Two open problems on $\mathbb{S}^4$ with Einstein metric of positive scalar curvature

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The little result of the present paper is inspired by some deep works of Schoen and Schoen and Yau. Completion of the two remaining tasks presented as open problems can turn it highly significant. In a 4-dimensional sphere with Einstein metric having positive scalar curvature we find many complete embedded hypersurfaces which admit Green's functions of the conformal Laplacian such that relative to the conformal metric on each hypersurface it is asymptotically flat at the pole of the Green's function and has scalar curvature zero. If one can choose the pole at some suitable point so that the mass of the asymptotically flat 3-manifold can be shown to be zero then the positive mass theorem would render the hypersurface conformally flat. At present it is not known whether a 4-dimensional sphere with Einstein metric having positive scalar curvature is isometric to the standard metric on $\mathbb{S}^4$.

Keywords: Einstein metric, constant scalar curvature Riemannian metric

I. INTRODUCTION

By smooth $\mathbb{S}^4$ we shall mean a manifold diffeomorphic to a standard 4-dimensional sphere. We shall consider a Riemannian Einstein metric having positive scalar curvature on $\mathbb{S}^4$. It is not known whether such a metric is isometric to the standard metric on $\mathbb{S}^4$ (p13 [1]). There is however a result showing the finite dimensionality of the moduli space of Einstein metrics (Chapter 12, Besse [2]). The use of the stability inequality of minimal submanifold theory to investigate the uniqueness question of Einstein metric on S^4 with positive scalar curvature is usually deemed hopeless because of Simons' result (Corollary 3.6.1. in [7]) which states that if M has positive Ricci curvature, any co-dimension 1 closed minimal variety immersed in M is deformable to a closed manifold of smaller area. However it appears that nobody has spent enough time combining the minimal submanifold method with the positive mass type arguments applied to the conformal metric where the conformal function is the Green's function of the conformal Laplacian. We show that the stability inequality for the stable minimal hypersurface in the asymptotically flat multi-end conformal metric implies that for the original induced metric on the hypersurface there exists positive Green's function of the hypersurface conformal Laplacian. Hope is that by choosing a nice point and calculating mass in suitable coordinates one can first show that the hypersurface is conformally flat. Then possibly because of too many conformally flat hypersurfaces one can show the 4-manifold is conformally flat.

Suppose on a smooth manifold M diffeomorphic to $\mathbb{S}^4$ we have the Riemannian metric g having the Ricci curvature

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta}, \qquad \Lambda > 0. \tag{I.1}$$

The scalar curvature is then $R = 4\Lambda$. We want to show that g is isometric to an usual round metric on $\mathbb{S}^4$.

Let $G_1, G_2, \ldots G_n$ be Green's functions of the operator $-\Delta + (1/6)R$ on (M, g) with poles at finite numbers of distinct points $P_A, A = 1, 2, \ldots, n$. Then on $M \setminus \{P_1, \ldots P_n\}$, $G = c_A \sum G_A$ satisfies the equation

$$\Delta G = (2/3)\Lambda G \tag{I.2}$$

where c_i's are positive constant.

In dimension 4, a Green's function, say, G_1 has the following expansion (page 214 [5],[6]) relative to the geodesic normal coordinates $\{x^i\}$ of some conformal metric

$$G_1 = r^{-2} + c + \alpha \tag{I.3}$$

where $r = \sqrt{\sum (x^i)^2}$, c is a constant, $\alpha = P\ln(r) + \alpha_0 = O(r)$. $P = O(r^2)$ and $\alpha_0 \in C^{2,\mu}$.

