INTERSECTION THEORY AND ENUMERATIVE GEOMETRY

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Contents

We work over an algebraically closed filed.

1. Chow ring

1.1. **Chow group.**

Definition 1.1 (Cycle)**.** Let X be a variety (or scheme). The *group of cycles* $Z(X)$ is the free abelian group generated by subvarieties of X. Let $Z_k(X)$ be the group of cycles that are linear combinations of k-dimensional subvarieties. A cycle $Z = \sum a_i Y_i$ is *effective* if all $a_i \geq 0$. It is a (Weil) *divisor* if it is an $(n-1)$-cycle on an n-dimensional variety X.

Definition 1.2 (Rational equivalence)**.** Let $\mathrm{Rat}(X) \subset Z(X)$ be the subgroup generated by cycles of the form

$$V \cap (\{t_0\} \times X) - V \cap (\{t_1\} \times X),$$

where $t_0, t_1 \in \mathbb{P}^1$ and V is a subvariety of $\mathbb{P}^1 \times X$ not contained in any fiber $\{t\} \times X$. Two cycles are *rationally equivalent* if their difference is in $\mathrm{Rat}(X)$.

Definition 1.3 (Chow group)**.** The *Chow group* of X is the quotient $A(X) = Z(X)/\mathrm{Rat}(X)$. We write $[Y]$ to denote the equivalence class of a cycle Y.

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There is another description of rational equivalence as follows. For a rational function f on X, write the associated divisor of f as

$$\operatorname{div}(f) = \sum_Y \operatorname{ord}_Y(f) Y,$$

where the summation ranges over zero and pole divisors of f. This can be first defined on open affine subsets, and then glued together. A divisor of type $\operatorname{div}(f)$ is called a *principal divisor*.

Proposition 1.4. $\operatorname{Rat}(X)$ *is generated by all divisors of rational functions on all subvarieties of* X*.*

Proof. Let f be a rational function on a subvariety Y of X. Then f induces a map $f : Y \dashrightarrow \mathbb{P}^1$ well-defined in codimension one. Take $V \subset \mathbb{P}^1 \times X$ to be the graph of f. Then $\operatorname{div}(f)$ is the difference of the intersection of V with the two fibers over 0 and ∞. Therefore, $\operatorname{div}(f) \in \operatorname{Rat}(X)$.

Conversely, let V be a subvariety of $\mathbb{P}^1 \times X$ not contained in any fiber over $\mathbb{P}^1$. Let $\alpha : V \to \mathbb{P}^1$ be the projection map regarded as a rational function on V. Then $\operatorname{div}(\alpha) = V_0 - V_\infty$, where $V_t = V \cap (\{t\} \times X)$. Denote by β the projection map to X and set $V' = \beta(V)$. If $\dim V' < \dim V$, then $V = \mathbb{P}^1 \times V'$, and hence $\operatorname{div}(\alpha) = 0 \in \operatorname{Rat}(X)$. Now assume that $\beta : V \to V'$ is generically finite. Then $V_0 - V_\infty = \beta_*(\operatorname{div}(\alpha)) = \operatorname{div}(N(\alpha))$, where $N : K(V) \to K(V')$ is the norm function. In other word, the push-forward of a principal divisor under a generically finite morphism remains to be a principal divisor. □

Remark 1.5. Let L/K be a finite field extension and let $n = [L : K]$. For any $\alpha \in L$, let $m_\alpha : L \to L$ be the multiplication map by α. Then m_α is a K-linear map. So we can choose a K-basis of L and write m_α as an $n \times n$ matrix. The *norm function* $N : L \to K$ is defined by $N(\alpha) = \det(m_\alpha)$.

Since the divisor of a rational function is a sum of cycles of codimension one, $A(X)$ is graded by dimension as

$$A(X) = \bigoplus_k A_k(X),$$

where $A_k(X)$ is called the k-dimensional Chow group. If $\dim X = n$, sometimes we also write $A^{n-k}(X) = A_k(X)$.

Example 1.6. Let C be a curve. Then two points in C are rationally equivalent if and only if C is rational, i.e., C is birational to $\mathbb{P}^1$.

Example 1.7. Let X be a smooth variety and $\operatorname{Pic}(X)$ its Picard group parameterizing isomorphic classes of line bundles on X. Sending a line bundle to its first Chern class induces a homomorphism:

$$c_1 : \operatorname{Pic}(X) \to A^1(X).$$

If a line bundle is in $\ker(c_1)$, it is associated to a principal divisor, hence it is trivial. On the other hand, since X is smooth, any Weil divisor is Cartier, and hence c_1 is surjective. We thus conclude that c_1 is an isomorphism.

Remark 1.8. If X is not smooth, c_1 may fail to be an isomorphism. For instance, let X be a rational nodal curve, say $y^2 - x^2(x+1) = 0$. Let p and q be two smooth points of X. Then p and q are rationally equivalent, since X is rational. However,

$\mathscr{O}_X(p) \not\sim \mathscr{O}_X(q)$. Otherwise there would exist a regular map $X \to \mathbb{P}^1$ of degree one, which is impossible.

Proposition 1.9. *We have the following properties of Chow group.*

(1) If X is irreducible of dimension n, then $A_n(X) \cong \mathbb{Z}$ generated by $[X]$. We call $[X]$ the fundamental class of X.

(2) (Excision) If $Y \subset X$ is a subvariety and $U = X \backslash Y$, then we have a right exact sequence

$$A(Y) \to A(X) \to A(U) \to 0,$$

where the maps are induced by inclusion and restriction, respectively.

(3) (Mayer?Vietoris Sequence) If Y_1 and Y_2 are two subvarieties of X, then we have a right exact sequence

$$A(Y_1 \cup Y_2) \to A(Y_1) \oplus A(Y_2) \to A(Y_1 \cap Y_2) \to 0.$$

Proof. (1) is obvious. For (2), define $\partial : \mathbb{P}^1 \times X \to X$ by $\partial(W) = 0$ if W is contained in any fiber $\{t\} \times X$ and $\partial(W) = W_0 - W_\infty$ otherwise. It follows that $\mathrm{Rat}(X) = \mathrm{Im}(\partial)$. Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & Z(\mathbb{P}^1 \times Y) & \to & Z(\mathbb{P}^1 \times X) & \to & Z(\mathbb{P}^1 \times U) & \to & 0 \\
 & & \downarrow^{\partial} & & \downarrow^{\partial} & & \downarrow^{\partial} & & \\
0 & \to & Z(Y) & \to & Z(X) & \to & Z(U) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A(Y) & \to & A(X) & \to & A(U) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The first two rows are exact and the three columns are right exact. A diagram chasing implies that $A(X) \to A(U)$ is surjective and the bottom row is right exact. The proof of (3) can be done by using a similar diagram. □

1.2. **Intersection product.**

Definition 1.10 (Intersection Transversality)**.** Let A and B be two subvarieties of X. We say that A and B intersect *transversally* at p if A, B and X are smooth at p and $T_pA + T_pB = T_pX$. We say that A and B are *generically transverse*, if they intersect transversally at a general point of each component C of $A \cap B$.

Equivalently, generic transversality says that $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$ and C is generically reduced. In particular, if $\operatorname{codim} A + \operatorname{codim} B > \dim X$, A and B are generically transverse if and only if they are disjoint.

If A and B are generically transverse, define the intersection product of their cycle classes as

$$[A][B] = [A \cap B].$$

We would like to generalize this to arbitrary cycle classes in $A(X)$, so that the intersection product makes $A(X)$ an associative, commutative ring graded by codimension, called the *Chow ring* of X. The following result plays a key role in this process.

Theorem 1.11 (Moving Lemma)**.** *Let X be a smooth variety. For every $\alpha, \beta \in A(X)$, there exist cycles $A = \sum m_i A_i$ and $B = \sum n_j B_j$ in $Z(X)$ such that they* *represent* α, β*, respectively, and* A_i, B_j *are generically transverse for each* i *and* j*. Moreover, the class*

$$\sum_{i,j} m_i n_j [A_i \cap B_j]$$

in $A(X)$ *is independent of the choice of* A *and* B*.*

We leave the proof as a reading assignment for the reader.

Remark 1.12. The moving lemma may fail if X is not smooth. For instance, let X be a quadric cone in $\mathbb{P}^3$ and α be a ruling class passing through the vertex of X. Then any divisor A representing α must have a component passing through the vertex as well, for otherwise A would be Cartier but α is not. In general if X is singular, the intersection product can be defined for locally complete intersection subvarieties of X.

Definition 1.13 (Intersection Multiplicity)**.** Suppose A and B are of codimension a and b in X. We say that their intersection is *dimensionally proper*, if every component Z of $A \cap B$ has codimension $a + b$. Then for each component Z, there is an *intersection multiplicity* $m_Z(A, B) \in \mathbb{Z}^+$ such that

$$[A][B] = \sum m_Z(A, B)[Z]$$

in $A^{a+b}(X)$.

Example 1.14. The parabola $y = x^2$ and the line $y = 0$ have intersection multiplicity 2 at the origin.

We would like to establish some functorial properties of Chow groups via the pull-back and push-forward induced by a morphism, analogous to the case of (co)homology. Let $f : Y \to X$ be a proper morphism (i.e., separated, finite type, universally closed). In particular, a subvariety V in Y maps to a subvariety $f(V)$ in X. If $\dim V = \dim f(V)$, then $f|_V$ is generically finite, and its *degree* $n = [K(A) : K(f(A))]$.

Definition 1.15 (Push-forward of Cycles)**.** Under the above assumption, we define $f([V]) \in A(X)$ as follows:
(1) If $\dim f(V) < \dim V$, set $f([V]) = 0$.
(2) If $\dim f(V) = \dim V$, set $f([V]) = n[f(V)]$.

Theorem 1.16. *The map defined above induces a map* $f_* : A(Y) \to A(X)$*.*

We leave the proof as a reading assignment for the reader.

