TSINGHUA LECTURE NOTES ON MAXWELL-KLEIN-GORDON EQUATIONS

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This lecture note is based on my joint work [1] with Lydia Bieri and Sohrab Shahshahani.

1. Preliminary

1.1. Complex scalar field. We first introduce the *complex scalar field*. Let M be the $3+1$ Minkowski spacetime with metric $g = (-1, 1, 1, 1)$. We have the standard Levi-Civita connection ∇, which is actually the usual partial derivative. Consider the complex line bundle:

$$V := M \times \mathbb{C} \tag{1.1}$$

over M with Hermitian product $<, \cdot, >$. Let D be the compatible connection on V, that is, for any vectorfield X on M and $\phi, \psi \in \Gamma(V)$ we have

$$X < \phi, \psi > = < D_X\phi, \psi > + < \phi, D_X\psi > . \tag{1.2}$$

Obviously, such a connection exists. For example, the usual partial differentiation for complex functions.

Note that the number "1" is a natural basis for V. Let's denote it by "1_V". The connection coefficient for D is given by

$$D_\alpha 1_V = iA_\alpha.$$

In view of (1.2),

$$0 = iA_\alpha + \overline{iA_\alpha},$$

which implies that A_α is real. Therefore, for $\phi \in \Gamma(V)$, we have

$$D_\alpha\phi = \partial_\alpha\phi + iA_\alpha\phi.$$

The gauge transformation which leaves $<, \cdot, >$ invariant is given by

$$1_V \rightsquigarrow \tilde{1}_V := e^{i\chi} 1_V, \tag{1.3}$$

where χ is some real-valued function on M. Then we have

$$i\widetilde{A}_\alpha \cdot \tilde{1}_V = \left(ie^{i\chi}\partial_\alpha\chi + e^{i\chi} iA_\alpha\right) 1_V, \quad \Rightarrow \quad \widetilde{A}_\alpha = A_\alpha + \partial_\alpha\chi.$$

The Hessian associated to D is defined as

$$D^2(X, Y)\phi = D_X D_Y\phi := D_X(D_Y\phi) - D_{\nabla_X Y}\phi \tag{1.4}$$

By writing

$$D_X(D_Y\phi) = X(Y\phi) + iX(Y^\alpha)A_\alpha\phi + iY^\alpha X(A_\alpha)\phi + iY^\alpha A_\alpha X\phi + iX^\alpha A_\alpha Y\phi + iX^\alpha A_\alpha(iY^\beta A_\beta\phi),$$

we show

$$D_X D_Y\phi - D_Y D_X\phi = iF(X,Y)\phi,$$

where

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{1.5}$$

is the curvature tensor of D. Note that F is a 2-form satisfying

$$dF = \partial_{[\alpha}F_{\beta\gamma]} = 0.$$

A section $\phi \in \Gamma(V)$ is called *a complex scalar field* if it satisfies the equation

$$D^\mu D_\mu\phi = 0. \tag{1.6}$$

The MKG equations describe an electromagnetic field represented by a two form $F_{\mu\nu}$ and a charged scalar field represented by a complex valued function ϕ, and are given by

$$\begin{aligned} &\partial^\mu F_{\mu\nu} = \mathrm{Im}\,(\overline{\phi}D_\nu\phi) =: J_\nu(\phi) \\ &\partial^{\mu*}F_{\mu\nu} = 0 \\ &D^\mu D_\mu\phi = 0, \end{aligned} \tag{1.7}$$

Here $*$ the Hodge star operator. In fact the second equation of (1.7) is a consequence of the other two and the relation between F and A. From a variational point of view these equations arise as the Euler-Lagrange equations associated to the Lagrangian

$$\mathcal{L}_{MKG} := \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi\overline{D^\mu\phi}.$$

1.2. **Coordinates and weights.** We use two different sets of coordinates. The rectangular coordinates are $x^0, \ldots, x^3$, but we also use t for x^0. When using rectangular coordinates Greek indices $(\alpha,\ \beta,\ ...)$ can take any value between 0 and 4 and lower case Roman indices $(i,\ j,\ ...)$ correspond only to spatial variables and thus take values between 1 and 3. We will also use null coordinates defined as

$$u = t - r, \qquad \underline{u} = t + r.$$

In these coordinates capital Roman indices $(A,\ B,\ ...)$ correspond to spherical variables. We also introduce the weights

$$\tau_+^2 = 1 + \underline{u}^2, \qquad \tau_-^2 = 1 + u^2.$$

The null derivatives L and $\underline{L}$ are defined as

$$L := \partial_t + \partial_r, \qquad \underline{L} = \partial_t - \partial_r.$$

Given a 2-form $G_{\mu\nu}$ we define its null-decomposition as

$$\underline{\alpha}_A(G) := G_{A\underline{L}}, \quad \alpha_A(G) := G_{AL}, \quad \rho(G) := \frac{1}{2}G_{L\underline{L}}, \quad \sigma(G)\epsilon_{AB} := G_{AB},$$

where ϵ_{AB} is the the volume form on $S_{t,r}$ (the sphere of radius r on the time-slice $x^0 \equiv t$, see below). If there is no risk of confusion we simply write α instead of $\alpha(G)$ and similarly for the other components.

1.3. **Regions.** We denote by Σ_t the spatial hypersurface $x^0 \equiv t$. C_u and $\underline{C}_{\underline{u}}$ denote the outgoing and incoming null cones $t-r \equiv u$ and $t+r \equiv \underline{u}$ respectively. $S_{t,r}$, $\tilde{S}_{u,r}$ and $\tilde{S}_{\underline{u},r}$ are the spheres of constant radius r on Σ_t, C_u and $\underline{C}_{\underline{u}}$ respectively. Due to the presence of charge we need separate estimates in the interior and exterior of a fixed outgoing null cone. For this we introduce the following regions

$$V_T := \{t \leq T,\ u \leq -1\}, \qquad V_T^O := \{t \leq T,\ u \geq -1\}.$$

We also define

$$\begin{aligned} &\Sigma_t(T) = \Sigma_t \cap V_T, \qquad C_u(T) = C_u \cap V_T, \qquad \underline{C}_{\underline{u}}(T) = \underline{C}_{\underline{u}} \cap V_T, \\ &\Sigma_t^O(T) = \Sigma_t \cap V_T^O, \qquad C_u^O(T) = C_u \cap V_T^O, \qquad \underline{C}_{\underline{u}}^O(T) = \underline{C}_{\underline{u}} \cap V_T^O. \end{aligned}$$

1.4. **Vectorfields and a modified Lie derivative.** For a vector field X we denote by L_X the usual Lie derivative with respect to X. Moreover, as stated earlier, D_X is defined as $\nabla_X + iX^\alpha A_\alpha$, where ∇_X is the Levi-Cevita covariant differentiation operator and A_α is defined by $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$. Operators on the spheres $S_{t,r}$ are written as $\nabla\!\!\!/$, $D\!\!\!/$, ... If X is conformal Killing we denote by Ω_X its conformal factor, $L_X g = \Omega_X g$. We have

$$\Omega_{T_\mu} = \Omega_{\Omega_{\mu\nu}} = 0, \quad \Omega_S = 2, \quad \Omega_{\overline{K}_0} = 4t.$$

If ϕ, J and G are a scaler field, one-form, and two-form respectively, we define the modified Lie derivatives

$$\begin{aligned} \mathcal{L}_X \phi &= D_X \phi + \Omega_X \phi, \\ \mathcal{L}_X J &= L_X J + \Omega_X J, \\ \mathcal{L}_X G &= L_X G. \end{aligned}$$

We also define $\mathcal{L}$ for bundle-valued tensors by requiring it to be a derivation and that it agree with D on sections. In particular we have

$$\begin{aligned} &\mathcal{L}_X D_\mu \phi := X^\nu D_\nu D_\mu \phi + \Omega_X D_\mu \phi + \nabla_\mu X^\nu D_\nu \phi, \\ &\mathcal{L}_X D_\mu D_\nu \phi := X^\alpha D_\alpha D_\mu D_\nu \phi + \Omega_X D_\mu D_\nu \phi + \nabla_\mu X^\alpha D_\alpha D_\nu \phi + \nabla_\nu X^\alpha D_\mu D_\alpha \phi. \end{aligned}$$

The next lemmas summarize some commutation properties of the modified Lie derivative.