Relative to coordinates $\left\{X^i = x^i r^{-2}\right\}$, $G_1^2 g$ has expansion (page 218 last paragraph [5])

$$G_1^2 g_{ij} = \left(1 + 2c\Re^{-2}\right)\delta_{ij} + O(\Re^{-3}) \tag{I.4}$$

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where $\Re = r^{-1}$. The constant c is proportional to the the mass of the 4-dimensional metric $G_1^2 g$. $c \geq 0$ by the positive mass theorem of Schoen and Yau [4]. In case $c = 0$, g is conformally flat. Actual value of the mass of the 4-metric is not important for this note. We have somewhat similar expansions for the hypersurface Green's function

$$\mathcal{G} = \bar{r}^{-1} + m/2 + O(\bar{r}). \tag{I.5}$$

Now the 3-metric is

$$\mathcal{G}^4 \bar{g}_{ij} = \left(1 + 2m\bar{\Re}^{-1}\right)\delta_{ij} + O(\bar{\Re}^{-2}) \tag{I.6}$$

where the bar refers to 3-dimensional quantities and m is the mass of the 3-metric.

II. STABLE MINIMAL HYPERSURFACE IN $G^2 g$

Since the metric $\hat{g} = G^2 g$ is asymptotically flat at all ends, at all ends the asymptotic 3-spheres are mean convex. Since $\hat{g}$ is asymptotically flat with at least two ends and the asymptotic 3-spheres are mean convex, by applying a lemma of Meeks, Simon and Yau [3] (Lemma 4 of Section 11) we get an embedded stable minimal 3-hypersurface Σ in $(M \setminus \{P_1 \cup P_2 \cup P_3 \ldots\}, \hat{g})$. We have the stability inequality (Eq. 3.2.2 in Simons [7])

$$\int\limits_{\Sigma,\hat{g}} \left(|\hat{\nabla}_\Sigma \zeta|^2_{\hat{g}} - \zeta^2 |\hat{A}|^2_{\hat{g}} - \zeta^2 Ric(\hat{g})(\hat{n},\hat{n})\right) \geq 0 \tag{II.1}$$

for some function ζ having compact support in Σ. Here $\hat{n}^\alpha = G^{-1} n^\alpha$ and n^α are unit normal vectors on Σ relative to $\hat{g}$ and g respectively. Since Σ is minimal relative to $\hat{g}$, on Σ, the scalar curvature of the induced 3-metric $\bar{\hat{g}}$ (bar is dropped when there is no confusion) is

$$\overline{R}_{\hat{g}} = -2Ric(\hat{g})(\hat{n},\hat{n}) - |\hat{A}|^2_{\hat{g}}. \tag{II.2}$$

Thus stability inequality becomes

$$\int\limits_{\Sigma,\hat{g}} \left(|\hat{\nabla}_\Sigma \zeta|^2_{\hat{g}} - (1/2)\zeta^2 |\hat{A}|^2_{\hat{g}} + (1/2)\zeta^2 \overline{R}_{\hat{g}}\right) \geq 0. \tag{II.3}$$

III. CONFORMAL LAPLACIAN IN 3-DIMENSION

In 3-dimension the conformal Laplacian L on $(\Sigma, \hat{g})$ is

$$L = -\hat{\Delta}_\Sigma + (1/8)\overline{R}_{\hat{g}}.$$

Let μ_1 be the first eigenfunction of the conformal Laplacian L,

$$Lf = \mu_1 f. \tag{III.1}$$

We have the following theorem.
Theorem 1. *μ_1 is positive.*

Proof. Integrating Eq. (III.1) we get

$$-\int\limits_{\Sigma,\hat{g}} \left((1/8)\overline{R}_{\hat{g}} - \mu_1\right) f^2 = \int\limits_{\Sigma,\hat{g}} |\nabla_\Sigma f|^2_{\hat{g}}. \tag{III.2}$$

Thus

$$\mu_1 \int\limits_{\Sigma,\hat{g}} f^2 \geq \int\limits_{\Sigma,\hat{g}} (1/8)\overline{R}_{\hat{g}} f^2. \tag{III.3}$$