Corollary 1.17 (Degree Map)**.** *Let* X *be an* n*-dimensional variety. Using the map* X *to a single point, we have a map* $\deg : A^n(X) \to \mathbb{Z}$*.*

Next, we consider the pull-back. Let $f : Y \to X$ be a generically separable map between two smooth varieties, i.e., at a generic point y, $T_y Y \to T_{f(y)} f(Y)$ is surjective. For instance, a dominant morphism between two smooth varieties over a field of characteristic 0 is generically separable, so is a finite map of degree not divisible by the characteristic.

Theorem 1.18 (Refined Moving Lemma)**.** *Under the above assumption, for any* $\alpha \in A^k(X)$*, there exists a cycle* A *representing* α *with* $A = \sum n_i A_i \in Z^k(X)$ *such that* $f^{-1}(A_i)$ *is generically reduced of codimension* k *in* Y *for all* i*. Moreover, the class* $\sum n_i [f^{-1}(A_i)] \in A^k(Y)$ *is independent of the choice of* A*.*

We thus define $f^*\alpha = \sum n_i[f^{-1}(A_i)] \in A^k(Y)$, which induces a ring homomorphism $f^* : A(X) \to A(Y)$.

Theorem 1.19 (Push-Pull Formula)**.** *If $\alpha \in A^k(X)$ and $\beta \in A_l(Y)$, then*

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta \in A_{l-k}(X).$$

Proof. This comes from a refinement of the set-theoretic equality $f(f^{-1}A \cap B) = A \cap f(B)$ by applying appropriate multiplicities. □

1.3. **Examples of the Chow ring.** Let us first consider the Chow ring of an open subset of $\mathbb{A}^n$.

Theorem 1.20. *If $U \subset \mathbb{A}^n$ is open, then $A(U) = A_n(U) = \mathbb{Z} \cdot [U]$.*

Proof. By the excision sequence, $A(\mathbb{A}^n) \to A(U)$ is surjective. Therefore, it suffices to show that for any proper subvariety $Y \subset \mathbb{A}^n$, $[Y] = 0 \in A(\mathbb{A}^n)$.

Without loss of generality, choose coordinates $z = (z_1, \ldots, z_n)$ for $\mathbb{A}^n$ and assume that the origin is not contained in Y. We can construct a family by gradually pushing Y away from the origin to ∞. It follows that Y is equivalent to the empty set. More precisely, define $W^\circ \subset (\mathbb{A}^1\backslash 0) \times \mathbb{A}^n$ as

$$\begin{aligned} W^\circ &= \{(t, z) : \frac{z}{t} \in Y\} \\ &= V(\{f(z/t) : f(z) \in I_Y\}), \end{aligned}$$

where I_Y is the ideal of functions vanishing on Y and $V(f_1, \ldots, f_m)$ is the subvariety cut out by $f_1, \ldots, f_m$. Evidently W° is an iso-trivial family with fiber Y over $\mathbb{A}^1\backslash 0$, hence it is irreducible. Let W be its closure in $\mathbb{P}^1 \times \mathbb{A}^n$, and W is also irreducible. Denote by W_t the fiber of W over $t \in \mathbb{P}^1$. We have $W_1 = Y$. Since Y does not contain the origin, there exists a function $g(z) \in I_Y$ such that it has a nonzero constant term c. Note that $G(t, z) = g(z/t)$ extends to a regular function on $(\mathbb{P}^1\backslash 0) \times \mathbb{A}^n$. As $t \to \infty$ it becomes c. Hence $W_\infty = \varnothing$, and consequently $Y \sim 0$. □

Next, we consider varieties that can be built by using affine strata.

Definition 1.21 (Stratification)**.** We say that a scheme X is has a *stratification* if it is a disjoint union of irreducible, locally closed subschemes U_i and in addition, if $\overline{U}_i$ meets U_j, then $\overline{U}_i$ contains U_j. Those U_i are called *strata* and $Y_i = \overline{U}_i$ are called *closed strata*.

A stratification can be recovered from its closed strata:

$$U_i = Y_i \backslash \bigcup_{Y_j \subset Y_i} Y_j.$$

Definition 1.22 ((Quasi)-affine Stratification)**.** If each U_i is isomorphic to some $\mathbb{A}^k$ (resp. an open subset of $\mathbb{A}^k$), we say that X has an *affine* stratification (resp. *quasi-affine* stratification).

Example 1.23. $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$ is an affine stratification of $\mathbb{P}^n$ with $U_i = \mathbb{P}^i \backslash \mathbb{P}^{i-1} \cong \mathbb{A}^i$.

Theorem 1.24. *If X has a quasi-affine stratification, then $A(X)$ is generated by the classes of its closed strata.*

Proof. Do induction on the number of strata. Let U_0 be a minimal stratum, i.e., its closure does not meet any other stratum. It implies that $U_0 = \overline{U}_0$, hence it is closed. Let $U = X \backslash U_0$, which is stratified by the strata of X other than U_0. By Theorem 1.20, $A(U_0) \cong \mathbb{Z} \cdot [U_0]$ for U_0 being quasi-affine, now the conclusion follows from the excision sequence

$$A(U_0) \to A(X) \to A(U) \to 0$$

by induction. □

Remark 1.25. In the above the classes of the quasi-affine strata may not be independent. For instance, take $\mathbb{A}^1$ stratified by a point and its complement. However, if X has an affine stratification, then the classes of the affine strata are independent in $A(X)$, due to a recent non-trivial result proved by Burt Totaro.

Proposition 1.26 (Chow Ring of $\mathbb{P}^n$)**.** *Let $\zeta \in A^1(\mathbb{P}^n)$ be the hyperplane class. Then*

$$A(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/\zeta^{n+1}.$$

Proof. Since $\mathbb{P}^n$ has an affine stratification by a complete flag of linear subspaces, we know $A(\mathbb{P}^n)$ is generated by h_i for $0 \le i \le n$, where h_i is the class of an i-dimensional linear subspace, and hence $A_i(\mathbb{P}^n) \cong \mathbb{Z}$ generated by h_i over $\mathbb{Z}$ with $h_i = \zeta^{n-i}$. Finally, $\zeta^{n+1} = 0$ is obvious since $\mathbb{P}^n$ is n-dimensional. □

Remark 1.27. The above implies that $A^k(\mathbb{P}^n)$ is dual to $A_k(\mathbb{P}^n)$ via the intersection product, analogous to Poincaré duality. This is only a special case. In general, such duality does not hold. For instance, let X be a smooth curve of positive genus. Then $A^0(X) \cong \mathbb{Z}$ but $A_0(X)$ is not finitely generated.

Remark 1.28. Moreover, a subvariety $X \subset \mathbb{P}^n$ of dimension $n-k$ and degree d intersect a general k-plane at d points by the degree map and Bertini's theorem. We conclude that $\deg([X][h_k]) = d$, and hence $[X] = dh_{n-k} = d\zeta^k$.

Corollary 1.29. *Let $\phi : \mathbb{P}^n \to X$ be a morphism to a quasi-projective variety X of dimension less than n. Then ϕ is constant.*

Proof. Without loss of generality, suppose ϕ is surjective. If ϕ is not constant, let $p \in X$ be a general point and $H \in X$ a general hyperplane section such that $p \notin H$. Then $\phi^{-1}(p) \cap \phi^{-1}(H) = \varnothing$. On the other hand, $\dim \phi^{-1}(p) > 0$ and $\dim \phi^{-1}(H) = n-1$. Two such subvarieties in $\mathbb{P}^n$ have nonzero intersection by the structure of $A(\mathbb{P}^n)$, leading to a contradiction. □

Corollary 1.30 (Bézout's Theorem)**.** *Let $Z_1, \ldots, Z_k \subset \mathbb{P}^n$ be subvarieties of codimension $c_1, \ldots, c_k$ with $\sum c_i \le n$ and intersect generically transversally, then*

$$\deg(Z_1 \cap \cdots \cap Z_k) = \prod_{i=1}^{k} \deg(Z_i).$$

In particular, two subvarieties $X, Y \subset \mathbb{P}^n$ with complementary dimension that intersect transversally will intersect at $\deg(X) \cdot \deg(Y)$ *distinct points.*

This is a generalization of Bézout's Theorem for plane curves, which says that two transversal plane curves of degree m and n intersect at mn points.

Definition 1.31 (Veronese map)**.** Let $\phi_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$ be the *d-th Veronese map*:

$$[Z_0, \dots, Z_n]) \mapsto [\cdots, Z^I, \cdots],$$

where Z^I ranges over all degree d monomials in $n+1$ variables and $N = \binom{n+d}{n} - 1$.

Example 1.32. The map $v_{1,2} : \mathbb{P}^1 \to \mathbb{P}^2$ by $[Z_0, Z_1] \mapsto [Z_0^2, Z_0 Z_1, Z_1^2]$ has image as a plane conic.

In general, the degree of the image of a Veronese map can be calculated as follows.

Proposition 1.33. *Let $\Phi_{n,d}$ be the image of $\phi_{n,d}$. Then* $\deg(\Phi_{n,d}) = d^n$.

Proof. Take n general hyperplanes $H_1, \dots, H_n$ in $\mathbb{P}^N$. Then $\phi_{n,d}^{-1}(H_i)$ is a general d hypersurface in $\mathbb{P}^n$. Since $\phi_{n,d}$ is one-to-one, we have

$$\deg(\Phi_{n,d}) = \prod_{i=1}^{n} \phi_{n,d}^{-1}(H_i) = d^n$$

based on Bézout's Theorem. □

Definition 1.34 (Dual hypersurface)**.** Let X be a (generically) smooth hypersurface in $\mathbb{P}^n$. Define its *dual* $X^* \subset \mathbb{P}^{n*}$ by the (closure of) collections of T_pX at a smooth point $p \in X$.