Lemma 1.1. *Let X be a conformal Killing vector field with constant conformal factor Ω_X. Then*

$$\begin{aligned} &D^\mu D_\mu \mathcal{L}_X \phi - \mathcal{L}_X D^\mu D_\mu \phi = i\left(2X^\alpha F_{\mu\alpha} D^\mu \phi + \nabla^\mu (X^\alpha F_{\mu\alpha})\phi\right), \\ &\nabla^\mu \mathcal{L}_X G_{\mu\nu} - \mathcal{L}_X(\nabla^\mu G_{\mu\nu}) = 0, \\ &{}^*\mathcal{L}_X G_{\mu\nu} - \mathcal{L}_X {}^*G_{\mu\nu} = 0. \end{aligned} \tag{1.8}$$

Proof. We only prove the first two statements. First note that

$$\begin{aligned} D_\mu \mathcal{L}_X \phi = \Omega_X D_\mu \phi + D_\mu(D_X \phi) &= X^\alpha D_\alpha D_\mu \phi + \Omega_X D_\mu \phi + \nabla_\mu X^\alpha D_\alpha \phi + iX^\alpha F_{\mu\alpha} \phi \\ &= \mathcal{L}_X D_\mu \phi + iX^\alpha F_{\mu\alpha} \phi. \end{aligned} \tag{1.9}$$

It follows that

$$\begin{aligned} D_\nu D_\mu \mathcal{L}_X \phi =& \nabla_\nu X^\alpha D_\alpha D_\mu \phi + X^\alpha D_\alpha D_\nu D_\mu \phi + iX^\alpha F_{\nu\alpha} D_\mu \phi + \Omega_X D_\nu D_\mu \phi \\ &+ \nabla_\nu \nabla_\mu X^\alpha D_\alpha \phi + \nabla_\mu X^\alpha D_\nu D_\alpha \phi + i\nabla_\nu (X^\alpha F_{\mu\alpha})\phi + iX^\alpha F_{\mu\alpha} D_\nu \phi. \end{aligned}$$

Since $\nabla^2 X = 0$ for the vector-fields under consideration, the first statement of the lemma follows by contracting in the μ and ν coordinates. For the second statement note that

$$\begin{aligned}\nabla_\rho \mathcal{L}_X G_{\mu\nu} &= \nabla_\rho X^\alpha \nabla_\alpha G_{\mu\nu} + X^\alpha \nabla_\alpha \nabla_\rho G_{\mu\nu} + \nabla_\rho \nabla_\mu X^\alpha G_{\alpha\nu} + \nabla_\mu X^\alpha \nabla_\rho G_{\alpha\nu} \\ &\quad + \nabla_\rho \nabla_\nu X^\alpha G_{\mu\alpha} + \nabla_\nu X^\alpha \nabla_\rho G_{\mu\alpha} \\ &= X^\alpha \nabla_\alpha \nabla_\rho G_{\mu\nu} + \nabla_\rho X^\alpha \nabla_\alpha G_{\mu\nu} + \nabla_\mu X^\alpha \nabla_\rho G_{\alpha\nu} + \nabla_\nu X^\alpha \nabla_\rho G_{\mu\alpha} \\ &= L_X \nabla_\rho G_{\mu\nu}.\end{aligned}$$

Noting that $L_X g^{\mu\rho} = -\Omega_X g^{\mu\rho}$ we get

$$\nabla^\mu \mathcal{L}_X G_{\mu\nu} = L_X(\nabla^\mu G_{\mu\nu}) + \Omega_X \nabla^\mu G_{\mu\nu} = \mathcal{L}_X(\nabla^\mu G_{\mu\nu}).$$

□

Lemma 1.2. *Let X and Y be conformal Killing vectorfields with constant conformal factors. Then*

$$\begin{aligned}D^\mu D_\mu \mathcal{L}_Y \mathcal{L}_X \phi - \mathcal{L}_Y \mathcal{L}_X D^\mu D_\mu \phi =& i\left(2Y^\beta F_{\mu\beta} D^\mu \mathcal{L}_X \phi + \nabla^\mu (Y^\beta F_{\mu\beta}) \mathcal{L}_X \phi\right) \\ &+ i\left(2X^\alpha F_{\mu\alpha} D^\mu \mathcal{L}_Y \phi + \nabla^\mu (X^\alpha F_{\mu\alpha}) \mathcal{L}_Y \phi\right) \\ &+ i\left(2[Y,X]^\alpha F_{\mu\alpha} D^\mu \phi + \nabla^\mu ([Y,X]^\alpha F_{\mu\alpha}) \phi\right) \\ &+ i\left(2X^\alpha \mathcal{L}_Y F_{\mu\alpha} D^\mu \phi + \nabla^\mu (X^\alpha \mathcal{L}_Y F_{\mu\alpha}) \phi\right) \\ &+ 2X^\alpha Y^\beta F^\mu{}_\alpha F_{\mu\beta} \phi.\end{aligned}$$

Proof. From Lemma 1.1 we have

$$D^\mu D_\mu \mathcal{L}_Y \mathcal{L}_X \phi - \mathcal{L}_Y D^\mu D_\mu \mathcal{L}_X \phi = i\left(2Y^\nu F_{\mu\nu} D^\mu \mathcal{L}_X \phi + \nabla^\mu (Y^\nu F_{\mu\nu}) \mathcal{L}_X \phi\right). \tag{1.10}$$

Again applying Lemma 1.1 to $\mathcal{L}_Y D^\mu D_\mu \mathcal{L}_X \phi$ we get

$$\begin{aligned}\mathcal{L}_Y D^\mu D_\mu \mathcal{L}_X \phi - \mathcal{L}_Y \mathcal{L}_X D^\mu D_\mu \phi =& i\left(2X^\alpha F_{\mu\alpha} D^\mu \mathcal{L}_Y \phi + \nabla^\mu (X^\alpha F_{\mu\alpha}) \mathcal{L}_Y \phi\right) \\ &+ i\left(2[Y,X]^\alpha F_{\mu\alpha} D^\mu \phi + \nabla^\mu ([Y,X]^\alpha F_{\mu\alpha}) \phi\right) \\ &+ i\left(2X^\alpha \mathcal{L}_Y F_{\mu\alpha} D^\mu \phi + \nabla^\mu (X^\alpha \mathcal{L}_Y F_{\mu\alpha}) \phi\right) \\ &+ 2X^\alpha Y^\beta F^\mu{}_\alpha F_{\mu\beta} \phi.\end{aligned}$$

□

We use the following vector fields in this work

$$T_\mu = \partial_\mu, \tag{1.11}$$

$$S = x^\mu \partial_\mu, \tag{1.12}$$

$$\Omega_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad \mu < \nu, \tag{1.13}$$

$$K_0 = \frac{1}{2}(u^2 \underline{L} + \underline{u}^2 L), \tag{1.14}$$

$$\overline{K}_0 = T_0 + K_0 = \left(\frac{1+u^2}{2}\right) \underline{L} + \left(\frac{1+\underline{u}^2}{2}\right) L. \tag{1.15}$$

T_μ, S, and $\Omega_{\mu\nu}$ will be used as commutators, while T_0 and K_0 will be our multipliers.

2. Energy identities and related tools

2.1. **Energies.** To a two-form G we associate the following energy momentum tensor

$$T(G)_{\mu\nu} = G_{\mu\alpha}G_{\nu}{}^{\alpha} + {}^*G_{\mu\alpha}{}^*G_{\nu}{}^{\alpha}. \tag{2.1}$$

Similarly for a scalar field ϕ we define

$$T(\phi)_{\mu\nu} = \mathrm{Re}\,(\overline{D_\mu\phi}D_\nu\phi) - \frac{1}{2}g_{\mu\nu}\overline{D^\alpha\phi}D_\alpha\phi, \tag{2.2}$$

and let

$$T(G,\phi)_{\mu\nu} = T(G)_{\mu\nu} + T(\phi)_{\mu\nu}. \tag{2.3}$$

When there is no risk of confusion we write T instead of $T(G,\phi)$. The following lemma is standard.

