R}^3$:

$$\phi = \frac{n}{2|\vec{x} - \vec{x}_0|}, \quad F = \star \mathrm{d}\phi.$$

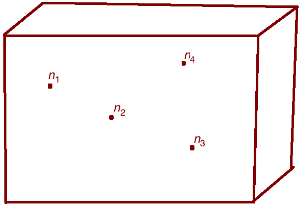
I have only defined F and not the connection A whose curvature is F or the line bundle $\mathcal{L}$ on which A is connection, but such an $\mathcal{L}$ and A exist (and are essentially unique) if $n \in \mathbb{Z}$.

For $G = U(1)$, since the Bogomolny equations are linear, they have a unique solution with singularities labeled by specified integers $n_1, n_2, \dots$ at specified points in $\mathbb{R}^3$:



We assume that $\sum_i n_i = 0$, which ensures that the given solution vanishes at infinity faster than $1/|\vec{x}|$.

We pick a decomposition $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ where we identify $\mathbb{R}^2$ as $\mathbb{C}$. Suppose that the singularities are at $y_i \times p_i$, with $y_i \in \mathbb{R}$, $p_i \in \mathbb{C}$:



For each $y \notin \{y_1, \ldots, y_n\}$, the indicated solution of the Bogomolny equations determines a holomorphic line bundle $\mathcal{L}_y \to \mathbb{C}$, and this naturally extends to $\mathcal{L}_y \to \mathbb{CP}^1$. $\mathcal{L}_y$ is constant up to isomorphism for y not equal to one of the y_i, but even when y crosses one of the y_i, $\mathcal{L}_y$ is constant when restricted to $\mathbb{CP}^1 \backslash p_i$. In crossing $y = y_i$, $\mathcal{L}_y$ undergoes a Hecke modification

$$\mathcal{L}_y \to \mathcal{L}_y \otimes \mathcal{O}(p_i)^{n_i}.$$

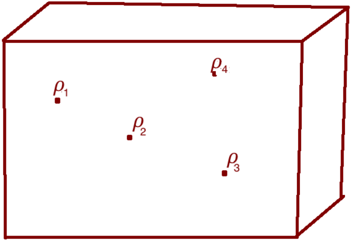
$\mathcal{L}_y$ is trivial for $y \to -\infty$ and for $y \to +\infty$. The solution describes a sequence of Hecke modifications mapping the trivial bundle to itself.

We can do something similar for any simple Lie group G. (This construction, introduced by 't Hooft in the late 1970's, is important in physical applications of quantum field theory.) Let T be the maximal torus of G and let $\mathfrak{t}$ be its Lie algebra. Pick a homomorphism $\rho : \mathfrak{u}(1) \to \mathfrak{t}$. Up to a Weyl transformation, such a ρ is equivalent to a dominant weight of the dual group $G^\vee$, so it corresponds to a representation $R^\vee$ of $G^\vee$. We turn the singular solution of the $U(1)$ Bogomolny equations that we already used (with $n = 1$) into a singular solution for G simply by

$$(A, \phi) \to (\rho(A), \rho(\phi)).$$

Then we look for solutions of the Bogomolny equations for G with singularities of this type at specified points $y_i \times p_i \in \mathbb{R}^3$.

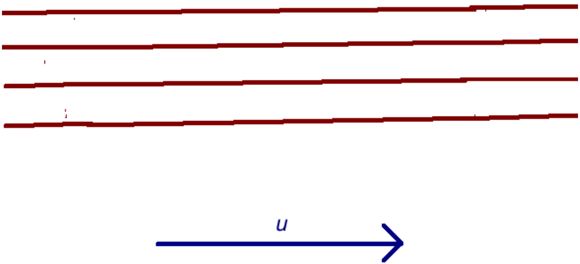
The picture is the same as before



except that now the points $y_i \times p_i$ are labeled by homomorphisms $\rho_i : \mathfrak{u}(1) \to \mathfrak{t}$, or in other words by representations $R_i^\vee$ of the dual group $G^\vee$, rather than by integers n_i. Also, now we must specify that the solution should go to 0 at infinity faster than $1/r$. Given this, such a solution describes a sequence of Hecke modifications at p_i of type ρ_i, mapping a trivial G-bundle $E \to \mathbb{CP}^1$ to itself.

The moduli space $\mathcal{M}$ of solutions of the Bogomolny equations on $\mathbb{R}^3$ with the indicated singularities and vanishing at infinity faster than $1/r$ is actually a hyper-Kahler manifold, essentially first studied by P. Kronheimer in the 1980's. If we pick a decomposition $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$, this picks one of the complex structures on the hyper-Kahler manifold and in that complex structure, $\mathcal{M}$ is the moduli space $\mathcal{M}_{p_1,\rho_1;p_2,\rho_2;\dots}$ of all Hecke modifications of the indicated types at the indicated points, mapping a trivial bundle over $\mathbb{CP}^1$ to itself.

This construction can be used to account for a number of properties of spaces of Hecke modifications, but for today we want to focus on the fact that for application to knot theory, we want $\mathcal{M}$ to be the space of u-independent solutions of some equations:



We already described $\mathcal{M}$ via solutions of some PDE's on $\mathbb{R}^3$, so now we have to think of $\mathcal{M}$ as a space of u-independent solutions on $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, where the second factor is parametrized by u.