If X is the zero locus of a homogeneous polynomial $F(Z_0, \dots, Z_n)$, T_pX is a hyperplane with equation

$$\sum_{i=0}^{n} \frac{\partial F}{\partial Z_i} Z_i = 0.$$

Then X^* is the image of $G : X \to \mathbb{P}^{n*}$ by

$$G(p) = \left[\frac{\partial F}{\partial Z_0}, \dots, \frac{\partial F}{\partial Z_n} \right]_p.$$

Since X is smooth at p, the partials are not all zero, hence G is well-defined at the smooth locus of X.

Proposition 1.35. *Let X be a smooth hypersurface of degree $d > 1$ in $\mathbb{P}^n$. Then the degree of its dual $X^* \subset \mathbb{P}^{n*}$ is a hypersurface of degree $d(d-1)^{n-1}$.*

Proof. We invoke the fact that $X^{**} = X$ in this case, hence $G : X \to X^*$ is a birational morphism and $X^* \subset \mathbb{P}^{n*}$ is a hypersurface (though not always smooth). Then $\deg(X^*)$ is given by the degree of

$$[X^*] \cdot \prod_{i=1}^{n-1} [H_i] \in A_0(\mathbb{P}^{n*}),$$

where H_i are general hyperplanes in $\mathbb{P}^{n*}$. This further equals the degree of

$$[X] \cdot \prod_{i=1}^{n-1} [G^{-1}(H_i)] \in A_0(\mathbb{P}^n).$$

Since $G^{-1}(H_i)$ is a hypersurface of degree $d-1$ and X is of degree d, we thus obtain that this intersection number equals $d(d-1)^{n-1}$. □

Remark 1.36. In the above we cannot drop the smoothness assumption. For instance, if X is a quadric cone in $\mathbb{P}^3$, then X^* is indeed a conic curve.

Theorem 1.37 (Chow ring of $\mathbb{P}^m \times \mathbb{P}^n$)**.** *Let α and β denote the pullbacks of hyperplane class of $\mathbb{P}^m$ and $\mathbb{P}^n$, respectively, to the product $\mathbb{P}^m \times \mathbb{P}^n$. Then*

$$A(\mathbb{P}^m \times \mathbb{P}^n) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{m+1}, \beta^{n+1}).$$

Moreover, a hypersuface of bidegree (d, e) in $\mathbb{P}^m \times \mathbb{P}^n$ has class $d\alpha + e\beta$.

Proof. Let $\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m = \mathbb{P}^m$ be a complete flag of linear subspaces in $\mathbb{P}^m$, which gives an affine stratification of $\mathbb{P}^m$. Similarly let $\Sigma_0 \subset \Sigma_1 \subset \cdots \subset \Sigma_n = \mathbb{P}^n$ be a complete flat for $\mathbb{P}^n$. Then their products $\Lambda_{i,j} = \Gamma_i \times \Sigma_j$ provide a stratification for $\mathbb{P}^m \times \mathbb{P}^n$ with open strata

$$\Lambda^\circ_{i,j} = \Lambda_{i,j} \backslash (\Lambda_{i,j-1} \cup \Lambda_{i-1,j}) \cong \mathbb{A}^i \times \mathbb{A}^j,$$

which is affine. Therefore, we conclude that $A(\mathbb{P}^m \times \mathbb{P}^n)$ is generated by the classes $\lambda_{i,j}$ of $\Lambda_{i,j}$. Since $\Lambda_{i,j}$ is the transverse intersection of the pullbacks of $m - i$ hyperplanes from $\mathbb{P}^m$ and $n - j$ hyperplanes from $\mathbb{P}^n$, we have $\lambda_{i,j} = \alpha^{m-i}\beta^{n-j}$.

By dimensional reason, $\alpha^{m+1} = \beta^{n+1} = 0$ and $\alpha^m\beta^n = 1$. This shows that the monomials $\alpha^i\beta^j$ are linearly independent over $\mathbb{Z}$ for $0 \le i \le m$ and $0 \le j \le n$.

Finally, let Z be a hypersurface defined by a bihomogeneous polynomial $F(X, Y)$ of bidegree (d, e), where $X = [X_0, \ldots, X_m]$ and $Y = [Y_0, \ldots, Y_n]$ are coordinates of $\mathbb{P}^m$ and $\mathbb{P}^n$, respectively. Then $F(X, Y)/X_0^dY_0^e$ is a rational function on $\mathbb{P}^m \times \mathbb{P}^n$. Therefore, $[Z] - (d\alpha + e\beta) = 0 \in A(\mathbb{P}^m \times \mathbb{P}^n)$. □

Definition 1.38 (Segre variety)**.** Define $\Sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1}$ by

$$([X_0, \ldots, X_m], [Y_0, \ldots, Y_n]) \mapsto [\ldots, X_iY_j, \ldots].$$

The image of $\Sigma_{m,n}$ is called a *Segre variety*.

Lemma 1.39. *$\Sigma_{m,n}$ is an embedding.*

Proof. Without loss of generality, consider the open set U by setting $X_0 = Y_0 = 1$ and using affine coordinates $x_i = X_i/X_0$, $y_j = Y_j/Y_0$ for $1 \le i \le m$ and $1 \le j \le n$. Then $\Sigma|_U$ is given by

$$(x_1, \ldots, x_m, y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_n, x_1, \ldots, x_m, \ldots, x_iy_j, \ldots),$$

and the domain coordinates can be recovered from the image. □

Example 1.40. $\Sigma_{1,1}$ embeds $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ as a smooth quadric surface.

Proposition 1.41 (Degree of the Segre variety)**.** *The degree of the Segre embedding of $\mathbb{P}^m \times \mathbb{P}^n$ is $\binom{m+n}{n}$.*

Proof. Take $m + n$ general hyperplanes in $\mathbb{P}^{(m+1)(n+1)-1}$. The pull-back of a hyperplane via $\Sigma_{m,n}$ is a hypersurface of bidegree $(1, 1)$, hence it has class $\alpha + \beta$ in $A(\mathbb{P}^m \times \mathbb{P}^n) \cong \mathbb{Z}[\alpha, \beta](\alpha^{m+1}, \beta^{n+1})$. Then the desired degree is equal to that of

$$(\alpha + \beta)^{m+n},$$

which is $\binom{m+n}{n}$, since $\deg(\alpha^m\beta^n) = 1$ under the degree map $A_0(\mathbb{P}^m \times \mathbb{P}^n) \to \mathbb{Z}$. □

Next, denote by Δ the *diagonal* of $\mathbb{P}^n \times \mathbb{P}^n$, and set $\delta = [\Delta] \in A_n(\mathbb{P}^n \times \mathbb{P}^n)$. We still use α, β as above as the generators of $A(\mathbb{P}^n \times \mathbb{P}^n)$.

Proposition 1.42 (Degree of the diagonal)**.** *We have*

$$\delta = \sum_{i=0}^{n} \alpha^i \beta^{n-i} \in A^n(\mathbb{P}^n \times \mathbb{P}^n).$$

Proof. We use the method of *test families*. Suppose that

$$\delta = \sum_{i=0}^{n} c_i \alpha^i \beta^{n-i},$$

where $c_i \in \mathbb{Z}$ are to be determined.

Let $\Lambda^{n-i} \times \Gamma^i \subset \mathbb{P}^n \times \mathbb{P}^n$ be the product of two general subspaces of codimension $n-i$ and i. It has cycle class $\alpha^{n-i}\beta^i$. Moreover, $\Delta \cap (\Lambda^{n-i} \times \Gamma^i)$ consists of a reduced single point given by $\Lambda^{n-i} \cap \Gamma^i$, where the intersection transversality can be easily checked by using local coordinates. Intersecting both sides of the above equation with $\alpha^{n-i}\beta^i$ and taking the degree, we conclude that $c_i = 1$ for all i. □

Let $f : \mathbb{P}^n \to \mathbb{P}^m$ be a morphism given by $m+1$ homogeneous polynomials F_i of degree d:

$$[X_0, \ldots, X_n] \mapsto [F_0(X), \ldots, F_m(X)],$$

where F_i do not have a common zero. This assumption already implies that $m \geq n$. Let Γ_f be the *graph* of f defined as

$$\Gamma_f = \{(p, f(p)) \mid p \in \mathbb{P}^n\} \subset \mathbb{P}^n \times \mathbb{P}^m.$$

Note that $\Gamma_f \cong \mathbb{P}^n$ by the projection to $\mathbb{P}^n$. The diagonal Δ is a special case of Γ_f when $m = n$ and $d = 1$.

Below we will calculate the class $[\Gamma_f] \in A^m(\mathbb{P}^n \times \mathbb{P}^m) \cong \mathbb{Z}[\alpha, \beta](\alpha^{n+1}, \beta^{m+1})$.

Proposition 1.43 (The degree of a graph)**.** *Under the above notation, we have*

$$[\Gamma_f] = \sum_{i=0}^{m} d^i \alpha^i \beta^{m-i} \in A^m(\mathbb{P}^n \times \mathbb{P}^m).$$

Proof. Note that $\alpha^i = 0$ for $i \geq n+1$. Suppose that

$$[\Gamma_f] = \sum_{i=1}^{n} c_i \alpha^i \beta^{m-i}$$

with $c_i \in \mathbb{Z}$ undetermined.

Take $\Lambda^{n-i} \times \Gamma^i$ the product of two general subspaces of codimension $n-i$ and i in $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively. Then the degree of $[\Gamma_f] \cdot \alpha^{n-i}\beta^i$ is equal to the degree of the intersection of i hypersurfaces of degree d and $n-i$ hyperplanes in $\mathbb{P}^n$, hence is d^i. The intersection transversality is due to a Bertini type argument. Therefore, using $\Lambda^{n-i} \times \Gamma^i$ as a test family, we conclude that $c_i = d^i$. □

Below is a simple application of the class of a graph.