Lemma 2.1. *$T(G,\phi)$, $T(\phi)$, and $T(G)$ are symmetric and $T(G)$ is traceless. Moreover if (F,ϕ) is a solution of (1.7) then $\nabla^\mu T(F,\phi)_{\mu\nu} = 0$.*

The energy norms inside of V_T are defined as

$$\begin{aligned}
Q_0(\phi)(t,T)^2 &= \int_{\Sigma_t(T)} \left(\tau_+^2|(D_L+\frac{1}{r})\phi|^2 + \tau_-^2|(D_{\underline{L}}-\frac{1}{r})\phi|^2 + (\tau_+^2+\tau_-^2)(|\not{D}\phi|^2 + \frac{|\phi|^2}{r^2})\right) \\
Q_0(G)(t,T)^2 &= \int_{\Sigma_t(T)} \left(\tau_+^2|\alpha(G)|^2 + \tau_-^2|\underline{\alpha}(G)|^2 + (\tau_+^2+\tau_-^2)(|\rho(G)|^2+|\sigma(G)|^2)\right) \\
Q_{out}(\phi)(u,T)^2 &= \int_{C_u(T)} \left(\tau_+^2|(D_L+\frac{1}{r})\phi|^2 + \tau_-^2(|\not{D}\phi|^2 + \frac{|\phi|^2}{r^2})\right) \\
Q_{out}(G)(u,T)^2 &= \int_{C_u(T)} \left(\tau_+^2|\alpha(G)|^2 + \tau_-^2(|\rho(G)|^2+|\sigma(G)|^2)\right) \\
Q_{in}(\phi)(\underline{u},T)^2 &= \int_{\underline{C}_{\underline{u}}(T)} \left(\tau_-^2|(D_{\underline{L}}-\frac{1}{r})\phi|^2 + \tau_+^2(|\not{D}\phi|^2 + \frac{|\phi|^2}{r^2})\right) \\
Q_{in}(G)(\underline{u},T)^2 &= \int_{\underline{C}_{\underline{u}}(T)} \left(\tau_-^2|\underline{\alpha}(G)|^2 + \tau_+^2(|\rho(G)|^2+|\sigma(G)|^2)\right).
\end{aligned} \tag{2.4}$$

We also let $Q_0(G,\phi) = Q_0(G) + Q_0(\phi)$ and define $Q_{in}(G,\phi)$ and $Q_{out}(G,\phi)$ analogously. The supremum of these energies in a region of spacetime is encoded in the following energies

$$\begin{aligned}
Q^*(\phi)(T) &= \sup_{0\le t\le T} Q_0(\phi)(t,T) + \sup_{-1\le u\le\infty} Q(\phi)(u,T) + \sup_{0<\underline{u}<\infty} Q(\phi)(\underline{u},T) \\
Q^*(\phi)(T) &= \sup_{0\le t\le T} Q_0(G)(t,T) + \sup_{-1\le u\le\infty} Q(G)(u,T) + \sup_{0<\underline{u}<\infty} Q(G)(\underline{u},T) \\
Q^*(G,\phi)(T) &= Q^*(G)(T) + Q^*(\phi)(T).
\end{aligned} \tag{2.5}$$

Finally the higher energy norms are defined as

$$Q_{0|I|}(G)(t,T) = \sum_{\Gamma_I\in\mathbb{L}^{|I|}} Q_0(\mathcal{L}_{\Gamma_I}^{|I|}G)(t,T), \tag{2.6}$$

with similar formulas for the other energies. We will eventually be interested in bounding $Q(\mathcal{L}_\Gamma^I F, \mathcal{L}_\Gamma^I\phi)$. We will next prove some energy estimates for the linear inhomogeneous equations

$$\nabla_{[\alpha}G_{\beta\gamma]} = 0, \quad \nabla^\mu G_{\mu\nu} = J_\nu, \quad D^\alpha D_\alpha\phi = f. \tag{2.7}$$

Proposition 2.1. *For the first two equations in* (2.7)*, we have the following estimate*

$$\begin{aligned}
&\int_{\Sigma_t(T)} \left(\tau_+^2|\alpha|^2 + \tau_-^2|\underline{\alpha}|^2 + (\tau_+^2+\tau_-^2)(|\rho|^2+|\sigma|^2)\right) \\
&+ \int_{\underline{C}_{\underline{u}}(T)\bigcap\{0\le t'\le t\}} \left(\tau_-^2|\underline{\alpha}|^2 + \tau_+^2(|\rho|^2+|\sigma|^2)\right) \\
&+ \int_{C_u(T)\bigcap\{0\le t'\le t\}} \left(\tau_+^2|\alpha|^2 + \tau_-^2(|\rho|^2+|\sigma|^2)\right) \\
&\le C\int_{C_{-1}\bigcap\{0\le t'\le t\}} \left(\tau_+^2|\alpha|^2 + \tau_-^2(|\rho|^2+|\sigma|^2)\right) + \int_0^t\int_{\Sigma_{t'}(T)} \left|\overline{K}_0^\nu G_{\mu\nu}J^\mu\right| dxdt' \\
&\quad + C\int_{\Sigma_0(T)} r^2\left(|\alpha|^2+|\underline{\alpha}|^2+|\rho|^2+|\sigma|^2\right)dx.
\end{aligned} \tag{2.8}$$

Here $\alpha, \underline{\alpha}, \rho, \sigma$ are associated to G.

Proof. The proof is a standard application of the divergence theorem to $T^{\mu\nu}\overline{K}_{0\nu}$,and we just need to observe that

$$Q(G)(\underline{L},\underline{L}) = 2|\underline{\alpha}|^2, \quad Q(G)(L,L) = 2|\alpha|^2, \quad Q(G)(\underline{L},L) = 2(\rho^2+\sigma^2), \quad \nabla^\mu(Q_{\mu\nu}\overline{K}_0^\nu) = \overline{K}_0^\nu\nabla^\mu Q_{\mu\nu}.$$

□

Proposition 2.2. *For the third equation of* (2.7)*, and $\Omega = r$ or $u\underline{u}$*

$$\begin{aligned}
&\int_{\Sigma_t(T)} \left(\underline{u}^2|\frac{1}{\Omega}D_L(\Omega\phi)|^2 + u^2|\frac{1}{\Omega}D_{\underline{L}}(\Omega\phi)|^2 + (\underline{u}^2+u^2)(|\not{D}\phi|^2 + \frac{|\phi|^2}{r^2})\right) \\
&+ \int_{\underline{C}_{\underline{u}}(T)\cap\{0\le t'\le t\}} \left(u^2|(\frac{1}{\Omega}D_{\underline{L}}(\Omega\phi)|^2 + \underline{u}^2|\not{D}\phi|^2 + \frac{\underline{u}^2}{r^2}|\phi|^2\right) \\
&+ \int_{C_u(T)\cap\{0\le t'\le t\}} \left(\underline{u}^2|\frac{1}{\Omega}D_L(\Omega\phi)|^2 + u^2|\not{D}\phi|^2 + \frac{u^2}{r^2}|\phi|^2\right) \\
&\le C\int_0^t\int_{\Sigma_t(T)} \left(|f\cdot\frac{1}{r}D_{K_0}(r\phi)| + |f\cdot\frac{1}{u\underline{u}}D_{K_0}(u\underline{u}\phi)| + |K_0^\beta F_{\beta\gamma}\mathrm{Im}\,\left(\phi\overline{D^\gamma\phi}\right)|\right)dxdt' \\
&\quad + CQ_0(\phi)(0,T)^2 + \int_{C_{-1}(T)\cap\{0\le t'\le t\}} \left(\underline{u}^2|\frac{1}{\Omega}D_L(\Omega\phi)|^2 + u^2|\not{D}\phi|^2 + \frac{u^2}{r^2}|\phi|^2\right).
\end{aligned} \tag{2.9}$$

Proof. We follow the argument in [3] to prove (2.9). We denote the Minkowski metric by $g_{\alpha\beta}$ and consider its conformal metric

$$\widetilde{g}_{\alpha\beta} = \frac{1}{\Omega^2}g_{\alpha\beta}$$

for some weight function Ω on $\mathbb{R}\times\mathbb{R}^3$. Then $\Omega\phi$ satisfies

$$\widetilde{D}^\alpha\widetilde{D}_\alpha(\Omega\phi) - \Omega f\Omega^3\nabla^\alpha\nabla_\alpha\left(\frac{1}{\Omega}\right) = \Omega^3 f.$$

