Coupled KPZ equations

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Plan of the talk

1. KPZ equation [1]
   - Ill-posedness, Renormalization
   - KPZ approximating equation-1: simple
   - Cole-Hopf solution, linear stochastic heat equation
   - KPZ approximating equation-2: suitable for finding invariant measures
   - Invariant measures of Cole-Hopf solution of KPZ eq
2. Coupled KPZ equation [2]
   - Coupled KPZ equation
   - Two approximating equations
   - Expansions
   - Convergence results due to paracontrolled calculus
   - Difference of two limits

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[1] F-Quastel, Stoch. PDE: Anal. Comp., **3** (2015)

[2] F, Seminaire de Probab., LNM, **2137** (2015), special issue for M. Yor

1. KPZ equation

- The KPZ (Kardar-Parisi-Zhang, 1986) equation describes the motion of growing interface with random fluctuation.
- It has the form for height function $h(t,x)$:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi(t,x), \quad x \in \mathbb{T} \ \text{(or } \mathbb{R}). \quad (1)$$

where $\mathbb{T} \equiv \mathbb{R}/\mathbb{Z} = [0,1)$.

- $\xi(t,x)$ is a space-time Gaussian white noise with mean 0 and correlation function:

$$E[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y). \quad (2)$$

Ill-posedness of the KPZ eq (1)

- The nonlinearity and roughness of the noise do not match.
- The linear SPDE:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \xi(t,x),$$

  obtained by dropping the nonlinear term has a solution $h \in C^{\frac{1}{4}-,\frac{1}{2}-}([0,\infty)\times\mathbb{R})$ a.s. Therefore, no way to define the nonlinear term $(\partial_x h)^2$ in (1) in a usual sense.
- Actually, the following Renormalized KPZ eq with compensator $\delta_x(x)$ $(=+\infty)$ has the meaning:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \xi(t,x).$$

KPZ approximating equation-1: Simple

- Symmetric convolution kernel Let $\eta \in C_0^\infty(\mathbb{R})$ s.t. $\eta(x) \geq 0$, $\eta(x) = \eta(-x)$ and $\int_{\mathbb{R}} \eta(x)dx = 1$ be given, and set $\eta^\varepsilon(x) := \frac{1}{\varepsilon}\eta(\frac{x}{\varepsilon})$ for $\varepsilon > 0$.
- Smeared noise $\xi^\varepsilon(t,x) = \xi(t) * \eta^\varepsilon(x)$
- Approximating Eq-1:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left((\partial_x h)^2 - c^\varepsilon\right) + \xi^\varepsilon(t,x),$$

where

$$c^\varepsilon = \int_{\mathbb{R}} \eta^\varepsilon(y)^2 dy \left(= \frac{1}{\varepsilon}\|\eta\|^2_{L^2(\mathbb{R})}\right).$$

Cole-Hopf solution to the KPZ equation

- Consider the Cole-Hopf transform: $Z = Z^\varepsilon := e^h$, then Z satisfies

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + Z\xi^\varepsilon(t,x).$$

(The product $Z\xi^\varepsilon$ is defined in Itô's sense.)

- Indeed, apply Itô's formula for $z = e^h$ to see

$$\begin{aligned}\partial_t Z &= Z\partial_t h + \tfrac{1}{2}Z(\partial_t h)^2 \\ &= \tfrac{1}{2}Z\{\partial_x^2 h + (\partial_x h)^2 - c^\varepsilon\} + Z\xi^\varepsilon + \tfrac{1}{2}Zc^\varepsilon \\ &= \tfrac{1}{2}\partial_x^2 Z + Z\xi^\varepsilon,\end{aligned}$$

since $Z\{\partial_x^2 h + (\partial_x h)^2\} = \partial_x^2 Z$.

- It is not difficult to show (Bertini-Giacomin '97) that $Z = Z^\varepsilon$ converges to the sol Z of the linear stochastic heat equation (SHE) (defined in Itô's sense):

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + Z\xi(t,x), \tag{3}$$

  with a multiplicative noise. This is a well-posed eq.