Stability inequality Eq. (II.3) gives

$$\int_{\Sigma,\hat{g}} \Big((1/8)(\overline{R}_{\hat{g}} - \mu_1)f^2 - (1/2)f^2|\hat{A}|^2_{\hat{g}} + (1/2)f^2\overline{R}_{\hat{g}}\Big) \geq 0$$

which gives

$$\int_{\Sigma,\hat{g}} (1/8)\overline{R}_{\hat{g}} f^2 \geq \int_{\Sigma,\hat{g}} (1/40)\mu_1 f^2 + \int_{\Sigma,\hat{g}} (1/10)f^2|\hat{A}|^2_{\hat{g}}. \tag{III.4}$$

Thus

$$\mu_1 \int_{\Sigma,\hat{g}} f^2 \geq (1/40)\mu_1 \int_{\Sigma,\hat{g}} f^2 + \int_{\Sigma,\hat{g}} (1/10)f^2|\hat{A}|^2_{\hat{g}}.$$

$$(39/4)\mu_1 \int_{\Sigma,\hat{g}} f^2 \geq \int_{\Sigma,\hat{g}} f^2|\hat{A}|^2_{\hat{g}}.$$

So either $\mu_1 > 0$, or $\mu_1 = 0$ and Σ is totally geodesic in $(M, \hat{g})$. Following lemma shows $\mu_1 \neq 0$. □

Lemma 2. *Suppose* $\mu_1 = 0$. *Then*

- *(i).* $\int_{\Sigma,\hat{g}} \overline{R}_{\hat{g}} = 0.$
- *(ii).* $\overline{R} > 0$ *where* $\overline{R}$ *is the scalar curvature of* Σ *in the metric induced from* g.
- *(iii).* *(i) gives a contradiction.*

Proof. For μ_1 =, 0 Eqs. (III.3,III.4) implies $\int_{\Sigma,\hat{g}} \overline{R}_{\hat{g}} f^2 = 0$. Then Eq. (III.2) says f is constant. Thus $\int_{\Sigma,\hat{g}} \overline{R}_{\hat{g}} = 0$. By the doubly contracted Gauss equation the scalar curvature of $(\Sigma, \overline{g})$ is $\overline{R} = 2\Lambda - |A|^2 + H^2$. Generally by the conformal transformation formula $\hat{A}_{ab} = GA_{ab} + G^{-2}\langle \nabla G, n\rangle \hat{g}_{ab}$ and (correct here) $H_{\hat{g}} = G^{-1}H + 3G^{-2}\langle \nabla G, n\rangle$. Σ is totally geodesic in $\hat{g}$. So we get $A_{ab} = (1/3)G^{-2}H\hat{g}_{ab}$. Thus $\overline{R} = 2\Lambda + (2/3)H^2 > 0$. (i) gives a contradiction because we have the conformal transformation formula $\overline{R}_{\hat{g}} = G^{-2}\overline{R} - 4G^{-1}\hat{\Delta}_\Sigma G + 6G^{-2}|\nabla_\Sigma G|^2_{\hat{g}}$. □

IV. HYPERSURFACE GREEN'S FUNCTION OF $(\Sigma, \overline{\hat{g}})$

For $u \in W^{1,2} \setminus \{0\}$, let

$$E(u) = \int_{\Sigma,\hat{g}} (|\nabla_\Sigma u|^2_{\hat{g}} + (1/8)\overline{R}_{\hat{g}} u^2).$$

Since $\mu_1 > 0$, following the arguments in (page 202 last paragraph [5]) we see that

$$\frac{E(u)}{||u||_2^2} \geq \mu_1 > 0.$$

This way however we get only the nonnegativity of the inf of $(E(u)/||u||_6^2)$. We need its positivity for the existence of the Green's function of L. That $E(u) \geq 0$ also follows directly from the stability inequality Eq. (II.3). We have

$$E(u) = \int_{\Sigma,\hat{g}} \Big[(1/4)(|\nabla_\Sigma u|^2_{\hat{g}} + (1/2)\overline{R}_{\hat{g}} u^2) + (3/4)|\nabla_\Sigma u|^2_{\hat{g}}\Big] \geq \int_{\Sigma,\hat{g}} (3/4)|\nabla_\Sigma u|^2_{\hat{g}} \Rightarrow$$

$$\frac{E(u)}{||u||_6^2} \geq \frac{CE(u)}{||\nabla_\Sigma u||_2^2} \geq (3/4)C.$$