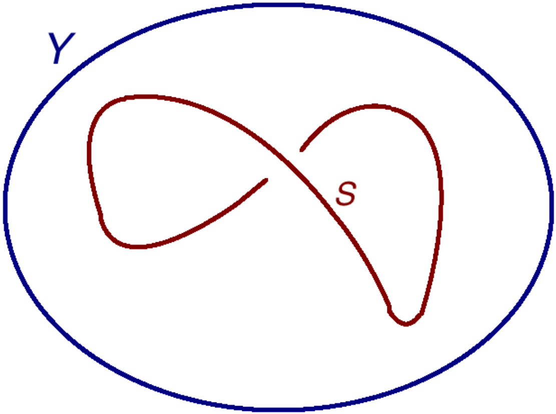
There actually are natural PDE's in four dimensions that work, sometimes called the KW equations (they appeared in my work on geometric Langlands with A. Kapustin). They are equations for a pair A, λ where A is a connection on $E \to Y_4$, Y_4 a four-manifold, and λ is a 1-form on Y_4 valued in $\mathrm{ad}(E)$:

$$F - \lambda \wedge \lambda = \star \mathrm{d}_A \lambda, \quad \mathrm{d}_A \star \lambda = 0.$$

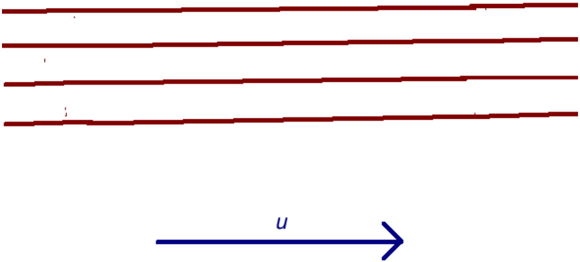
In a special case $Y_4 = W_3 \times \mathbb{R}$, with A a pullback from W_3 and $\lambda = \phi \, \mathrm{d}u$ (where ϕ is a section of $\mathrm{ad}(E)$ and u parametrizes the second factor in Y_4) these equations reduce to the Bogomolny equations:

$$F = \star \mathrm{d}_A \phi.$$

Therefore, the singular solution of the Bogomolny equations that we have already studied can be embedded as a singular solution of the KW equation, but now the singularity is along a line rather than a point. If Y_4 is a 4-manifold and $S \subset Y_4$ is an embedded 1-manifold, labeled by a homomorphism $\rho : \mathfrak{u}(1) \to \mathfrak{t}$ (or by a representation of $G^\vee$), then one can look for solutions of the KW equations with a singularity of the indicated type along S:



If we specialize to the case that $Y_4 = W_3 \times \mathbb{R}$, with $S = \cup_i S_i$, and $S_i = q_i \times \mathbb{R} \subset W_3 \times \mathbb{R}$ (q_i are points in $\mathbb{R}^3$)



then the u-independent solutions of the KW equations are parametrized by $\mathcal{M}$; and indeed one can show that these are all solutions of the KW equations in this situation with reasonable behavior at infinity.

So we have an elliptic PDE in four dimensions and we can specify in an interesting way what sort of singularity it should have on an embedded circle $S \subset Y_4$. But this sounds like a ridiculous framework for knot theory, because there is no knottedness of a 1-manifold in a 4-manifold!

A couple of things are missing from what I have said so far. There are a few directions that we could go next but I think I will head for categorification, which will also resolve the point I just mentioned.

Let us practice with an ordinary equation rather than a partial differential equation. Suppose that we are on a finite-dimensional compact oriented manifold N with a real vector bundle $V \to N$ with rank(V)=dimension(N). Suppose also we are given a section s of V. We can define an integer by counting, with multiciplities (and in particular with signs) the zeroes of s. This integer is the Euler class $\int_M \chi(V)$.

In general as far as I know, there is no way to categorify the Euler class of a vector bundle. However, suppose that $V = T^*N$ and that $s = \mathrm{d}h$ where h is a Morse function. Then the zeroes of s, which are critical points of h, have a natural "categorification" described in Morse homology. One defines a complex $\mathcal{V}$ with a basis vector ψ_p for each critical point p of h. The complex is $\mathbb{Z}$-graded by assigning to ψ_p the "index" of the critical point p, and it has a natural differential that is defined by counting gradient flow lines between different critical points.

Concretely the differential is defined by

$$\mathrm{d}\psi_p = \sum_q n_{pq}\psi_q$$

where the sum runs over all critical points q whose Morse index exceeds by 1 that of p, and the integer n_{pq} is defined by counting flows from p to q:

A "flow" is a solution of the gradient flow equation

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = -\vec{\nabla}h.$$

(To define this equation, one has to pick a Riemannian metric on the manifold N. The complex that one gets is independent of the metric up to quasi-isomorphism. What one actually counts are 1-parameter families of flow, related by time translations.)