- This implies that the solution $h = h^\varepsilon$ of the approximating KPZ equation-1 converges to the Cole-Hopf solution of the KPZ eq. defined by

$$h_{CH}(t,x) := \log Z(t,x). \tag{4}$$

KPZ approximating equation-2: Suitable for studying inv meas

- We consider another KPZ approximating equation:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left((\partial_x h)^2 - c^\varepsilon\right) * \eta_2^\varepsilon + \xi^\varepsilon(t,x), \quad (5)$$

where $\eta_2(x) = \eta * \eta(x)$, $\eta_2^\varepsilon(x) = \eta_2(x/\varepsilon)/\varepsilon$.

- (F-Quastel [1]) The distribution of $B * \eta^\varepsilon(x)$, where B is the periodic Brownian motion (in case $\mathbb{T}$) or the two-sided Brownian motion (in case $\mathbb{R}$), is invariant for h^ε determined by (5).

Cole-Hopf transform for SPDE (5)

- The goal is to pass to the limit $\varepsilon \downarrow 0$ in the KPZ approximating equation (5):

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\left((\partial_x h)^2 - c^\varepsilon\right) * \eta_2^\varepsilon + \xi^\varepsilon(t,x).$$

- F-Quastel [1] considered its Cole-Hopf transform: $Z \,(\equiv Z^\varepsilon) := e^h$. Then, by Itô's formula, Z satisfies the SPDE:

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + A^\varepsilon(x,Z) + Z\xi^\varepsilon(t,x), \tag{6}$$

where

$$A^\varepsilon(x,Z) = \frac{1}{2}Z(x)\left\{\left(\frac{\partial_x Z}{Z}\right)^2 * \eta_2^\varepsilon(x) - \left(\frac{\partial_x Z}{Z}\right)^2(x)\right\}.$$

- The complex term $A^\varepsilon(x,Z)$ looks vanishing as $\varepsilon \downarrow 0$.

- But this is not true. Indeed, under the average in time t, $A^{\varepsilon}(x, Z)$ can be replaced by a linear function $\frac{1}{24}Z$.
- The limit as $\varepsilon \downarrow 0$ (under stationarity of tilt),

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + \frac{1}{24}Z + Z\xi(t, x).$$

- Or, heuristically at KPZ level,

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \frac{1}{24} + \xi(t, x).$$

- Or one can say that the limit $h(t, x)$ of the KPZ approximating eq-2 (5) is given by

$$h(t, x) = h_{CH}(t, x) + \frac{1}{24}t,$$

where $h_{CH}(t, x)$ denotes the Cole-Hopf solution.

Taking the limit $\varepsilon \downarrow 0$ (Similar to Boltzmann-Gibbs principle)

- Asymptotic replacement of $A^\varepsilon(x, Z^\varepsilon(s))$ by $\frac{1}{24} Z^\varepsilon(s, x)$.
- To avoid the complexity arising from the infiniteness of invariant measures, we view $h^\varepsilon(t, \rho) = \int h^\varepsilon(t, x)\rho(x)dx$ (height averaged by $\rho \in C_0^\infty(\mathbb{R}), \geq 0, \int \rho(x)dx = 1$) in modulo 1 (called wrapped process).

Theorem 1

For every $\varphi \in C_0(\mathbb{R})$ satisfying supp $\varphi \cap$ supp $\rho = \emptyset$, we have that

$$\lim_{\varepsilon \downarrow 0} E^{\pi \otimes \nu^\varepsilon} \left[\left\{ \int_0^t \tilde{A}^\varepsilon(\varphi, Z^\varepsilon(s)) ds \right\}^2 \right] = 0,$$

where π is the uniform measure for $h^\varepsilon(0, \rho) \in [0, 1)$,

$$\tilde{A}^\varepsilon(\varphi, Z) = \int_{\mathbb{R}} \tilde{A}^\varepsilon(x, Z)\varphi(x)dx$$

$$\tilde{A}^\varepsilon(x, Z) = A^\varepsilon(x, Z) - \frac{1}{24} Z(x).$$

Invariant measures of Cole-Hopf solution

As a byproduct, one can give a class of invariant measures for the Cole-Hopf solution of the KPZ equation. (We state results only on $\mathbb{R}$.)