Thus there is exists an unique positive Green's function $\mathcal{G}$ on Σ on $\Sigma \setminus \{q\}$ where q is the pole of $\mathcal{G}$ such that

$$\hat{\Delta}_\Sigma \mathcal{G} - (1/8)\overline{R}_{\hat{g}} \mathcal{G} = 0. \tag{IV.1}$$

The metric $h = \mathcal{G}^4\overline{\hat{g}}$ has 0 scalar curvature and is asymptotically flat on $\Sigma \setminus \{q\}$. Since $h = \mathcal{G}^4 G^{-2}\overline{g}$, we also have

$$\Delta_\Sigma(\mathcal{G}G^{-1/2}) - (1/8)\overline{R}(\mathcal{G}G^{-1/2}) = 0. \tag{IV.2}$$

V. REMAINING TASKS

Now one tries to use the method followed in [8]. Near a point q on the hypersurface one chooses the special coordinates such that $Ric(h)_{ij} = 0$ at q in these coordinates. Existence of such coordinates is proved in Schoen and Yau [5] (page 212). One chooses q suitably such that calculation of the asymptotic expansions of the metric up to certain order is easy. One calculates in two ways the expansions of $\Delta_E G_1$, where G_1 is a single pole Green's function of the 4-dimensional operator or a suitable function of it (analog of the static potential V in [8]). Here Δ_E is the Laplacian of $\mathbb{R}^3$ in the Euclidean coordinates $\{X^i = x^i / \sum_{k=1}^{3} (x^k)^2\}$, $\{x^i\}$ being the special coordinates mentioned above. In one way we use Eq. (I.2) transforming it conformally for the metric $h = \mathcal{G}^4 \bar{\hat{g}}$. We note that the RHS of Eq. (I.2) or its conformally transformed form does not contain explicitly the second partial derivatives of G_1. The passage from $\Delta_{\Sigma,h} G_1$ to $\Delta_E G_1$ involves second partial derivatives with coefficients having higher decay. In the other way we compute $\Delta_E G_1$ directly by computing the second partial derivatives relative to X^is of the expansion of G_1 centered at q. In both ways we shall need the expansion of the Christoffel symbols for which we exploit $Ric(h)_{ij} = 0$. Then one hopes that the two expressions are compatible only if $m = 0$, that is, if Σ is conformally flat. The analogue mentioned above may not be necessary but can be constructed by reverse engineering of round $\mathbb{S}^4$ and its totally geodesic round $\mathbb{S}^3$. We have not thought about the problem of proving the conformal flatness of the 4-dimension from the too many conformally flat hypersurfaces. One may ask what happens if P_1 and P_2 converge to q and whether that will give a relation between the 4-dimensional mass and the 3-dimensional mass. For this step we may need to keep in mind the known results on Einstein metrics on $\mathbb{S}^4$.

VI. APPENDIX: STANDARD $\mathbb{S}^4$

Because of symmetry for any two distinct points as the poles, the totally geodesic $\mathbb{S}^3$ symmetrically positioned between the two points will remain totally geodesic relative to the conformal metric $\hat{g}$. In the following we consider when the poles are antipodal pair in the standard metric.