Now we fix $\Omega = r$ or $u\underline{u}$, and note that for these choices $\nabla^\alpha\nabla_\alpha\left(\Omega^{-1}\right) = 0$ and $\mathcal{L}_{K_0}\widetilde{g} = 0$. We define the corresponding energy-momentum tensor associated to these two conformal factors as

$$\widetilde{Q}_{\alpha\beta}[\phi] = \mathrm{Re}\left(D_\alpha(\Omega\phi)\overline{D_\beta(\Omega\phi)}\right) - \frac{1}{2}\widetilde{g}_{\alpha\beta}\widetilde{D}^\gamma(\Omega\phi)\overline{D_\gamma(\Omega\phi)}.$$

Here we have used the notation $\widetilde{D}^\gamma = \widetilde{g}^{\alpha\gamma} D_\alpha$. The actual definition for $\widetilde{D}^\gamma$ is $\widetilde{D}^\gamma = \widetilde{g}^{\alpha\gamma}\widetilde{D}_\alpha$, but here we only apply $\widetilde{D}_\alpha$ to a scalar function, we can replace it by D_α. By direct calculation

$$\widetilde{\nabla}^\alpha \widetilde{Q}_{\alpha\beta}[\phi] = \Omega^4\Big(\mathrm{Re}\left(f\cdot\overline{\frac{1}{\Omega}D_\beta(\Omega\phi)}\right) + F_{\beta\gamma}\mathrm{Im}\left(\phi\overline{\frac{1}{\Omega}D^\gamma(\Omega\phi)}\right)\Big).$$

Here $F_{\beta\gamma}$ comes from the commutator $[D_\beta, D_\gamma]$. Applying the above identity to the multiplier $\overline{K}_0$ we have:

$$\widetilde{\nabla}^\alpha\big(\widetilde{Q}_{\alpha\beta}[\phi](K_0)^\beta\big) = \Omega^4\Big(\mathrm{Re}\left(f\cdot\overline{\frac{1}{\Omega}D_{K_0}(\Omega\phi)}\right) + K_0^\beta F_{\beta\gamma}\mathrm{Im}\left(\phi\overline{\frac{1}{\Omega}D^\gamma(\Omega\phi)}\right)\Big).$$

Integrating this identity over various space-time domains we obtain energy estimates involving space-time integrals of f. Since now we work with the conformal metric $\widetilde{g}$, the spacetime measure $d\widetilde{V}$ is given in terms of the measure in Minkowski spacetime dV by

$$d\widetilde{V} = \frac{1}{\Omega^4}dV.$$

Similarly, the measures associated to the conformal metric on the hypersurfsces Σ_t and C_u are

$$d\widetilde{\Sigma}_t = \frac{1}{\Omega^3}d\Sigma_t, \quad d\widetilde{C}_u = \frac{1}{\Omega^3}dC_u$$

On the other hand, associated to the conformal metric $\widetilde{g}$, the unit normal vector-fields to Σ_t, C_u, and $\underline{C}_{\underline{u}}$ are, up to multiplication by a constant, $\Omega(L+\underline{L})$, ΩL, and $\Omega\underline{L}$ respectively. Noting that ϕ vanishes on C_{-1}, for $\Omega = r,\ u\underline{u}$, we get

$$\begin{aligned}
&\int_{\Sigma_t(T)}\Big(\underline{u}^2|\frac{1}{\Omega}(D_L(\Omega\phi)|^2 + u^2|\frac{1}{\Omega}D_{\underline{L}}(\Omega\phi)|^2 + (\underline{u}^2+u^2)|\not{D}\phi|^2\Big)\\
&+\int_{\underline{C}_{\underline{u}}(T)\cap\{0\le t'\le t\}}\Big(u^2|\frac{1}{\Omega}D_{\underline{L}}(\Omega\phi)|^2 + \underline{u}^2|\not{D}\phi|^2\Big)\\
&+\int_{C_u(T)\cap\{0\le t'\le t\}}\Big(\underline{u}^2|\frac{1}{\Omega}D_L(\Omega\phi)|^2 + u^2|\not{D}\phi|^2\Big)\\
&\le \text{“initial data”} + C\int_0^t\int_{\Sigma_t(T)}\Big(|f\cdot\frac{1}{\Omega}D_{K_0}(\Omega\phi)| + |K_0^\beta F_{\beta\gamma}\mathrm{Im}\left(\phi\overline{D^\gamma\phi}\right)|\Big)dxdt'.
\end{aligned}$$

Schematically, here we have first used $\int_{\Sigma_t} + \int_{\underline{C}_{\underline{u}}} \le \int\int + \int_{\Sigma_0}$ and then $\int_{C_u} \le \int_{\Sigma_t} + \int_{\Sigma_0} + \int\int$. To complete the proof of the proposition we only need to control $\left(\frac{\underline{u}^2+u^2}{r^2}\right)|\phi|^2$, for which it suffices to note that

$$\frac{-u\phi}{r} = (\underline{u}D_L\phi + 2\phi) - \big(\underline{u}D_L\phi + \frac{u}{r}\phi\big), \quad \frac{\underline{u}\phi}{r} = (uD_{\underline{L}}\phi + 2\phi) - \big(uD_{\underline{L}}\phi - \frac{u}{r}\phi\big),$$

as well as

$$\begin{aligned}
&\underline{u}^2\big|\frac{1}{r}D_L(r\phi)\big|^2 = \big|\underline{u}D_L\phi + \frac{\underline{u}\phi}{r}\big|^2, \quad u^2\big|\frac{1}{r}D_{\underline{L}}(r\phi)\big|^2 = \big|uD_{\underline{L}}\phi - \frac{u\phi}{r}\big|^2\\
&\underline{u}^2\big|\frac{1}{u\underline{u}}D_L(u\underline{u}\phi)\big|^2 = |\underline{u}D_L\phi + 2\phi|^2, \quad u^2\big|\frac{1}{u\underline{u}}D_{\underline{L}}(u\underline{u}\phi)\big|^2 = |uD_{\underline{L}}\phi + 2\phi|^2.
\end{aligned}$$

□

Noting that ϕ is supported in V_T and combining Proposition 2.2 with the usual energy estimates associated to the multiplier $D_t\phi$ and Hardy's inequality

$$\int_{\Sigma_t} \left|\frac{\phi}{r}\right|^2 dx \lesssim \int_{\Sigma_t} |D\phi|^2 dx,$$

we get the following energy estimates for ϕ.

Corollary 2.1. *Under the same assumption of Proposition 2.2*

$$\begin{aligned}
&\int_{\Sigma_t(T)} \Big(\tau_+^2|(D_L+\frac{1}{r})\phi|^2 + \tau_-^2|(D_{\underline{L}}-\frac{1}{r})\phi|^2 + (\tau_+^2+\tau_-^2)(|\not{D}\phi|^2+\frac{|\phi|^2}{r^2})\Big) \\
&+\int_{\underline{C}_{\underline{u}}(T)\cap\{0\le t'\le t\}} \Big(\tau_-^2|(D_{\underline{L}}-\frac{1}{r})\phi|^2 + \tau_+^2(|\not{D}\phi|^2+\frac{|\phi|^2}{r^2})\Big) \\
&+\int_{C_u(T)\cap\{0\le t'\le t\}} \Big(\tau_+^2|(D_L+\frac{1}{r})\phi|^2 + \tau_-^2(|\not{D}\phi|^2+\frac{|\phi|^2}{r^2})\Big) \\
&\le CQ_0(\phi)(0,T)^2 + C\int_0^t\int_{\Sigma_t(T)} |f\cdot\frac{1}{r}D_{\overline{K}_0}(r\phi)| + |\overline{K}_0^\beta F_{\beta\gamma}\mathrm{Im}\left(\phi\overline{D^\gamma\phi}\right)|dxdt' \\
&+\int_{C_{-1}(T)\cap\{0\le t'\le t\}} \Big(\tau_+^2|(D_L+\frac{1}{r})\phi|^2 + \tau_-^2(|\not{D}\phi|^2+\frac{|\phi|^2}{r^2})\Big).
\end{aligned} \tag{2.10}$$

Note that we did not include the contribution from $|f\cdot\frac{1}{u\underline{u}}D_{K_0}(u\underline{u}\phi)|$ on the right hand side because

$$|\frac{1}{u\underline{u}}D_{K_0}(u\underline{u}\phi)| \lesssim \underline{u}^2|(D_L+\frac{1}{r})\phi| + u^2|(D_{\underline{L}}-\frac{1}{r})\phi|$$

which is already included in $|\frac{1}{r}D_{\overline{K}_0}(r\phi)|$.