- Let ν^c be the distribution of $B(x) + c\,x$, where $B(x)$ is the two-sided Brownian motion s.t. $\nu^c(B(0) \in dx) = dx$. Note that these are not probability measures.
- Then, $\{\nu^c\}_{c\in\mathbb{R}}$ are invariant under the Cole-Hopf solution of the KPZ equation.
- c means the average tilt of the interface. We have different invariant measures for different average tilts.

2. Coupled KPZ equation

- (Ferrari-Sasamoto-Spohn 2013) $\mathbb{R}^d$-valued coupled KPZ equation for $h(t,x) = (h^\alpha(t,x))_{\alpha=1}^d$ on $\mathbb{T}$ (or $\mathbb{R}$):

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \sigma^\alpha_\beta \xi^\beta, \ x \in \mathbb{T}. \tag{7}$$

- We use Einstein's convention.
- $\xi(t,x) = (\xi^\alpha(t,x))_{\alpha=1}^d$ is an $\mathbb{R}^d$-valued space-time Gaussian white noise with the covariance structure

$$E[\xi^\alpha(t,x)\xi^\beta(s,y)] = \delta^{\alpha\beta}\delta(x-y)\delta(t-s).$$

- $(\sigma^\alpha_\beta)_{1\le\alpha,\beta\le d}$, $(\Gamma^\alpha_{\beta\gamma})_{1\le\alpha,\beta,\gamma\le d}$ are given constants.
- From the form of the equation (7), the constants $\Gamma^\alpha_{\beta\gamma}$ ought to satisfy

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} \ \text{ for all } \ \alpha, \beta, \gamma. \tag{8}$$

Two approximating equations

- Symmetric convolution kernels $\eta \in C_0^\infty(\mathbb{R})$ and $\eta^\varepsilon, \eta_2^\varepsilon$ are given similarly as before.
- Simple approximating equation with smeared noise:

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon A^{\beta\gamma}) + \sigma^\alpha_\beta \xi^\beta * \eta^\varepsilon, \tag{9}$$

where $A^{\beta\gamma} = \sum_{\delta=1}^d \sigma^\beta_\delta \sigma^\gamma_\delta$ and $c^\varepsilon = \frac{1}{\varepsilon}\|\eta\|^2_{L^2(\mathbb{R})}$ is the same as before.

- Approximation suitable for studying invariant measures:

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon A^{\beta\gamma}) * \eta_2^\varepsilon + \sigma^\alpha_\beta \xi^\beta * \eta^\varepsilon, \tag{10}$$

- For the solution of (10), F [2] showed (on $\mathbb{R}$), under the additional condition:

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\gamma_{\alpha\beta} \tag{11}$$

and $\sigma^\alpha_\beta = \delta^\alpha_\beta$ (Kronecker's δ), the infinitesimal invariance of the distribution of $B * \eta^\varepsilon(x)$, where B is the $\mathbb{R}^d$-valued two-sided Brownian motion.

- When $d = 1$ and $\Gamma^{\alpha}_{\beta\gamma} = \sigma^{\alpha}_{\gamma} = 1$ for simplicity, the approximating equations (9) and (10) have the forms:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left((\partial_x h)^2 - c^{\varepsilon}\right) + \xi * \eta^{\varepsilon}, \qquad (12)$$

and

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left((\partial_x h)^2 - c^{\varepsilon}\right) * \eta_2^{\varepsilon} + \xi * \eta^{\varepsilon}, \qquad (13)$$

respectively, as we have already discussed.

Goal

- As we saw, the solution of (12) converges as $\varepsilon \downarrow 0$ to the Cole-Hopf solution $h_{CH}(t, x)$ of the KPZ equation, while the solution of (13) converges to $h_{CH}(t, x) + \frac{1}{24}t$ under the equilibrium setting (F-Quastel) and also under the non-equilibrium setting (Hoshino).
- The method of F-Quastel is based on the Cole-Hopf transform, which is not available for the coupled equation with multi-components in general.
- Our goal is to study the limits of the solutions of (9) and (10) as $\varepsilon \downarrow 0$ based on the paracontrolled calculus (Gubinelli and others).
- In particular, we study the difference between these two limits.