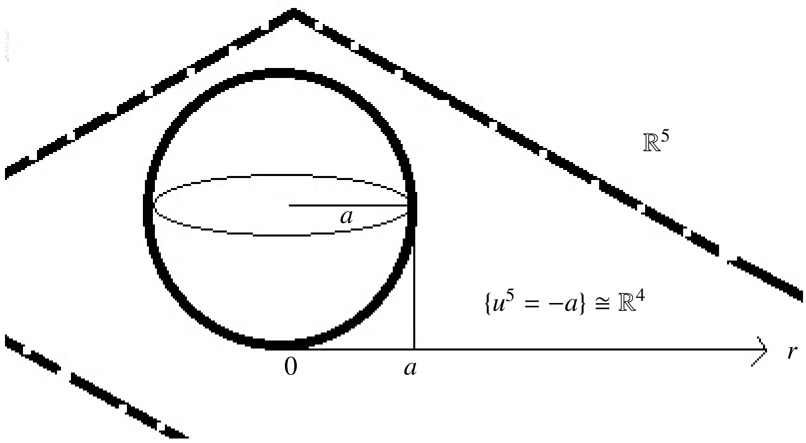
Let $\{u^1, \ldots u^5\}$ be a point in $\mathbb{R}^5$. We consider the sphere $M = \mathbb{S}^4$ of radius a,

$$(u^5)^2 + r^2 = a^2 \tag{VI.1}$$

where $r^2 = \sum_{i=1}^{4} (u^i)^2$. Metric g is on M is,

$$g = (1 - r^2/a^2)^{-1} dr^2 + r^2 d\Omega^2 \tag{VI.2}$$

Here $\eta = dr^2 + r^2 d\Omega^2$ is the Euclidean metric for $\mathbb{R}^4$ in a polar coordinates of $\mathbb{R}^4$.



We consider the conformal rescaling

$$\hat{g}_P = G_P^2 g = \psi_+^{-2}((1 - r^2/a^2)^{-1} dr^2 + r^2 d\Omega^2) \tag{VI.3}$$
$$\hat{g}_Q = G_Q^2 g = \psi_-^{-2}((1 - r^2/a^2)^{-1} dr^2 + r^2 d\Omega^2) \tag{VI.4}$$
$$\tag{VI.5}$$

where

$$\psi_\pm = 1 \pm \sqrt{1 - r^2/a^2} \tag{VI.6}$$

We got $\psi_\pm$ by solving the following two equations

$$\begin{aligned} r &= \psi_\pm \widetilde{r} \\ (1 - r^2/a^2)^{-1/2} dr &= \pm\psi_\pm d\widetilde{r} \end{aligned}$$

so that we get

$$\psi_\pm^{-2}((1 - r^2/a^2)^{-1} dr^2 + r^2 d\Omega^2) = d\widetilde{r}^2 + \widetilde{r}^2 d\Omega^2$$

It is useful to introduce the symbol V,

$$V = \pm\sqrt{1 - r^2/a^2} \tag{VI.7}$$

V is the analogue of the static potential in the de Sitter spacetime considered in [8]. However in that paper we had two 3-hemispheres and now we have two 4-hemispheres. We have

$$G_P = (1 + V)^{-1}, \qquad G_Q = (1 - V)^{-1}$$

Thus the two-pole Green's function for the antipodal pair P, Q are

$$G = 2(1 - V^2)^{-1}$$

Then

$$\begin{aligned} \hat{g} = G^2 g &= a^4 r^{-4}((1 - r^2/a^2)^{-1} dr^2 + r^2 d\Omega^2) \\ &= a^4 r^{-4}(1 \pm \sqrt{1 - r^2/a^2})^2 d\widetilde{r}^2 + r^2 d\Omega^2) \end{aligned}$$

3-volume of a r = constant hypersurface is constant times $a^4 r^{-1}$ and their minimum occurs at $r = a$ because $r \leq a$. This is the totally geodesic equatorial $\mathbb{S}^3$ in $\mathbb{S}^4$ with the standard metric. It satisfies the stability inequality in the corresponding $\hat{g}$ metric by virtue of symmetry and the fact the G is constant on it.

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