Proposition 1.44 (Fixed point formula)**.** *Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a morphism given by $n+1$ general homogeneous polynomials F_i of degree d. Then the number of fixed points of f is $d^n + d^{n-1} + \cdots + 1$.*

Proof. Since F_i are general, we can make sure that Γ_f and Δ intersect transversally in $\mathbb{P}^n \times \mathbb{P}^n$. Then the number of fixed points of f equals the degree of $[\Gamma_f][\Delta]$, which has class

$$\left(\sum_{i=0}^{n} d^i \alpha^i \beta^{n-i}\right) \cdot \left(\sum_{j=0}^{n} \alpha^j \beta^{n-j}\right),$$

and its degree is $d^n + d^{n-1} + \cdots + 1$. ☐

Corollary 1.45. *A general $(n+1) \times (n+1)$ matrix has $n+1$ eigenvalues.*

Proof. In the above proposition, take $d = 1$. ☐

Let $\widetilde{\mathbb{P}^n}$ be the *blowup* of $\mathbb{P}^n$ at a point p. Let $E \cong \mathbb{P}^{n-1}$ be the *exceptional divisor*. Geometrically, E separates lines in $\mathbb{P}^n$ passing through p so that $\widetilde{\mathbb{P}^n}$ is a $\mathbb{P}^1$-bundle over E. In terms of local coordinates, $\widetilde{\mathbb{A}^n} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ is defined by $(x_1, \ldots, x_n); [t_1, \ldots, t_n]$ satisfying $x_i t_j = x_j t_i$ for all i, j, where $p = (0, \ldots, 0)$ and $E = \{p\} \times \mathbb{P}^{n-1}$. The projection map $\pi : \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$ is an isomorphism away from E, and maps E to p. For a subvariety $Z \subset \mathbb{P}^n$, let $\widetilde{Z}$ be the closure of $\pi^{-1}(Z\backslash p)$ in $\widetilde{\mathbb{P}^n}$, called the *strict transform* of Z.

Example 1.46. Draw the picture of $\widetilde{\mathbb{P}^2}$.

Let $h \in A^1(\mathbb{P}^n)$ be the hyperplane class. Denote by $\lambda = \pi^* h$ and e the class of E in $A^1(\widetilde{\mathbb{P}^n})$.

Proposition 1.47 (Chow ring of $\widetilde{\mathbb{P}^n}$)**.** *We have*

$$A(\widetilde{\mathbb{P}^n}) \cong \frac{\mathbb{Z}[\lambda, e]}{(\lambda e, \lambda^n + (-1)^n e^n)}.$$

Proof. One can find an affine stratification of $\widetilde{\mathbb{P}^n}$ as in Eisenbud-Harris (reading assignment). Alternatively, we have the right exact sequences:

$$A_k(E) \to A_k(\widetilde{\mathbb{P}^n}) \to A_k(U) \to 0,$$
$$A_k(p) \to A_k(\mathbb{P}^n) \to A_k(U) \to 0,$$

where $U = \mathbb{P}^n \backslash p = \widetilde{\mathbb{P}^n} \backslash E$. We thus obtain that $A(U) = \mathbb{Z}[\lambda]/\lambda^n$ and the rank of $A^k(\widetilde{\mathbb{P}^n}) \leq 2$ for $0 < k < n$ and $A^0(\widetilde{\mathbb{P}^n}) = A^n(\widetilde{\mathbb{P}^n}) = \mathbb{Z}$ under the degree map.

First, let us figure out the relation between λ and e. Take a general hyperplane in $\mathbb{P}^n$ not containing p. Then its preimage will be disjoint with E. We thus conclude that

$$\lambda e = 0.$$

Moreover, among all hyperplanes containing p, take n general ones Γ_i for $1 \leq i \leq n$ such that they intersect transversally at p. The intersection of their proper transforms is the locus in E parameterizing lines contained in $\cap_{i=1}^n \Gamma_i = \{p\}$, which indeed does not contain any line. Note that $[\widetilde{\Gamma_i}] = \lambda - e$. Therefore, we obtain that

$$(\lambda - e)^n = 0,$$

i.e. $\lambda^n + (-1)^n e^n = 0$.

Finally, let us show that λ^k, e^k freely generate $A^k(\widetilde{\mathbb{P}^n})$ for $0 < k < n$ over $\mathbb{Z}$. By the projection formula, $\pi_* \lambda^n = \pi_*(\pi^* h)^n = h^n$ has degree 1. By the above relation, e^n has degree $(-1)^{n-1}$. If a class $z \in A^k(\widetilde{\mathbb{P}^n})$ for $0 < k < n$ can be written as $z = a\lambda^k + be^k$ for $a, b \in \mathbb{Q}$, intersecting with $\lambda n - k$ and e^{n-k} and taking the degree, we conclude that $a, b \in \mathbb{Z}$. If $a\lambda^k + be^k = 0$, the same argument implies that $a = b = 0$. □

Remark 1.48. We know that $e = \lambda - [\widetilde{\Gamma}]$, and hence

$$e^2 = [E \cap (\Lambda - \widetilde{\Gamma})] = -[E \cap \widetilde{\Gamma}],$$

which is a hyperplane in E parameterizing lines passing through p and contained in Γ. In other words, $[E]^2$ is the negative of a hyperplane class in E.

Example 1.49. $A(\widetilde{\mathbb{P}}^2) \cong \mathbb{Z}[\lambda, e]/(\lambda e, \lambda^2 + e^2)$. In particular, $\deg(e^2) = -1$ has negative self intersection.

Now let us consider plane cubics. The parameter space of plane cubics is $\mathbb{P}^9$. Let $R, T \subset \mathbb{P}^9$ be the closures of the loci parameterizing reducible cubics and cubics consisting of three lines, respectively. An easy dimension count shows that $\dim R = 7$ and $\dim T = 6$.

Proposition 1.50. *As subvarieties in* $\mathbb{P}^9$*, we have* $\deg R = 21$ *and* $\deg T = 15$.

Proof. A general point in R parameterizes a line union a conic. Therefore, R is the image of the birational map

$$f : \mathbb{P}^2 \times \mathbb{P}^5 \to \mathbb{P}^9,$$

where $(F, G) \mapsto FG$. Let ζ be the hyperplane class in $A(\mathbb{P}^9)$. Note that $f^*\zeta$ is the class of a hypersurface of bidgree $(1, 1)$, i.e.,

$$f^*\zeta = \alpha + \beta \in A^1(\mathbb{P}^2 \times \mathbb{P}^5),$$

where α, β are the standard generators of $A(\mathbb{P}^2 \times \mathbb{P}^5)$. Then we obtain that

$$\deg R = \deg(f^*\zeta)^7 = \deg(\alpha + \beta)^7 = \binom{7}{2} = 21.$$

Similarly, T is the image of

$$g : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9.$$

However, g is generically of degree $3! = 6$, since the three lines are not ordered. Therefore, we obtain that

$$\deg T = \frac{1}{6}\deg(g^*\zeta)^6 = \frac{1}{6}\deg(\alpha + \beta + \gamma)^6 = \frac{1}{6}\binom{6}{2, 2, 2} = 15.$$

□

1.4. **Canonical class and adjunction.** If X is a smooth n-dimensional variety, let $\omega_X = \wedge^n \Omega_X$ be the canonical line bundle and $K_X = c_1(\omega_X)$ denote the *canonical class* of X. Equivalently, ω_X is the locally free sheaf of holomorphic n-forms on X.

Example 1.51. If X is a smooth curve of genus g, then K_X is of degree $2g - 2$.

Let us calculate the canonical class of $\mathbb{P}^n$.

Proposition 1.52 (The class of $K_{\mathbb{P}^n}$)**.** *Let h be the hyperplane class in $A(\mathbb{P}^n)$. Then we have*

$$K_{\mathbb{P}^n} = -(n + 1)h.$$

Proof. One proof is based on the *Euler sequence*

$$0 \to \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0.$$

Alternatively, let $[X_0, \ldots, X_n]$ be the coordinates of $\mathbb{P}^n$. Set $x_i = X_i/X_0$ for $1 \le i \le n$. In the open set $X_0 \ne 0$,

$$\omega = dx_1 \wedge \cdots \wedge dx_n$$

is a holomorphic n-form. Consider another open set $X_1 \ne 0$. Set $y_1 = X_0/X_1$ and $y_i = X_i/X_1$ for $2 \le i \le n$. Then $x_1 = 1/y_1$ and $x_i = y_i/y_1$ for $2 \le i \le n$. The n-form ω in terms of the y-coordinates becomes

$$\omega = \frac{-dy_1}{y_1^2} \wedge \wedge_{i=2}^n \Big(\frac{y_i dy_1 - y_1 dy_i}{y_1^2}\Big) = (-1)^n \frac{dy_1 \wedge \cdots \wedge dy_n}{y_1^{n+1}}.$$

It implies that ω has a pole divisor along $y_1 = 0$, i.e. the hyperplane $X_0 = 0$ of order $n+1$. The desired claim thus follows. □

Let $Y \subset X$ be a smooth codimension-one subvariety of X. The *normal bundle* $N_{Y/X}$ is the dual of the line bundle $I_{Y/X}/I_{Y/X}^2 = \mathscr{O}_X(-Y)|_Y$, where $I_{Y/X}$ is the ideal sheaf of Y in X. This follows from the short exact sequence of differentials

$$0 \to I_{Y/X}/I_{Y/X}^2 \to \Omega_X|_Y \to \Omega_Y \to 0.$$

We thus conclude that

$$N_{Y/X} = \mathscr{O}_X(Y)|_Y.$$

Proposition 1.53 (Adjunction Formula)**.** *We have $\omega_Y = \omega_X \otimes \mathscr{O}_X(Y)|_Y$. In terms of divisor classes, this is $K_Y = (K_X + [Y])|_Y$. In particular, if Y is a curve in a surface X, we have $2g(Y) - 2 = \deg((K_X + [Y])[Y])$.*

Proof. We have the exact sequence (the dual of the sequence of differentials)

$$0 \to T_Y \to T_X|_Y \to N_{Y/X} \to 0.$$

Taking the top wedge product of the middle term, we obtain that

$$\omega_X^*|_Y = \wedge^{n-1} T_Y \otimes N_{Y/X} = \omega_Y^* \otimes \mathscr{O}_X(Y).$$