Corollary 2.2. *Under the assumptions of Theorem 4.1, and with $\Omega = r$ (or $u\underline{u}$)*

$$\begin{aligned}
Q^*(G,\phi)(T)^2 \lesssim I.D. + Q_{out}(G,\phi)(-1,T)^2 + \int_{V_T}\left|\frac{1}{\Omega}D_{\overline{K}_0}(\Omega\phi)\right||D^\alpha D_\alpha\phi| \\
+\int_{V_T}|\overline{K}_0^\nu(F_{\mu\nu}J^\mu(\phi) + G_{\nu\mu}\nabla^\alpha G_\alpha{}^\mu + {}^*G_{\nu\mu}\nabla^{\alpha *}G_\alpha{}^\mu)|.
\end{aligned}$$

2.2. **Isoperimetric, Poincaré, and Sobolev estimates.** In this subsection we record some standard estimates.

Lemma 2.2 (Isoperimetric Inequality). *Let S be a sphere and $\overline{f}$ be the average of f over S. Then*

$$\int_S |f-\overline{f}|^2 \le C\left(\int_S|\not\nabla f|\right)^2,$$

where $\not\nabla$ is the covariant differentiation on S, and C is independent of the radius of the sphere.

See for instance [4] for a proof. The proof of the following Poincaré estimates can be found in Section 3 of [2].

Lemma 2.3 (Poincaré Inequalities). *Let $F_{\mu\nu}$ be an arbitrary two form with null decomposition α, $\underline{\alpha}$, σ, ρ. With $\overline{\rho}, \overline{\sigma}$ denoting the averages of σ and ρ over $S_{t,r}$ respectively, we have*

$$\int_{S_{t,r}} |\alpha|^2 \leq \sum_{\Omega_{ij}} \int_{S_{t,r}} |\mathcal{L}_{\Omega_{ij}} \alpha|^2, \quad \int_{S_{t,r}} |\underline{\alpha}|^2 \leq \sum_{\Omega_{ij}} \int_{S_{t,r}} |\mathcal{L}_{\Omega_{ij}} \underline{\alpha}|^2,$$
$$\int_{S_{t,r}} \left(|\sigma - \overline{\sigma}|^2 + r^2 |\nabla\!\!\!/ \sigma|^2\right) \leq \sum_{\Omega_{ij}} \int_{S_{t,r}} |\mathcal{L}_{\Omega_{ij}} \sigma|^2,$$
$$\int_{S_{t,r}} \left(|\rho - \overline{\rho}|^2 + r^2 |\nabla\!\!\!/ \rho|^2\right) \leq \sum_{\Omega_{ij}} \int_{S_{t,r}} |\mathcal{L}_{\Omega_{ij}} \rho|^2.$$

Remark 2.1. Since in our case the magnetic charge vanishes, $\overline{\sigma} = 0$.

For completeness we also record the following standard estimate. See [2] for references.

Lemma 2.4. *For any $f \in C^\infty(\mathbb{R}^{1+3})$*

$$\tau_+ \tau_-^{\frac{1}{2}} |f(t,x)| \leq C \sum_{\substack{|I| \leq 2 \\ \Gamma \in \mathbb{L}}} \|\Gamma^I f(t, \cdot)\|_{L^2_x}.$$

We will also need the following L^p Sobolev estimates from [5].

Lemma 2.5. *Let $u \geq -1$ and denote by $\tilde{S}_{u,r}$ and $\tilde{S}_{\underline{u},r}$ the spheres of radius r on the null hypersurfaces C_u and $\underline{C}_{\underline{u}}$ respectively.*

$$\left(\int_{C_u(T)} r^6 |f|^6\right)^{1/6} + \sup_{\tilde{S}_{u,r} \subseteq C_u(T)} \left(\int_{\tilde{S}_{u,r}} r^4 |f|^4\right)^{1/4} \lesssim \left(\int_{C_u} (|f|^2 + r^2 |Lf|^2 + r^2 |\nabla\!\!\!/ f|^2)\right)^{1/2} + \left(\int_{\Sigma_0} (|f|^2 + (1+r^2)|\nabla f|^2)\right)^{1/2}$$

and

$$\left(\int_{\underline{C}_{\underline{u}}(T)} r^4 \tau_-^2 |f|^6\right)^{1/6} + \sup_{\tilde{S}_{u,r} \subseteq \underline{C}_{\underline{u}}(T)} \left(\int_{\tilde{S}_{u,r}} r^2 \tau_-^2 |f|^4\right)^{1/4} \lesssim \left(\int_{\underline{C}_{\underline{u}}} (|f|^2 + \tau_-^2 |\underline{L} f|^2 + r^2 |\nabla\!\!\!/ f|^2)\right)^{1/2} + \left(\int_{\Sigma_0} (|f|^2 + (1+r^2)|\nabla f|^2)\right)^{1/2}.$$

We next establish an improved decay estimate for $(D_L + \frac{1}{r})\phi$, Recall that for large r, C_0^∞ solutions of the wave equation $\Box f = 0$ in $\mathbb{R}^{1+3}$ satisfy the decay

$$|(L + \frac{1}{r}) f| \lesssim \tau_+^{-3} \tau_-^{1/2},$$

(of course assuming that appropriate energy norms of f are finite) which is faster than the $\tau_-^{-1/2} \tau_+^{-2}$ decay for Lf guaranteed by Lemma 2.4. This can be seen for example by writing $\Box f = 0$ as

$$0 = -\underline{L} L f + \nabla\!\!\!/^A \nabla\!\!\!/_A f + \frac{1}{r} L f - \frac{1}{r} \underline{L} f$$
$$= -\underline{L}((L + \frac{1}{r}) f) + \nabla\!\!\!/^A \nabla\!\!\!/_A f + \frac{1}{r} L f + \underline{L}(\frac{1}{r}) f.$$

Now in the expression above all the terms except the first enjoy the decay $\tau_+^{-3}\tau_-^{-1/2}$ in the region $r \gtrsim t$. Integrating in the u direction we get the claimed decay for $(L+\frac{1}{r})f$. Our goal in the next lemma is to show that the same conclusion holds for solutions of $D^\mu D_\mu\phi = 0$. While it is possible to derive this decay rate for solutions of $D^\mu D_\mu \phi = 0$, we will prove only the slower decay of $\tau_+^{-5/2}$, because this is the best possible rate when considering the inhomogeneous equation satisfied by $\mathcal{L}_\Omega\phi$, which is the relevant term when establishing higher regularity.

Lemma 2.6. *Suppose ϕ is a solution of $D^\mu D_\mu\phi = 0$, and that F satisfies*

$$\tau_+^{5/2}|\alpha| + \tau_-^{3/2}\tau_+|\underline{\alpha}| + \tau_+^2|\rho| \leq C.$$

Then in the region $r \geq t/2,\ t \geq 1$

$$|(D_L + \frac{1}{r})\phi| \lesssim \tau_+^{-5/2}$$

where the implicit constants depend only on $Q_3(\phi)$ and C.