This tells us what we need in order to be able to categorify a problem of counting solutions of the KW equations: we have to be able to write those equations as equations for a critical point of a functional $\Gamma(A, \phi)$:

$$\frac{\delta \Gamma}{\delta A} = \frac{\delta \Gamma}{\delta \lambda} = 0.$$

And the associated gradient flow equation, which will be a PDE in 5 dimensions on $X_5 = \mathbb{R} \times Y_4$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{\delta \Gamma}{\delta A}, \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}t} = -\frac{\delta \Gamma}{\delta \lambda}$$

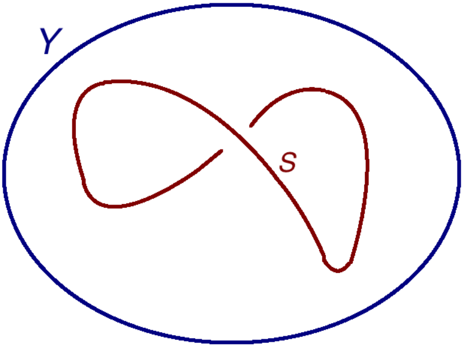
has to be elliptic, so that it will makes sense to try to count its solutions.

Generically, it is not true that the KW equations on a manifold Y_4 are equations for a critical point of some functional. However, this is true if $Y_4 = W_3 \times \mathbb{R}$ for some W_3. If singularities are present on an embedded 1-manifold $S \subset Y_4$ then there is a further condition: The KW equations in this situation are equations for critical points of a functional if and only if S is contained in a 3-manifold $W_3 \times q$, with q a point in $\mathbb{R}$. So to make categorification possible, we have to be in the situation that leads to knot theory: S is an embedded 1-manifold in a 3-manifold W_3.

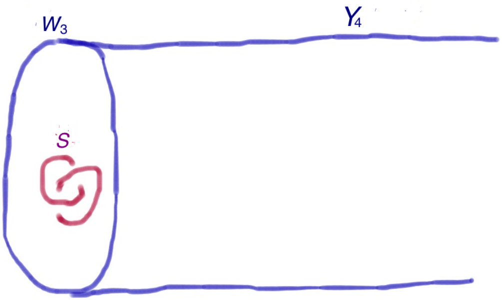
Naively, this leads to “categorified” knot invariants for any three-manifold W_3, but to justify this claim one needs some compactness properties for solutions of the equations under consideration. I think it is unclear in how much generality this will work, but I do expect it to work for knots in $\mathbb{R}^3$.

What I have described so far is supposed to correspond (for $W_3 = \mathbb{R}^3$, $G = PGL(2)$ and ρ corresponding to the 2-dimensional representation of $G^\vee = SL(2)$) to "singly-graded Khovanov homology." The only grading I've mentioned is the grading that is associated to the Morse index, or in other words to categorification. In the mathematical theory, one says that singly-graded Khovanov homology becomes trivial (it does not distinguish knots) if one "decategorifies" it and forgets the grading. In the approach I have described, this is true because in the uncategorified version, the embedded 1-manifold S is just a 1-manifold in a 4-manifold Y_4 (it has no reason to be embedded in the 3-manifold $W_3 \times q$) so there is no knottedness.

The physical picture makes clear where the additional "q"-grading of Khovanov homology would come from. It is supposed to come from the second Chern class, integrated over the 4-manifold Y_4. But for topological reasons, this q-grading cannot be defined in the construction as I have presented it so far. The second Chern class cannot be defined in the presence of the singularities that we've assumed:



The physical picture tells us what we have to do to get the q-grading: Y_4 should be a manifold with boundary, with the knot placed in its boundary:

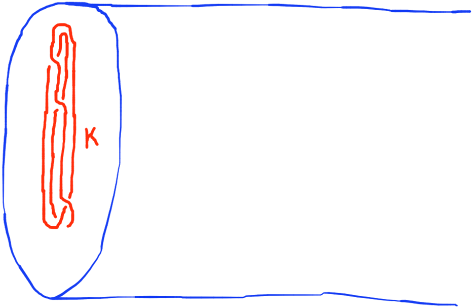


The boundary condition is subtle to describe, but has the property that the bundle is trivialized on the boundary, so the second Chern class can be defined. I will tell about this boundary condition in the next lecture.

In work I cited earlier, Gaiotto and I analyzed this situation (in the uncategorified situation, meaning that we counted solutions in 4 dimensions, not 5) with the aim of showing directly, without referring to the physical picture, that the Jones polynomial is

$$J(q) = \sum_n a_n q^n$$

where a_n is the number of solutions with second Chern class n. As usual, the starting point was to stretch the knot in one direction, reducing to equations in one dimension less:



The solutions in one dimension less that satisfied the boundary condition are related to a lot of interesting mathematical physics involving integrable systems, geometric Langlands, etc. We arrived at a description of the Jones polynomial that, as we now understand, is that of Bigelow (2001), following earlier work by R. Lawrence, and is closely related to the vertex model that I described at the beginning. What we added was to derive this description from a starting point with manifest 3-dimensional symmetry. The analog of this for the categorified version has not been done yet. In one version, it is expected to involve a Fukaya-Seidel category with a certain superpotential.