Expansion

- We think of the noise as the leading term and the nonlinear term as its perturbation by putting (small parameter) $a > 0$ in front of the nonlinear term, though we eventually take $a = 1$.

$$\mathcal{L}h^\alpha = \frac{a}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \sigma^\alpha_\beta \xi^\beta,$$

  where $\mathcal{L} = \partial_t - \frac{1}{2}\partial_x^2$.
- We expand the solution h of the coupled KPZ eq (7) in a: $h^\alpha = \sum_{k=0}^\infty a^k h^\alpha_k$. Then, we have

$$\sum_{k=0}^\infty a^k \mathcal{L}h^\alpha_k = \sigma^\alpha_\beta \xi^\beta + \frac{a}{2}\sum_{k_1,k_2=0}^\infty a^{k_1+k_2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta_{k_1}\partial_x h^\gamma_{k_2}.$$

- Comparing the terms of order $a^0, a^1, a^2, a^3, \dots$ in both sides and noting the condition (8), we obtain the followings:

$$
\begin{aligned}
\mathcal{L}h_0^\alpha &= \sigma_\beta^\alpha \xi^\beta, \\
\mathcal{L}h_1^\alpha &= \frac{1}{2}\Gamma_{\beta\gamma}^\alpha \partial_x h_0^\beta \partial_x h_0^\gamma, \\
\mathcal{L}h_2^\alpha &= \Gamma_{\beta\gamma}^\alpha \partial_x h_0^\beta \partial_x h_1^\gamma, \\
\mathcal{L}h_3^\alpha &= \Gamma_{\beta\gamma}^\alpha \partial_x h_0^\beta \partial_x h_2^\gamma + \frac{1}{2}\Gamma_{\beta\gamma}^\alpha \partial_x h_1^\beta \partial_x h_1^\gamma, \\
&\dots
\end{aligned}
$$

- By replacing ξ^β by $\xi^\beta * \eta^\varepsilon$ and taking care of the factor $-c^\varepsilon A^{\beta\gamma}$, we have the expansion of the solution of the equation (9) (simple approximating eq):

$$h^{\alpha,\varepsilon} = \sum_{k=0}^{\infty} a^k h_k^{\alpha,\varepsilon}$$

and the equations:

$$\begin{aligned}
\mathcal{L}h_0^\alpha &= \sigma_\beta^\alpha \xi^\beta * \eta^\varepsilon, \\
\mathcal{L}h_1^\alpha &= \frac{1}{2}\Gamma_{\beta\gamma}^\alpha (\partial_x h_0^\beta \partial_x h_0^\gamma - c^\varepsilon A^{\beta\gamma}), \\
\mathcal{L}h_2^\alpha &= \Gamma_{\beta\gamma}^\alpha \partial_x h_0^\beta \partial_x h_1^\gamma, \\
\mathcal{L}h_3^\alpha &= \Gamma_{\beta\gamma}^\alpha (\partial_x h_0^\beta \partial_x h_2^\gamma - B^{\beta\gamma,\varepsilon}) + \frac{1}{2}\Gamma_{\beta\gamma}^\alpha (\partial_x h_1^\beta \partial_x h_1^\gamma - C^{\beta\gamma,\varepsilon}), \\
&\cdots
\end{aligned}$$

- Furthermore, we have the expansion

$$\tilde{h}^{\alpha,\varepsilon} = \sum_{k=0}^{\infty} a^k \tilde{h}_k^{\alpha,\varepsilon}$$

of the solution of the equation (10) (approximation suitable for studying invariant measures) and the equations:

$$\mathcal{L}\tilde{h}_0^\alpha = \sigma_\beta^\alpha \xi^\beta * \eta^\varepsilon,$$

$$\mathcal{L}\tilde{h}_1^\alpha = \frac{1}{2}\Gamma_{\beta\gamma}^\alpha(\partial_x \tilde{h}_0^\beta \partial_x \tilde{h}_0^\gamma - c^\varepsilon A^{\beta\gamma}) * \eta_2^\varepsilon,$$

$$\mathcal{L}\tilde{h}_2^\alpha = \Gamma_{\beta\gamma}^\alpha \partial_x \tilde{h}_0^\beta \partial_x \tilde{h}_1^\gamma * \eta_2^\varepsilon,$$