Proof. We write the equation for ϕ as

$$\begin{aligned} 0 &= -\frac{1}{2}D_{\underline{L}}D_L\phi - \frac{1}{2}D_L D_{\underline{L}}\phi + D^B D_B\phi \\ &= -D_{\underline{L}}D_L\phi - i\rho(F)\phi + D^B D_B\phi. \end{aligned}$$

Now note that with χ and $\underline{\chi}$ denoting the second fundamental forms on the spheres with respect to L and $\underline{L}$ respectively

$$\begin{aligned} (DD\phi)(e_B, e_B) &= D_{e_B}(D_{e_B}\phi) - D_{\nabla_{e_B}e_B}\phi \\ &= \not{D}_{e_B}(\not{D}_{e_B}\phi) - \not{D}_{\not\nabla_{e_B}e_B}\phi - D_{(\chi_{BB}L + \underline{\chi}_{BB}\underline{L})}\phi = (\not{D}\not{D}\phi)(e_B,e_B) - \frac{1}{r}D_{\underline{L}}\phi + \frac{1}{r}D_L\phi, \end{aligned}$$

and therefore

$$D_{\underline{L}}((D_L + \frac{1}{r})\phi) = \not{D}^B\not{D}_B\phi - i\rho(F)\phi + \frac{1}{r}D_L\phi + \underline{L}(\frac{1}{r})\phi =: M. \tag{2.11}$$

Now we claim that for $r \geq t/2,\ t\geq 1$ we have the bound

$$|M| \lesssim \tau_+^{-3}\tau_-^{-1/2}. \tag{2.12}$$

First we show how the claim proves the lemma. Assuming the claim, from (2.11)

$$\nabla_{\underline{L}}\left|(D_L + \frac{1}{r})\phi\right| \leq \left|D_{\underline{L}}((D_L + \frac{1}{r})\phi)\right| \leq M.$$

Now given any point P in our region of consideration, we integrate this equation along a straight line in $\{\underline{u} = \text{constant}\}$ connecting P to a point on the initial hyper surface Σ_0. Since $\tau_+ \sim \tau_- \sim r$ on Σ_0, $(D_L + \frac{1}{r})\phi$ satisfies the desired decay on Σ_0 and therefore we get the desired bound on $(D_L+\frac{1}{r})\phi$ using the fundamental theorem of calculus. It remains to prove (2.12). The following bounds follow directly from the assumptions of F :

$$\begin{aligned} |\rho(F)\phi| &\lesssim \tau_+^{-3}\tau_-^{-1/2}, \\ \left|\underline{L}(\frac{1}{r})\phi\right| &\lesssim \tau_+^{-3}\tau_-^{-1/2}. \end{aligned}$$

Also from the definition of $Q_2(\phi)$ and in view of Lemmas 2.4 $\tau_+^2\tau_-^{1/2}|D_L\phi| \lesssim 1$, and therefore

$$\left|\frac{1}{r}D_L\phi\right| \lesssim \tau_+^{-3}\tau_+^{-1/2}.$$

Finally for $\not{D}^B\not{D}_B\phi$ we observe that for appropriate constants c_B^{ij} we can write $e_B = \frac{c_B^{ij}}{r}\Omega_{ij}$ and therefore

$$|\not{D}_A\not{D}_B\phi| = \frac{c_B^{ij}}{r}|\not{D}_A D_{\Omega_{ij}}\phi| = \frac{c_B^{ij}}{r}|\not{D}_A\mathcal{L}_{\Omega_{ij}}\phi| \lesssim \tau_+^{-3}\tau_-^{-1/2}Q_3(\phi).$$

Here to get the last inequality we have again used Lemmas 2.4. □

3. Estimates in $\{u \leq -1\}$

3.1. **Compactly supported scalar fields.** In this first subsection we consider the case where ϕ is compactly supported. The general case is treated in the next subsection. Recall that V_T is the spacetime domain enclosed by the outgoing cone C_{-1} and the spacelike hypersurfaces Σ_0 and Σ_T. In this section we use the phrase "outside of V_T" to refer to the region $\{u \leq -1,\ t \leq T\}$ (see Figure 1). Since $\phi(0,\cdot)$ is supported in $\{r \leq 3/4\}$, by the finite speed propagation of the equation

$$D^\alpha D_\alpha\phi = 0$$

the scalar field ϕ vanishes outside V_T. One of our goals in this section is to prove peeling estimates for F

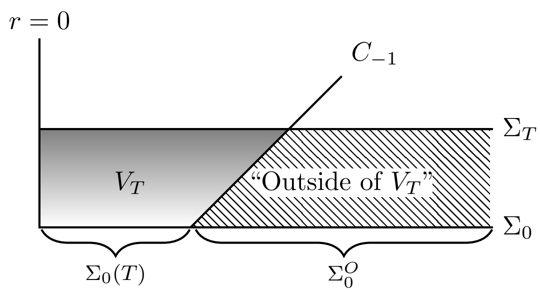

Figure 1

and ϕ outside of V_T. Moreover, we will see in the next section that in order to complete the energy estimates inside V_T, one needs to be given the data on C_{-1}. More precisely, the flux

$$\int_{C_{-1}} \tau_+^2|\alpha|^2 + \tau_-^2(\rho^2+\sigma^2) \tag{3.1}$$

needs to be bounded in terms of the initial data on Σ_0. Showing why this is true is the second goal of this section. Since the electric charge

$$e(t) := \lim_{r\to\infty}\int_{S_{t,r}} \frac{x^i}{r}E_i dS_r, \quad E_i := F_{0i}$$

is non-zero, we cannot assume that the initial energy for electric field E_i

$$\int_{\Sigma_0}(1+r^2)|E|^2dx$$

is finite. This means that although outside V_T equation (1.7) becomes the following homogeneous linear Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 0, \quad \nabla_\mu ({}^*F)^{\mu\nu} = 0, \tag{3.2}$$

one cannot use the multiplier $\overline{K}_0$ to get an estimate like (2.8) and use it to derive an estimate for the flux on C_{-1}. To recover finiteness, instead of F we will work with $\mathcal{L}_\Omega F$ which is charge free, where Ω is any Ω_{ij}. To see that $\mathcal{L}_\Omega F$ is chargeless note that the charge associated to $\mathcal{L}_\Omega F$ is

$$\begin{aligned}\lim_{r\to\infty}\int_{S_{t,r}} \frac{x^i}{r}\big(\mathcal{L}_\Omega F\big)_{0i} dS_r = \lim_{r\to\infty}\int_{S_{t,r}} \big(\mathcal{L}_\Omega F\big)_{0r} dS_r &= \lim_{r\to\infty}\int_{S_{t,r}} \Omega\big(F_{0r}\big) dS_r \\ &= -\lim_{r\to\infty}\int_{S_{t,r}} \operatorname{d\!\!/iv}\Omega \cdot F_{0r} dS_r.\end{aligned}$$

Here $\operatorname{d\!\!/iv}$ represents the divergence operator on $S_{t,r}$. By the definition of $\Omega_{ij} := x^i\partial_j - x^j\partial_i$, we have:

$$\operatorname{div}_{\mathbb{R}^3}\Omega = 0.$$

On the other hand, since Ω is tangential to $S_{t,r}$, we have

$$\operatorname{d\!\!/iv}\Omega = \operatorname{div}_{\mathbb{R}^3}\Omega = 0$$

which implies that $\mathcal{L}_\Omega F$ is charge free. Moreover by the conformal invariance of Maxwell's equations (see [2]), $\mathcal{L}_\Omega F$ also satisfies the linear homogeneous Maxwell equations

$$\nabla^\mu \big(\mathcal{L}_\Omega F\big)_{\mu\nu} = 0, \quad \nabla^\mu \big({}^*\left(\mathcal{L}_\Omega F\right)\big)_{\mu\nu} = 0. \tag{3.3}$$

It follows that $\nabla^\mu (Q(\mathcal{L}_\Omega F)_{\mu\nu}\overline{K}_0^\nu) = 0$, and applying the divergence theorem to this identity outside of V_T (see Figure 1) yields the following energy estimate

$$\begin{aligned}&\int_{\Sigma_t^O} \big(\tau_+^2|\alpha(\mathcal{L}_\Omega F)|^2 + \tau_-^2|\underline{\alpha}(\mathcal{L}_\Omega F)|^2 + (\tau_+^2+\tau_-^2)(\rho(\mathcal{L}_\Omega F)^2 + \sigma(\mathcal{L}_\Omega F)^2)\big) \\ &+ \int_{C_{-1}\bigcap\{0\le t'\le t\}} \tau_+^2|\alpha(\mathcal{L}_\Omega F)|^2 + \tau_-^2\big(\rho(\mathcal{L}_\Omega F)^2 + \sigma(\mathcal{L}_\Omega F)^2\big) \\ &\le C\int_{\Sigma_0^O} (1+r^2)\big(|\alpha(\mathcal{L}_\Omega F)|^2 + |\underline{\alpha}(\mathcal{L}_\Omega F)|^2 + \rho(\mathcal{L}_\Omega F)^2 + \sigma(\mathcal{L}_\Omega F)^2\big).\end{aligned} \tag{3.4}$$

Here we are assuming that

$$\int_{\Sigma_0^O} (1+r^2)\big(|E(\mathcal{L}_\Omega F)|^2 + |H(\mathcal{L}_\Omega F)|^2\big) dx$$

is finite, which is consistent with the chargelessness of $\mathcal{L}_\Omega F$.