The desired formula follows right away. □

Proposition 1.54 (Canonical class of a hypersurface)**.** *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. Then $\omega_X = \mathscr{O}_X(d-n-1)$.*

Proof. By adjunction,

$$\omega_X = \omega_{\mathbb{P}^n} \otimes \mathscr{O}_{\mathbb{P}^n}(d)|_X = \mathscr{O}_{\mathbb{P}^n}(d-n-1)|_X.$$

□

Proposition 1.55 (Canonical class of a complete intersection)**.** *Let $X \subset \mathbb{P}^n$ be the complete intersection of k hypersurfaces $Z_1, \ldots, Z_k$ with $\deg Z_i = d_i$. Then $\omega_X = \mathscr{O}_X(\sum_{i=1}^k d_i - n - 1)$.*

Proof. Apply adjunction repeatedly to the partial intersections $Z_1 \cap \cdots \cap Z_i$ for $i = 1, \ldots, k$. □

Proposition 1.56 (Genus of a plane curve)**.** *Let $C \subset \mathbb{P}^2$ be a plane curve of degree d. Then the (arithmetic) genus of C is*

$$g = \frac{(d-1)(d-2)}{2}.$$

Proof. By numerical adjunction, we have

$$2g - 2 = (d-3)d.$$

Another proof, which also works for C singular, is to consider the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-d) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_C \to 0.$$

It follows that

$$g = h^1(C, \mathscr{O}) = h^2(\mathbb{P}^2, \mathscr{O}(-d)) = h^0(\mathbb{P}^2, \mathscr{O}(d-3)) = \binom{d-1}{2}.$$

□

Let $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a quadric surface in $\mathbb{P}^3$ via the Segre embedding $\Sigma_{1,1}$. Let α, β be the two ruling classes. The canonical class of Q is

$$K_Q = -2\alpha - 2\beta,$$

which follows from adjunction or pulling back $K_{\mathbb{P}^1}$ from each factor.

Proposition 1.57 (Curves on a quadric)**.** *Let $C \subset Q$ be a curve with class $a\alpha + b\beta$ for $a, b \in \mathbb{Z}$. Then the genus and degree of C is*

$$g = (a-1)(b-1), \quad d = a + b.$$

In particular, C is a smooth rational curve if and only if $a = 1$ or $b = 1$; C is a smooth elliptic curve if and only if $[C] = 2\alpha + 2\beta$ and its degree is 4 (elliptic quartic curve in $\mathbb{P}^3$).

Proof. By adjunction,

$$2g - 2 = \deg((K_Q + [C])[C]) = 2ab - 2a - 2b.$$

Moreover, let H be a hyperplane in $\mathbb{P}^3$, and then $[H]|_Q = \alpha + \beta$. We thus obtain that

$$d = \deg([C][H]|_Q) = a + b.$$

□

Finally, let us consider the blowup of a surface. Let $\widetilde{S}$ be the blowup of a smooth surface S at a point p. Let $E \cong \mathbb{P}^1$ be the exceptional divisor parameterizing tangent lines of S at p, and $e = [E] \in A(\widetilde{S})$. Denote by $\pi : \widetilde{S} \to S$ the blowdown map, which maps E to p. For $C \subset S$ a curve, we use $\widetilde{C}$ to denote its strict transform.

Proposition 1.58 (Chow ring of $\widetilde{S}$)**.**

(1) $\deg(e^2) = -1$.

(2) $A_1(\widetilde{S}) \cong A_1(S) \oplus \mathbb{Z}$, where the first summand is by π^* and the second is generated by e.

(3) For $\alpha, \beta \in A_1(S)$, $\pi^*\alpha \cdot \pi^*\beta = \pi^*(\alpha\beta)$.

(4) For $\alpha \in A_1(S)$, $e \cdot \pi^*\alpha = 0$.

(5) $K_{\widetilde{S}} = \pi^* K_S + e$.

Proof. (1) Take a smooth curve $C \subset S$ passing through p. Then $\pi^*[C] = [\widetilde{C}] + e$. By the projection formula, we have

$$\deg(\pi_*(\pi^*[C] \cdot e) = \deg([C] \cdot \pi_* e) = 0.$$

Moreover, $\widetilde{C}$ and E intersect transversally at the point in E parameterizing the tangent line T_pC. It follows that $\deg(e^2) = -1$.

(2) For $\alpha \in A_1(S)$, by the Moving Lemma, we can choose $A \in Z_1(S)$ representing α such that the support of A is away from p. Then $\pi_*\pi^*A = A$, and hence $\pi_*\pi^*$ is the identity map of $A_1(S) \to A_1(S)$.

Consider $\pi_* : A_1(\widetilde{S}) \to A_1(S)$. It is surjective and the kernel is generated by e. Therefore, we have

$$0 \to \langle e \rangle \to A_1(\widetilde{S}) \to A_1(S) \to 0.$$

Since $\deg(e^2) = -1$, e is not torsion in $A_1(\widetilde{S})$. Moreover, π^* splits this sequence. Therefore, we obtain that $A_1(\widetilde{S}) \cong A_1(S) \oplus \mathbb{Z}e$.

(3) By the (refined) Moving Lemma, we can take A, B representing α, β such that both of them are away from p. Then $\pi^{-1}A$ and $\pi^{-1}B$ are disjoint with E.

(4) By the Moving Lemma, we can take A representing α and away from p. Then $\pi^{-1}A$ is disjoint with E. Alternatively, this follows from the projection formula.

(5) Suppose $K_{\widetilde{S}} = \pi^*K_S + ne$ for $n \in \mathbb{Z}$ to be determined. Since E is of genus 0, by adjunction, we have

$$-2 = e \cdot (K_{\widetilde{S}} + e) = e \cdot (\pi^*K_S + (n+1)e) = -n - 1,$$

and hence $n = 1$. □

Let $C \subset S$ be a curve passing through p. Suppose x, y are the local coordinates at p and C is locally defined by $f(x,y) = \sum_i f_i(x,y) \in k[[x,y]]$, where f_i is homogeneous of degree i. Define the *multiplicity* of C at p by the minimum of i among all nonzero f_i.

Example 1.59. A nodal curve $xy = 0$ has multiplicity 2 at the node. A cuspidal curve $y^2 - x^3 = 0$ has multiplicity 2 at the cusp. If C has local equation $(x - a_1y) \cdots (x - a_my)$ with distinct $a_1, \ldots, a_m \in k$, we say that C has an *m-fold point*.

Lemma 1.60. *If C has an m-fold point at $p \in S$, then $\pi^*[C] = [\widetilde{C}] + me$.*

Proof. Let x, t denote a local coordinate system for $\widetilde{S}$ such that $y/x = t$, where t is the slope of tangent lines at p. Then $\pi^*f = f_m(x, tx) + \sum_{i>m} f_i(x, tx)$ is the local defining equation of $\pi^{-1}C$. Therefore, the vanishing locus of π^*f consists of $\widetilde{C}$ with m copies of E. □

Proposition 1.61 (Genus of a singular curve)**.** *In the above setting, the (arithmetic) genus of $\widetilde{C}$ is*

$$\widetilde{g} = \frac{[C]^2 + K_S \cdot [C]}{2} + 1 - \binom{m}{2}.$$

Proof. By adjunction and the preceding proposition,

$$\begin{aligned} 2\widetilde{g} - 2 &= (K_{\widetilde{S}} + [\widetilde{C}]) \cdot [\widetilde{C}] \\ &= \big(\pi^*K_S + \pi^*[C] + (1-m)e\big) \cdot \big(\pi^*[C] - me\big) \\ &= K_S \cdot [C] + [C]^2 - m(m-1). \end{aligned}$$

The desired formula thus follows. □

2. Chern class

Chern classes are objects associated to vector bundles. They can be defined in various ways. Here our approach is to treat them as elements in the Chow ring.

2.1. **Formal calculation of Chern classes.** Let E be a vector bundle of rank m on X. Assign to E an element $c_k(E) \in A^k(X)$ for $1 \leq k \leq m$. Call $c_k(E)$ the *kth Chern class* of E and define the *total Chern class* by

$$c(E) = 1 + c_1(E) + \cdots + c_m(E) \in A(X).$$

We want these assignments to satisfy the following properties:

(1) If L is a line bundle associated to a divisor D, then $c_1(L) = [D]$.

(2) If there exists an exact sequence $0 \to E \to F \to G \to 0$, then

$$c(F) = c(E)c(G).$$

(3) If $f : Y \to X$ is a map and E a vector bundle on a variety X, then $c(f^*E) = f^*c(E)$.

The actual definition of Chern classes will be introduced later by degeneracy loci. Here we do some calculation first.