$$\mathcal{L}\tilde{h}_3^\alpha = \Gamma_{\beta\gamma}^\alpha(\partial_x \tilde{h}_0^\beta \partial_x \tilde{h}_2^\gamma - \tilde{B}^{\beta\gamma,\varepsilon}) * \eta_2^\varepsilon + \frac{1}{2}\Gamma_{\beta\gamma}^\alpha(\partial_x \tilde{h}_1^\beta \partial_x \tilde{h}_1^\gamma - \tilde{C}^{\beta\gamma,\varepsilon}) * \eta_2^\varepsilon,$$

$$\ldots$$

- After defining $h_0^\alpha, \dots, h_3^\alpha$ (actually, $+$one more term h_4^α) in the above way, we consider the equation for extra term and solve it by fixed point theorem in a suitable space (controlled by these driving terms). Similar for $\tilde{h}$.
- Our notation and those in [Hairer, Gubinelli] studying the case $d = 1$ correspond with each other as follows:

$$h_0 = X_\epsilon^{\mathrm{I}}, h_1 = X_\epsilon^{\mathrm{Y}}, h_2 = X_\epsilon^{\mathrm{YY}}, h_3 = X_\epsilon^{\vee\vee} + X_\epsilon^{\mathrm{YY}\,\mathrm{Y}},$$

$$c^\varepsilon A^{\beta\gamma} = c_\epsilon^{\vee}, C^{\beta\gamma,\varepsilon} = c_\epsilon^{\vee\vee}, B^{\beta\gamma,\varepsilon} = c_\epsilon^{\mathrm{YY}\,\mathrm{Y}}.$$

Convergence results due to paracontrolled calculus
(rough formulation)

- Convergence of driving terms:
  $^\exists h_i^\alpha$ ($i = 0, 1, \dots, 4$; indeed, two terms in h_3^α should be considered separately) s.t.

$$h_i^{\alpha,\varepsilon} \underset{\varepsilon\downarrow 0}{\to} h_i^\alpha \text{ and } \tilde{h}_i^{\alpha,\varepsilon} \underset{\varepsilon\downarrow 0}{\to} h_i^\alpha \text{ in } C([0,T], C^{\kappa_i}(\mathbb{T})),$$

  where $\kappa_0 = \mu$, $\kappa_1 = 2\mu$, $\kappa_2 = \mu + 1$, $\kappa_3 = 2\mu + 1$, $\kappa_4 = 2\mu - 1 (< 0)$ with $\mu \in (\frac{1}{3}, \frac{1}{2})$.
- If driving terms $(h_i^{\alpha,\varepsilon})$ converges to (h_i^α), then the solutions of the KPZ equations with these driving terms converge in $C([0,T], C^\mu(\mathbb{T}))$.

Example

- Assume the condition (11) (the condition we assumed for studying the stationarity) in addition to (8) and $(\sigma^\alpha_\beta) = \sigma I$ with $\sigma \in \mathbb{R}$ and a unit matrix I.
- In this case, we have

$$\tilde{h}^\alpha(t,x) = h^\alpha(t,x) + c^\alpha t, \quad 1 \le \alpha \le d,$$

in the limit, where

$$c^\alpha = \frac{\sigma^4}{24} \sum_{\beta,\gamma,\gamma_1,\gamma_2} \Gamma^\alpha_{\beta\gamma} \Gamma^\beta_{\gamma_1\gamma_2} \Gamma^\gamma_{\gamma_1\gamma_2}.$$

Summary of the talk.

1. KPZ equation:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}(\partial_x h)^2 + \xi(t,x), \quad x \in \mathbb{R}.$$

2. KPZ approximating equation with $\xi^\varepsilon(t,x) = \xi(t) * \eta^\varepsilon(x)$:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\big((\partial_x h)^2 - c^\varepsilon\big) * \eta_2^\varepsilon + \xi^\varepsilon(t,x)$$

   has invariant measure ν^ε (=distribution of $B * \eta^\varepsilon$).

3. As $\varepsilon \downarrow 0$, h^ε converges to $h_{CH}(t,x) + \frac{1}{24}t$.
4. To study the limits of two types of coupled KPZ approximating equations, we apply Gubinelli's paracontrolled calculus.

Thank you for your attention!