Lemma 3.1. *Under the assumptions of the Theorem 4.1, (i)*

$$\sum_{j\le k-1} Q_{out}(\mathcal{L}^j F)(-1,T) + Q_{out}(\mathcal{L}_\Omega\mathcal{L}^{k-1}F)(-1,T) \le C\epsilon.$$

Proof. Equation (3.4), Lemma 2.3 and Remark 2.1 give the following estimate on C_{-1}

$$\begin{aligned}&\int_{C_{-1}} \big(\tau_+^2|\alpha|^2 + \tau_-^2(|\sigma|^2 + |\rho-\overline{\rho}|^2)\big) \\ &\quad\le C\int_{\Sigma_0^O} (1+r^2)\big(|\alpha(\mathcal{L}_\Omega F)|^2 + |\underline{\alpha}(\mathcal{L}_\Omega F)|^2 + \rho(\mathcal{L}_\Omega F)^2 + \sigma(\mathcal{L}_\Omega F)^2\big).\end{aligned}$$

We still need to get an estimate for $\int_{C_{-1}} \tau_-^2 \rho^2$. Now

$$\int_{C_{-1}} \tau_-^2 |\rho|^2 \le C \int_{C_{-1}} \tau_-^2 |\rho - \overline{\rho}|^2 + C \int_{C_{-1}} \tau_-^2 |\overline{\rho}|^2.$$

From the previous estimate and (3.4) we already have the estimate for the first term on the right hand side, so we only consider (note that $\tau_- \sim 1$ on C_{-1})

$$\int_{C_{-1}} \tau_-^2 |\overline{\rho}|^2 \sim \int_1^\infty \int_{S_{t,r}} |\overline{\rho}|^2 dS_r d\underline{u}.$$

We first show that the integral $\int_{S_{t,r}} \rho dS_r$ is bounded by a constant independent of t and r. Writing equations (3.2) in the null frame we see that

$$-\not{\mathrm{d}}\mathrm{iv}\alpha - L\rho - 2r^{-1}\rho = 0.$$

Integrating this equation on $S_{t,r}$ gives

$$\int_{\mathbb{S}^2} r^2 \big(L\rho + 2r^{-1}\rho\big) d\mu_{\mathbb{S}^2} = \int_{S_{t,r}} \big(L\rho + 2r^{-1}\rho\big) dS_r = 0,$$

which means

$$L\Big(\int_{S_{t,r}} \rho dS_r\Big) = L\Big(\int_{\mathbb{S}^2} r^2 \rho(t,\rho,\cdot) d\mu_{\mathbb{S}^2}\Big) = \int_{\mathbb{S}^2} L\big(r^2\rho(t,\rho,\cdot)\big) d\mu_{\mathbb{S}^2} = 0.$$

This implies that the integral $\int_{S_{t,r}} \rho dS_r$ is preserved along each outgoing cone C_u

$$\int_{S_{t,r}} \rho dS_r = \int_{S_{0,r-t}} \rho dS_{r-t}.$$

Therefore we only need to show that on the initial surface Σ_0, the integral $\int_{S_{0,r}} \rho dS_r$ is bounded by a constant independent of r which follows from

$$\left|\int_{S_{0,r}} \rho\right| = \left|\int_{\Sigma_0 \cap \{|x| \le r\}} \mathrm{div} E\right| \le \Big(\int_{\Sigma_0} |\phi|^2\Big)^{1/2} \Big(\int_{\Sigma_0} |D\phi|^2\Big)^{1/2} \le \epsilon.$$

We have proved

$$|\overline{\rho}| \le C \left|\frac{1}{r^2} \int_{S_{t,r}} \rho dS_r\right| \le \epsilon r^{-2},$$

which in turn implies

$$\int_{C_{-1}} \tau_-^2 |\overline{\rho}|^2 = \int_1^\infty \int_{S_{t,r}} |\overline{\rho}|^2 dS_r d\underline{u} \le \epsilon \int_1^\infty \frac{1}{r^2} d\underline{u} \le \epsilon \int_1^\infty \frac{1}{\underline{u}^2} d\underline{u} < \epsilon.$$

This completes the proof of the boundedness of the flux on C_{-1} by a constant which can be made small by taking the initial data to be small. To bound the flux for $\mathcal{L}^k F$ it suffices to note that by the conformal equivariance of Maxwell's equations $\mathcal{L}^k F$ also solves

$$\nabla^\mu \big(\mathcal{L}^k F\big)_{\mu\nu} = 0, \quad \nabla^\mu \big({}^*\mathcal{L}^k F\big)_{\mu\nu} = 0.$$

This means that the analogue of (3.4) holds for $\mathcal{L}^k F$ and therefore we can use the same proof as above to estimate the flux for the derivatives of F. Since the last vector field we commute is a rotational vector field Ω, the term with the maximum number of derivatives is automatically chargeless so we do not need an extra derivative to estimate the average. □

4. The Main theorem

The following theorem is the main result of this work.

Theorem 4.1. *Suppose that one of the following holds:*

(i) $\phi(0,\cdot)$ *and* $D_t\phi(0,\cdot)$ *are supported on* $\{r \le 3/4\}$*, and that the initial data for the Cauchy problem (1.7) satisfy the following smallness conditions*

$$\begin{aligned}
&\sum_{\substack{\Gamma\in\mathbb{L}\\ k\le 7}} \int_{\Sigma_0} (|D_{t,x}\mathcal{L}_\Gamma^k\phi|^2 + |\mathcal{L}_\Gamma^k\phi|^2) r^2 dx \le \epsilon^2,\\
&\sum_{\substack{\Gamma\in\mathbb{L}\\ k\le 7}} \int_{\Sigma_0} \left(|\alpha(\mathcal{L}_\Gamma^k F)|^2 + |\sigma(\mathcal{L}_\Gamma^k F)|^2 + |\underline{\alpha}(\mathcal{L}_\Gamma^k F)|^2 + r^{-2}|\rho(\mathcal{L}_\Gamma^k F)|^2\right) r^2 dx \le \epsilon^2,\\
&\sum_{\substack{\Gamma\in\mathbb{L}\\ k\le 6}} \int_{\Sigma_0} |\rho(\mathcal{L}_{\Omega_{ij}}\mathcal{L}_\Gamma^k F)|^2 r^2 dx \le \epsilon^2.
\end{aligned} \tag{4.1}$$

(ii) $\phi(0,\cdot)$ *and* $D_t\phi(0,\cdot)$ *are not compactly supported but (4.1) holds with the following modification for the scalar field energies:*

$$\sum_{\substack{\Gamma\in\mathbb{L}\\ k\le 7}} \int_{\Sigma_0} (|D_{t,x}\mathcal{L}_\Gamma^k\phi|^2 + |\mathcal{L}_\Gamma^k\phi|^2) r^{2+2\gamma} dx \le \epsilon^2, \tag{4.2}$$

for some $\gamma \in (0, 1/2)$*.*

Then if ϵ *is sufficiently small, the locally defined solution can be extended globally in time and the following decay estimates hold*

$$\begin{aligned}
&|\phi| \lesssim \epsilon\tau_+^{-1}\tau_-^{-1/2}, && |\not{D}\phi| \lesssim \epsilon\tau_+^{-2}\tau_-^{-1/2},\\
&|D_L\phi| \lesssim \epsilon\tau_+^{-2}\tau_-^{-1/2}, && |D_{\underline{L}}\phi| \lesssim \epsilon\tau_+^{-1}\tau_-^{-3/2},\\
&|\alpha| \lesssim \epsilon\tau_+^{-5/2}, && |\underline{\alpha}| \lesssim \epsilon\tau_+^{-1}\tau_-^{-3/2},\\
&|\sigma| \lesssim \epsilon\tau_+^{-2}\tau_-^{-1/2}, && |\rho| \lesssim \begin{cases} \epsilon\tau_+^{-2}\tau_-^{-1/2} & \text{if } u \ge -1\\ \epsilon r^{-2} & \text{if } u \le -1. \end{cases}
\end{aligned} \tag{4.3}$$