Example 2.1. Let $E_i = \mathscr{O}(d_i)$ be a line bundle of degree d_i on $\mathbb{P}^n$ for $i = 1, 2$. Let h be the hyperplane class of $\mathbb{P}^n$. We have

$$\begin{aligned} c(E_1 \oplus E_2) &= c(E_1)c(E_2) \\ &= (1 + d_1 h)(1 + d_2 h) \\ &= 1 + (d_1 + d_2)h + d_1 d_2 h^2. \end{aligned}$$

Hence we conclude that

$$c_1(E_1 \oplus E_2) = (d_1 + d_2)h,$$
$$c_2(E_1 \oplus E_2) = d_1 d_2 h^2.$$

Example 2.2. Let us compute the Chern class of the tangent bundle on $\mathbb{P}^n$. Let S be the *tautological line bundle* such that the fiber $S|_{[L]}$ represents the 1-dimensional subspace $L \subset \mathbb{C}^{n+1}$. Then S^* is the *universal line bundle* such that the fiber $S^*|_{[L]}$ is the space of linear functionals $\mathrm{Hom}(L, \mathbb{C})$. Take a non-zero linear map $\phi : \mathbb{C}^{n+1} \to \mathbb{C}$ and define $\Phi([L], v) = \phi(v)$, where v is a vector in L. Then Φ can be regarded as a section of S^*. Moreover, Φ is the zero map at $[L]$ if and only if $L \subset \ker(\phi)$, i.e. the zero locus of Φ in $\mathbb{P}^n$ is a hyperplane given by the projectivization of $\ker(\phi)$. It implies

$$S^* = \mathscr{O}_{\mathbb{P}^n}(1), \quad S = \mathscr{O}_{\mathbb{P}^n}(-1).$$

We have an exact sequence

$$0 \to S \to V \to Q \to 0,$$

where $V = \mathbb{C}^{n+1} \times X$ is the trivial bundle of rank $n+1$, $S \to V$ is the natural inclusion $L \subset \mathbb{C}^{n+1}$ fiber by fiber and Q is the quotient bundle with fiber $Q|_{[L]} = \mathbb{C}^{n+1}/L$. Tensor it with S^* and apply the identification $S = \mathscr{O}_{\mathbb{P}^n}(-1)$. We thus obtain

$$0 \to \mathscr{O}_{\mathbb{P}^n} \to \bigoplus_{i=1}^{n+1} \mathscr{O}_{\mathbb{P}^n}(1) \to S^* \otimes Q \to 0.$$

Note that the fiber of the tangent bundle $T_{\mathbb{P}^n}|_{[L]}$ is canonically isomorphic to $\mathrm{Hom}(L, \mathbb{C}^{n+1}/L)$. More precisely, given $\phi \in \mathrm{Hom}(L, \mathbb{C}^{n+1})$ and $v \in L$, one can define an arc $v(t) = v + t\phi(v) \in \mathbb{C}^{n+1}$. The part $\phi(v)$ modulo L is a tangent vector at $[L]$. Conversely, one can lift a tangent vector as an equivalence class of arcs through $v \in L$ and deduce ϕ accordingly. It implies that

$$T_{\mathbb{P}^n} \cong S^* \otimes Q.$$

Hence we obtain the following *Euler sequence*

$$0 \to \mathscr{O}_{\mathbb{P}^n} \to \bigoplus_{i=1}^{n+1} \mathscr{O}_{\mathbb{P}^n}(1) \to T_{\mathbb{P}^n} \to 0.$$

The total Chern class of $T_{\mathbb{P}^n}$ is

$$\begin{aligned} c(T_{\mathbb{P}^n}) &= c(\mathscr{O}_{\mathbb{P}^n}(1))^{n+1} \\ &= (1+h)^{n+1} \\ &= 1 + (n+1)h + \binom{n+1}{2}h^2 + \cdots + \binom{n+1}{n}h^n. \end{aligned}$$

Example 2.3. For $n = 2$, $c(T_{\mathbb{P}^2}) = 1 + 3h + 3h^2$ is not a product of two linear forms of h. Hence $T_{\mathbb{P}^2}$ cannot be an *extension* of two line bundles on $\mathbb{P}^2$, i.e. there cannot exist two line bundles L and M on $\mathbb{P}^2$ such that they fit in an exact sequence

$$0 \to L \to T_{\mathbb{P}^2} \to M \to 0.$$

2.2. **The splitting principle.** In an ideal world, a vector bundle would split as a direct sum of line bundles. Then we could calculate its Chern class based on that of line bundles. Well, in real world there exist tons of non-splitting vector bundles. But for computing the Chern classes only, one can still pretend that a vector bundle splits and carry out the calculation formally.

To illustrate what this means, consider a vector bundle E of rank n. Suppose E splits as a direct sum of line bundles

$$E = L_1 \oplus \cdots \oplus L_n$$

and let $a_i = c_1(L_i)$. Then we have

$$\begin{aligned} c(E) &= \prod_{i=1}^{n}(1 + a_i) \\ &= 1 + \Big(\sum_{i=1}^{n} a_i\Big) + \Big(\sum_{i \neq j} a_i a_j\Big) + \cdots \end{aligned}$$

hence we conclude that

$$c_1(E) = \sum_{i=1}^{n} a_i, \quad c_2(E) = \sum_{i \neq j} a_i a_j, \quad \ldots$$

Suppose L is a line bundle with first Chern class $c_1(L) = b$. Then

$$\begin{aligned} c(E \otimes L) &= c((L_1 \otimes L) \oplus \cdots \oplus (L_n \otimes L)) \\ &= (1 + a_1 + b) \cdots (1 + a_n + b) \\ &= 1 + \Big(\sum_{i=1}^{n} a_i + nb\Big) + \left(\sum_{i \neq j} a_i a_j + \binom{n}{2} b^2 + (n-1)\Big(\sum_{i=1}^{n} a_i\Big) b\right) + \cdots \\ &= 1 + (c_1(E) + nc_1(L)) + \left(c_2(E) + \binom{n}{2} c_1^2(L) + (n-1)c_1(E)c_1(L)\right) + \cdots \end{aligned}$$

Hence we conclude that

$$c_1(E \otimes L) = c_1(E) + nc_1(L),$$
$$c_2(E \otimes L) = c_2(E) + \binom{n}{2} c_1^2(L) + (n-1)c_1(E)c_1(L)$$

and etc.

The splitting principle says that even if E is not decomposable as a direct sum of line bundles, the above calculation still holds. In other words, one can write

$$c(E) = \prod_{i=1}^{n} (1 + a_i),$$

where a_i's are *formal roots* of the *Chern polynomial*

$$c_t(E) = 1 + c_1(E)t + \cdots c_n(E)t^n.$$

We call $a_1, \ldots, a_n$ the *Chern roots* of E. If E splits as a direct sum of line bundles, then a_i's are just the first Chern classes of the summands, hence belong to $A^1(X)$. In general, each individual a_i may not correspond to a divisor class. But the Chern classes of E are represented by symmetric functions of a_i's, hence eventually one can get rid of a_i's and express the final result of this kind of calculation by the Chern classes of the original vector bundle.

Let us use the splitting principle to recover the Chern class of the canonical line bundle.

Proposition 2.4. *Let K be the canonical divisor class and T the tangent bundle on a smooth variety X. Then we have*

$$K = -c_1(T).$$

Proof. If T splits as $\bigoplus_{i=1}^n L_i$, then

$$\bigwedge^n T = \bigotimes_{i=1}^n L_i.$$

Hence we conclude that

$$\begin{aligned} c_1\Big(\bigwedge^n T\Big) &= \sum_{i=1}^n c_1(L_i) \\ &= c_1(T). \end{aligned}$$

Since K is the first Chern class of T^*, we obtain that $K = -c_1(T)$. The splitting principle ensures that it holds regardless of whether or not T is decomposable. □

Corollary 2.5. *On $\mathbb{P}^n$, we have $K = -(n+1)h$.*

Proof. We have seen that

$$c(T) = (1 + h)^{n+1}.$$

In particular, $c_1(T) = (n+1)h$. Then the claim follows from the above proposition. □

Finally let us say a word about why the splitting principle works. Roughly speaking, for a vector bundle E on X, one can always find $f : \tilde{X} \to X$ such that f^*E splits and $A^*(X) \to A^*(\tilde{X})$ is injective. Then one can carry out the calculation in $A^*(X)$ regarded as a subring of $A^*(\tilde{X})$.

2.3. **Determinantal varieties.** Let $M = M(m,n)$ be the space of $m \times n$ matrices. As a variety, M is isomorphic to $\mathbb{A}^{mn}$. For $0 \leq k \leq \min(m,n)$, denote by $M_k = M_k(m,n)$ the locus of matrices of rank at most k, that is, cut out by all $(k+1) \times (k+1)$ minors. We say that M_k is the *kth generic determinantal variety.*

Proposition 2.6. *M_k is an irreducible subvariety of codimension $(m-k)(n-k)$ in M.*

Proof. Define the incidence correspondence

$$\tilde{M}_k = \{(A, W) \mid A \cdot W = 0\} \subset M \times G(n-k, n).$$

Then $\tilde{M}_k$ admits two projections p_1 and p_2 to M and $G(n-k,n)$, respectively. The map p_1 is onto M_k and generically one to one. In other words, p_1 has a positive dimensional fiber over A if and only if the rank of A is at most $k-1$. On the other hand, for a fixed $W \in G(n-k,n)$, the fiber of p_2 over W is isomorphic to $\mathbb{A}^{mk}$. Hence we conclude that

$$\begin{aligned} \dim M_k &= \dim \tilde{M}_k \\ &= \dim G(n-k,n) + mk \\ &= (m+n)k - k^2. \end{aligned}$$

Therefore, the codimension of M_k in M is equal to

$$mn - (m+n)k + k^2 = (m-k)(n-k).$$

Moreover, $\tilde{M}_k$ is a vector bundle on $G(n-k,n)$, hence is irreducible. So is M_k. □

Remark 2.7. As a vector bundle, $\tilde{M}_k$ is smooth, but the contraction M_k may not be smooth. With a more detailed study of the tangent space, one can show that the singular locus of M_k is exactly M_{k-1}.

Determinantal varieties are useful to the study of maps between vector bundles. Let E and F be two vector bundles on X of rank n and m, respectively. Suppose $\phi : E \to F$ is a bundle map. Choose a local trivialization of E and F. Then ϕ locally corresponds to an $m \times n$ matrix, whose entries are holomorphic functions. Denote by Φ_k the locus in X where the rank of ϕ is at most k. Then Φ_k is the inverse image of the generic determinantal variety M_k, hence it has expected codimension $(m-k)(n-k)$ in X.

2.4. **Degeneracy loci.** Let E be a vector bundle of rank n on a projective variety X. Moreover, suppose E is *globally generated*, i.e. for any $p \in X$, the fiber $E|_p$ is fully spanned by its global sections. We will define Chern classes for such E. In general, one can twist a vector bundle by a very ample line bundle to obtain a globally generated vector bundle, define its Chern classes and then twist back to the original bundle.