Moreover in case (ii) above the scalar field enjoys the stronger decay estimates

$$\begin{aligned}
&|\phi| \lesssim \epsilon\tau_+^{-1}\tau_-^{-1/2-\gamma}, && |\not{D}\phi| \lesssim \epsilon\tau_+^{-2}\tau_-^{-1/2-\gamma},\\
&|D_L\phi| \lesssim \epsilon\tau_+^{-2}\tau_-^{-1/2-\gamma}, && |D_{\underline{L}}\phi| \lesssim \epsilon\tau_+^{-1}\tau_-^{-3/2-\gamma}.
\end{aligned}$$

The number of derivatives is not optimal, but regularity is not our concern here. By carefully studying our proof the minimum possible order can be investigated. Note also that in the proof we commute k derivatives with $k \ge 7$ and the last derivative that we commute will always be a rotational vector-field. This has to do with using the Poincaré inequality to deal with the contribution of the charge, and is explained during the proof. The proof follows the classical energy method used in [2]. We use the Morawetz vector field K_0 and the time translation vector field ∂_t as multipliers and ∂_μ, S, and $\Omega_{\mu\nu}$ as commutators. In implementing the energy method we face two major challenges. First, since the equation is not conformally Killing invariant, we will need to bound error terms when using the energy method. It turns out that to obtain optimal decay of the error terms, we need to carefully take into account the structure of the equations to observe certain

cancelations. While some of these cancelations were already observed in [6], there are many more which arise in the terms not considered there. These cancelations are especially important in view of our use of the double null foliation in the error analysis.
The second difficulty is due to the non-vanishing of the electric charge (defined below). Specifically, the multiplier K_0 contributes boundary terms which, in the presence of a non-zero charge, are not finite. In particular, with $E_i := F_{0i}$, it can be seen from the divergence theorem that if the electric charge

$$e(t) = \frac{1}{4\pi} \lim_{r\to\infty} \int_{S_{t,r}} E_r$$

is non-zero (say for $t = 0$), then the following integral cannot be convergent

$$\int_{\Sigma_0} (1 + r^2)|E|^2 dx.$$

In the null decomposition of F this is reflected in the fact that the energy for $\rho := \frac{1}{2} F_{L\underline{L}}$ on Σ_t is infinite. To salvage the situation, we divide $\mathbb{R}^{3+1}$ into the two regions $\{u \geq -1\}$ and $\{u \leq -1\}$. Inside the null-cone $u = -1$ the space-like hypersurface $t = 0$ has finite radius and the integral above is convergent. To bound the energies outside the $u = -1$ cone and the flux along C_{-1} we make crucial use of the Poincaré inequality

$$\int_{S_{t,r}} |\rho - \overline{\rho}|^2 \leq C \sum_{1\leq i<j\leq 3} \int_{S_{t,r}} |\mathcal{L}_{\Omega_{ij}} \rho|^2.$$

The point here is that $\mathcal{L}_{\Omega_{ij}} F$ is automatically chargeless and therefore by assuming sufficient decay on the initial data for $\mathcal{L}_{\Omega_{ij}} F$ we are able to obtain the required estimates for the null decomposition of $\mathcal{L}_{\Omega_{ij}} F$. On the other hand, due to the finiteness of the charge, the average of ρ on Σ_0 decays like r^{-2}, and by considering the propagation equation for ρ we are able to show that this decay in fact holds uniformly in time. Combining this with the estimates for $\mathcal{L}_{\Omega_{ij}} \rho$ we are able to prove the decay rate of r^{-2} for ρ itself outside the cone $u = -1$ (compare with $\tau_+^{-2}\tau_-^{-1/2}$ inside the cone), which is consistent with the result obtained in [3]. With these observations we are able to complete the proof of our estimates in $\{u \leq -1\}$ when the scalar field is compactly supported, because the equations degenerate to the free Maxwell equations there. However, when the scalar field is not compactly supported we will still have error terms involving ρ (and not $\overline{\rho}$), coming from commuting Lie derivatives with the equation for the scalar field. Here we need to use fractional Morawetz estimates for the scalar field to compensate for the loss of decay in ρ. These estimates are proven under stronger initial decay assumptions on the scalar field. We should note that for the proof of our Morawetz estimates for ϕ (both inside and outside the null cone $u = -1$) we use the elegant argument presented by Lindblad and Sterbenz in [3]. Besides the modification of F described above, and careful exploitation of the structure of the equation suitable for analysis using the double null foliation, the other key ingredient which allows us to avoid fractional Morawetz estimates for the electromagnetic field is use of the L^4 Sobolev estimates from Lemma 2.5, which are taken from [5]. We set up a continuity argument as follows to prove the error estimates inside V_T. Assume that $Q_k^*(T) \leq C\epsilon$. We then show that if ϵ is sufficiently small, then $Q_k^*(T) \leq C\epsilon/2$. The proof is divided into three steps corresponding to commuting zero, one, or more vector fields.

4.1. **A rough idea for the proof of main theorem.**

Step 1: $Q_0^*(T) \leq C\epsilon/2$. The bound on $Q_0^*(T)$ simply follows from the flux estimates on C_{-1} from the previous section and Corollary 2.2, and by taking the initial data sufficiently small.

Step 2: $Q_1^*(T) \leq C\epsilon/2$. In light of (1.8), the flux estimates on C_{-1}, and Corollary 2.2 we have

$$
\begin{aligned}
Q_1^*(T)^2 \leq\ & C\epsilon_0^2 \\
&+ C\sum_\Gamma \int_{V_T} |\frac{1}{\Omega}D_{\overline{K}_0}(\Omega\mathcal{L}_\Gamma\phi)||2\Gamma^\nu F_{\mu\nu}D^\mu\phi + \nabla^\mu\Gamma^\nu F_{\mu\nu}\phi| && (4.4)\\
&+ C\sum_\Gamma \int_{V_T} |\frac{1}{\Omega}D_{\overline{K}_0}(\Omega\mathcal{L}_\Gamma\phi)||\Gamma^\mu J_\mu(\phi)||\phi| && (4.5)\\
&+ C\sum_\Gamma \int_{V_T} |\overline{K}_0^\mu F_{\mu\nu}J^\nu(\mathcal{L}_\Gamma\phi)| && (4.6)\\
&+ C\sum_\Gamma \int_{V_T} |\overline{K}_0^\mu \mathcal{L}_\Gamma F_{\mu\nu}\mathcal{L}_\Gamma J^\nu(\phi)| && (4.7)\\
&=: C\epsilon_0^2 + \sum_\Gamma\sum_{i=1}^4 \mathscr{E}_i^1(\Gamma).
\end{aligned}
$$

4.2. **An example to use L^4 estimates.** There are some error integrals for which we are not able to use $L^\infty - L^2$ argument to estimate. Here is an example: We denote by $r_m(u)$ and $r_M(u)$ the minimum and maximum radii on $C_u(T)$ respectively. Then

$$
\begin{aligned}
\int_{V_T\bigcap\{r\geq 1+\frac{t}{2}\}} \tau_+^3|(D_L+\frac{1}{r})\mathcal{L}^k\phi||\alpha||(D_{\underline{L}}-\frac{1}{r})\phi| &\lesssim \int_{V_T} r^3|(D_L+\frac{1}{r})\mathcal{L}^k\phi||\alpha||(D_{\underline{L}}-\frac{1}{r})\phi| \\
&\lesssim \int_{-1}^\infty \Big(\int_{C_u} r^2|(D_L+\frac{1}{r})\mathcal{L}^k\phi|^2\Big)^{1/2} \\
&\quad \cdot \Big[\int_{r_m(u)}^{r_M(u)}\Big(\int_{\widetilde{S}_{u,r}} r^2\tau_-^6|(D_{\underline{L}}-\frac{1}{r})\phi|^4\Big)^{1/2}\Big(\int_{\widetilde{S}_{u,r}} r^6|\alpha|^4\Big)^{1/2}dr\Big]^{1/2}\tau_-^{-3/2}du.
\end{aligned}
$$

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