For $1 \leq i \leq \min\{n, \dim X\}$, let $s_1, \ldots, s_i$ be general sections of E. At a point $p \in X$, we expect these sections to be linearly independent. Hence we denote by D_i the *degeneracy locus* in X where $s_1, \ldots, s_i$ are linearly dependent. Equivalently, D_i is the locus of p in X such that

$$s_1(p) \wedge \cdots \wedge s_i(p) = 0.$$

Note that D_i possesses a determinantal structure. Take a trivialization of E in a local neighborhood U and let $e_1, \ldots, e_n$ be sections that generate $E|_U$. Then one can write

$$s_j = \sum_{l=1}^{n} a_{jl} e_l$$

for $j = 1, \ldots, i$, where $a_{jl} \in \mathscr{O}(U)$. Then $D_i|_U$ is defined by all $i \times i$ minors of the $i \times n$ matrix (a_{jl}). Adapting the dimension of the corresponding determinantal variety, we conclude that the (expected) codimension of D_i in X is $n - i + 1$. We thus define the Chern class

$$c_{n-i+1}(E) := [D_i] \in A^{n-i+1}(X).$$

Remark 2.8. If the sections $s_1, \ldots, s_i$ are general, then D_i has the right codimension as expected. Moreover, different choices of general sections give rise to rationally equivalent degeneracy loci.

The definition by degeneracy loci satisfies the formal properties of Chern classes. For instance, if $E = L_1 \oplus L_2$ such that L_i is a very ample line bundle, take a general section s_i of L_i such that $(s_i) = S_i$ is an effective divisor. Then D_1 is the locus where (s_1, s_2) is zero, i.e.

$$\begin{aligned} c_2(E) &= [D_1] \\ &= [S_1] \cdot [S_2] \\ &= c_1(L_1) c_1(L_2). \end{aligned}$$

Moreover, take $(s_1, 0)$ and $(0, s_2)$ as two sections of E. Then D_2 is the locus where either s_1 or s_2 is zero, i.e.

$$\begin{aligned} c_1(E) &= [D_2] \\ &= [S_1] + [S_2] \\ &= c_1(L_1) + c_1(L_2). \end{aligned}$$

Hence we recover that $c(E) = c(L_1)c(L_2)$ as required by the formal properties of Chern classes.

Let us consider some applications along this circle of ideas.

Example 2.9 (Bézout's Theorem)**.** Consider the vector bundle $E = \mathscr{O}(a) \oplus \mathscr{O}(b)$ on $\mathbb{P}^2$ with $a, b > 0$. We know

$$c(E) = 1 + (a+b)l + abl^2,$$

where l is the line class of $\mathbb{P}^2$. Let F, G be two general homogenous polynomials in three variables of degree a and b, respectively. Then (F, G) is a general section of E. The degeneracy locus D_1 associated to (F, G) consists of points that are common zeros of F and G. In other words, let C and D be the corresponding plane curves defined by F and G, respectively. Then we conclude that

$$\begin{aligned} \#(C \cap D) &= \deg(c_2(E)) \\ &= ab. \end{aligned}$$

3. Grassmannians

Consider the Grassmanian $\mathbb{G}(k,n)$ parameterizing k-dimensional subspaces of $\mathbb{P}^n$, or equivalently $(k+1)$-dimensional subspaces of $\mathbb{C}^{n+1}$. Recall that $\dim \mathbb{G}(k,n) = (k+1)(n-k)$. There is an exact sequence on $\mathbb{G}(k,n)$:

$$0 \to S \to V \to Q \to 0,$$

where S is the tautological bundle of rank $k+1$ whose fiber over $[W] \in \mathbb{G}(k,n)$ represents the subspace W, V is a trivial vector bundle of rank $n+1$ and Q is the quotient bundle of rank $n-k$. The tangent bundle $T_{\mathbb{G}}$ is isomorphic to $S^* \otimes Q$, hence it fits in the Euler sequence

$$0 \to S^* \otimes S \to \bigoplus^{n+1} S^* \to T_{\mathbb{G}} \to 0.$$

In order to compute the Chern class of $T_{\mathbb{G}}$, we need to know the Chow ring of $\mathbb{G}(k,n)$. In general, the Chow ring of a Grassmanian is generated by its subvarieties parameterizing k-dimensional linear subspaces satisfying certain interpolation conditions. Below we will use the Grassmannian of lines in $\mathbb{P}^3$ to illustrate the idea.

3.1. **Chow ring of $\mathbb{G}(1,3)$.** Consider $\mathbb{G}(1,3)$ parameterizing lines in $\mathbb{P}^3$. It is a 4-dimensional variety. Below we will use Λ to denote lines in $\mathbb{P}^3$. Fix a *flag* $p \in L \subset H$, where p is a point, L is a line and H is a plane in $\mathbb{P}^3$. Define the *Schubert cycles* of $\mathbb{G}(1,3)$ as follows:

$$\Sigma = \mathbb{G}(1,3);$$

$$\Sigma_1 = \{\Lambda \mid \Lambda \cap L \neq \varnothing\};$$

$$\Sigma_2 = \{\Lambda \mid p \in \Lambda\};$$

$$\Sigma_{1,1} = \{\Lambda \mid \Lambda \subset H\};$$

$$\Sigma_{2,1} = \{\Lambda \mid p \in \Lambda \subset H\};$$

$$\Sigma_{2,2} = \{\Lambda \mid \Lambda = L\}.$$

It is easy to verify that $\Sigma_{a,b}$ has codimension $a+b$ in $\mathbb{G}(1,3)$. Moreover, $\Sigma_{a,b}$ is contained in $\Sigma_{c,d}$ if and only if $(a,b) \geq (c,d)$, i.e. $a \geq c$ and $b \geq d$. Then the $\Sigma_{a,b}$ provide a stratification of $\mathbb{G}(1,3)$. Let $\widetilde{\Sigma}_{a,b}$ be the complement in $\Sigma_{a,b}$ of all the other Schubert cycles properly contained in $\Sigma_{a,b}$.

Lemma 3.1. *The $\Sigma_{a,b}$ form an affine stratification of $\mathbb{G}(1,3)$.*

Proof. We need to prove that the $\widetilde{\Sigma}_{a,b}$ are affine. Let us do the hardest case

$$\widetilde{\Sigma}_1 = \{\Lambda \mid \Lambda \cap L \neq \varnothing;\ p \notin \Lambda;\ \Lambda \not\subset H\}.$$

Take a plane N such that $N \cap H = L'$ containing p. For $\Lambda \in \widetilde{\Sigma}_1$, let $q = \Lambda \cap L$. Since $N \cap L = p \neq q$ and $\Lambda \not\subset H$, we have $\Lambda \not\subset N$ and hence $\Lambda \cap N = r \notin L'$. Conversely, given $q \in L \backslash p$ and $r \in N \backslash L'$, the line Λ spanned by q and r is contained in $\widetilde{\Sigma}_1$. We thus obtain that

$$\widetilde{\Sigma}_1 \cong (L\backslash p) \times (N \backslash L') \cong \mathbb{A}^1 \times \mathbb{A}^2 \cong \mathbb{A}^3.$$

□

Let $\sigma_{a,b} = [\Sigma_{a,b}] \in A(\mathbb{G}(1,3))$. As a corollary, the $\sigma_{a,b}$ generate $A(\mathbb{G}(1,3))$. Next, we will use these Schubert cycles to determine the ring structure of $A(\mathbb{G}(1,3))$. In order to deal with intersection multiplicity, let us invoke a result for Grassmannians as homogeneous varieties. We assume that the ground field is of characteristic 0.

Proposition 3.2. *The algebraic group* $\mathrm{GL}(n+1)$ *acts transitively on* $\mathbb{G}(k,n)$*. Given two subvarieties* $A, B \subset \mathbb{G}(k,n)$*, there is an open subset* $U \subset \mathrm{GL}(n+1)$ *such that for* $g \in U$ *the subvariety* gA *is rationally equivalent to* A *and generically transverse to* B*.*

Theorem 3.3 (Chow ring of $\mathbb{G}(1,3)$)**.** *The Schubert cycles* $\sigma_{a,b} \in A^{a+b}(\mathbb{G}(1,3))$ *freely generate* $A(\mathbb{G}(1,3))$ *as a graded abelian group. Moreover, the multiplicative structure is given by*

$$\sigma_1^2 = \sigma_{1,1} + \sigma_2; \quad (A^1 \times A^1 \to A^2)$$

$$\sigma_1\sigma_{1,1} = \sigma_1\sigma_2 = \sigma_{2,1}; \quad (A^1 \times A^2 \to A^3)$$

$$\sigma_1\sigma_{2,1} = \sigma_{2,2}; \quad (A^1 \times A^3 \to A^4)$$

$$\sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2}; \quad \sigma_{1,1}\sigma_2 = 0. \quad (A^2 \times A^2 \to A^4)$$

As a result,

$$A(\mathbb{G}(1,3)) = \frac{\mathbb{Z}[\sigma_1, \sigma_2]}{(\sigma_1^3 - 2\sigma_1\sigma_2,\ \sigma_1^2\sigma_2 - \sigma_2^2)}.$$

Proof. We have already seen that the $\sigma_{a,b}$ generate $A(\mathbb{G}(1,3))$. Since $\deg(\sigma_{2,2}) = 1$, it follows that $A^4(\mathbb{G}(1,3)) \cong \mathbb{Z}$ is freely generated by $\sigma_{2,2}$. Assuming the intersection formulae, by $\sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2}$ and $\sigma_{1,1}\sigma_2 = 0$, we conclude that $\sigma_{1,1}$ and σ_2 freely generate $A^2(\mathbb{G}(1,3))$. Similarly by $\sigma_1\sigma_{2,1} = \sigma_{2,2}$, we conclude that σ_1 freely generates $A^1(\mathbb{G}(1,3))$ and $\sigma_{2,1}$ freely generates $A^3(\mathbb{G}(1,3))$.

Next, we will verify some of the intersection formulae and leave the others to the reader. Note that we can vary a defining flag of a Schubert cycle Σ to obtain another cycle Σ' such that $[\Sigma] = [\Sigma']$. For instance for σ_2^2, by generic transversality we have

$$\sigma_2^2 = \#(\Sigma_2 \cap \Sigma_2') \cdot \sigma_{2,2} = \sigma_{2,2}.$$

All the other formulae are obvious except σ_1^2. Let us use the method of test families. Suppose we have

$$\sigma_1^2 = a\sigma_2 + b\sigma_{1,1}$$

for undetermined coefficients $a, b \in \mathbb{Z}$. Multiply $\sigma_{1,1}$ to both sides and take degree. We obtain that $b = 1$. Multiply σ_2 to both sides and take degree. We obtain that $a = 1$. □

Let us calculate the Chern class of the tangent bundle of $\mathbb{G}(1,3)$. Recall the exact sequence

$$0 \to S \to V \to Q \to 0$$

over $\mathbb{G}(1,3)$. We know that $S^*|_{[W]} = \mathrm{Hom}(W, \mathbb{C})$, where $W \subset \mathbb{C}^4$ is a 2-dimensional linear subspace. Let us use the degeneracy locus definition to calculate the Chern class of S^*.

Take two general linear maps $\phi_i : \mathbb{C}^4 \to \mathbb{C}$ for $i = 1, 2$. They induce two general sections Φ_i of S^* by $\Phi_i([W])(w) = \phi_i(w)$ for $w \in W$, $i = 1, 2$. By definition, the vanishing locus of Φ_1 (or Φ_2) is $c_2(S^*)$. Note that Φ_1 is the zero map on W if and only if $W \subset \ker(\phi_1)$, i.e. if and only if the line $[W]$ is contained in the plane $[\ker(\phi_1)]$. We thus obtain that

$$c_2(S^*) = \sigma_{1,1}.$$

Similarly the locus where Φ_1 and Φ_2 fail to be linearly independent is $c_1(S^*)$. This happens if and only if $W \cap (\ker(\phi_1) \cap \ker(\phi_2)) \neq 0$, i.e. $[W]$ intersects the line $[\ker(\phi_1) \cap \ker(\phi_2)]$. We thus obtain that

$$c_1(S^*) = \sigma_1.$$

It follows that

$$c(S^*) = 1 + \sigma_1 + \sigma_{1,1},$$
$$c(S) = 1 - \sigma_1 + \sigma_{1,1},$$
$$c(Q) = \frac{1}{c(S)} = 1 + \sigma_1 + \sigma_2,$$

hence one can calculate $c(T_{\mathbb{G}}) = c(S^* \otimes Q)$ by the formal definition of Chern classes.

3.2. **Lines and surfaces in $\mathbb{P}^3$.** In this section we will treat a number of classical enumerative problems regarding lines and curves in $\mathbb{P}^3$.

Example 3.4. Let $L_1, \dots, L_4$ be four general lines in $\mathbb{P}^3$. Then the number of lines L that meet all the L_i is determined by

$$\sigma_1^4 = 2.$$

An alternative (heuristic) argument can be given by a specialization technique, by letting two of the L_i meet.

What happens if we replace the four lines by four curves in $\mathbb{P}^3$? First, let us determine the Schubert class of lines meeting a space curve of degree d.

Proposition 3.5. *Let C be a reduced curve of degree d in $\mathbb{P}^3$. Then the locus $\Gamma \subset \mathbb{G}(1,3)$ parameterizing lines meeting C has class $\gamma = d\sigma_1$.*

Proof. First, let us verify that Γ is a divisor in $\mathbb{G}(1,3)$. Consider the incidence correspondence

$$\widetilde{\Gamma} = \{(p, L) \mid p \in C,\ p \in L\} \subset C \times \mathbb{G}(1,3).$$

It admits two projections π_1 and π_2 to C and Γ, respectively. A fiber of p_1 is a Schubert cycle of type Σ_2, hence $\dim \widetilde{\Gamma} = 1 + 2 = 3$. On the other hand, p_2 is generically one-to-one, because the secant variety of C has dimension $\leq 2 < \dim \widetilde{\Gamma}$. We thus conclude that $\dim \Gamma = 3$.

Next, suppose $\gamma = a\sigma_1 \in A^1(\mathbb{G}(1,3))$ for undetermined $a \in \mathbb{Z}$. Intersect both sides by $\sigma_{2,1}$. Geometrically, $\Gamma \cap \Sigma_{2,1}$ consists of lines contained in a general plane H, containing a general point $p \in H$ and meeting C. Let $q_1, \dots, q_d$ be the d intersection points of C and H. Then there are d such lines spanned by p and each of the q_i. We thus obtain that $a = d$. □

Example 3.6. Let $C_1, \dots, C_4$ be general curves of degree $d_1, \dots, d_4$ in $\mathbb{P}^3$. Then the number of lines meeting all the C_i is

$$\deg\Big(\prod_{i=1}^4 \gamma_i\Big) = \Big(\prod_{i=1}^4 d_i\Big) \deg(\sigma_1^4) = 2\prod_{i=1}^4 d_i.$$

This number can also be obtained by a (heuristic) specialization argument, by taking each C_i as the union of d_i general lines.

Let C be a smooth and non-degenerate curve in $\mathbb{P}^3$ of degree d and g. Let $S \subset \mathbb{G}(1,3)$ be the *Secant variety* of C, defined as the closure of the locus of secant lines of C. Note that $\operatorname{Sym}^2 C$ admits a rational map to S, which is generically finite since C is not a line. We know that $\dim S = 2$. Let $s = [S] \in A^2(\mathbb{G}(1,3))$.

Proposition 3.7. *In the above setting, we have*

$$s = \left(\binom{d-1}{2} - g \right) \sigma_2 + \binom{d}{2} \sigma_{1,1}.$$

Proof. Let $s = a\sigma_2 + b\sigma_{1,1}$ with undetermined $a, b \in \mathbb{Z}$. Intersect both sides with σ_2. Geometrically, $S \cap \Sigma_2$ consists of secant lines of C containing a fixed point $p \in \mathbb{P}^3$. Project C to a plane curve C' from p. Such secant lines correspond to nodes of C', whose number is determined by the genus formula. We thus obtain that

$$a = \#(S \cap \Sigma_2) = \binom{d-1}{2} - g.$$

Next, intersect both sides with $\sigma_{1,1}$. Geometrically, $S \cap \Sigma_{1,1}$ consists of secant lines of C contained in a fixed plane $H \subset \mathbb{P}^3$. Since $H \cap C$ consists of d distinct points for a general H, we obtain that

$$b = \#(S \cap \Sigma_{1,1}) = \binom{d}{2}.$$

□

Example 3.8. Let C_1 and C_2 be two general twisted cubics in $\mathbb{P}^3$. Then the number of lines that are secant lines of both C_1 and C_2 is given by

$$\deg(\sigma_2 + 3\sigma_{1,1})^2 = 10.$$

Let Q be a smooth quadric surface in $\mathbb{P}^3$. We have seen that Q possesses two $\mathbb{P}^1$-families of lines in $\mathbb{P}^3$. Let $F \subset \mathbb{G}(1,3)$ be the locus of lines contained in Q. Then F is a curve. Let $f = [F] \in A^3(\mathbb{G}(1,3))$.

Proposition 3.9. *In the above setting, we have* $f = 4\sigma_{2,1}$.

Proof. Suppose $f = a\sigma_{2,1}$ with undetermined $a \in \mathbb{Z}$. Intersect both sides with σ_1. Geometrically, $F \cap \Sigma_1$ consists of lines in Q that intersect a fixed line $L \subset \mathbb{P}^3$. Note that $L \cap Q = \{p_1, p_2\}$. There are two lines in Q through each p_i. We thus obtain that $a = \#(F \cap \Sigma_1) = 4$. □

Remark 3.10. Indeed, F consists of two disjoint curves, each of which has class $2\sigma_{2,1}$, corresponding to the two ruling classes of Q.

Let S be a smooth surface of degree $d > 1$ in $\mathbb{P}^3$. Let $T \subset \mathbb{G}(1,3)$ be the locus of tangent lines of S. Consider the incidence correspondence

$$\widetilde{T} = \{(p, L) \mid p \in S,\ p \in L,\ L \subset T_p(S)\} \subset S \times T.$$

Let π_1 and π_2 be the two projections from $\widetilde{T}$ to S and T, respectively. A fiber of π_1 is a Schubert cycle of type $\Sigma_{2,1}$, hence $\dim \widetilde{T} = 2 + 1 = 3$. The map π_2 is generically finite. Otherwise every tangent line of S will be contained in S, contradicting that S is not a plane. It implies that $\dim T = 3$. Let $t = [T] \in A^1(\mathbb{G}(1,3))$.

Proposition 3.11. *In the above setting, we have* $t = d(d-1)\sigma_1$.

Proof. Suppose $t = a\sigma_1$ with undetermined $t \in \mathbb{Z}$. Intersect both sides with $\Sigma_{2,1}$. Geometrically, $T \cap \Sigma_{2,1}$ consists of tangent lines of S that are contained in a plane H and contain a point $p \in H$. Let $C = H \cap S$ be a plane curve of degree d. The number of tangent lines of C through p is the degree of the dual curve C^*, which is $d(d-1)$. We thus obtain that

$$a = \#(T \cap \Sigma_{2,1}) = d(d-1).$$

□

Example 3.12. Let $Q_1, \ldots, Q_4$ be four general quadric surfaces in $\mathbb{P}^3$. The number of lines tangent to all the Q_i is

$$\deg(2\sigma_1)^4 = 32.$$

Finally, let us study lines on a cubic surface in $\mathbb{P}^3$. Consider the incidence correspondence

$$\Sigma = \{(L, X) \mid L \subset X\} \subset \mathbb{G}(1,3) \times \mathbb{P}^{19},$$

where $\mathbb{P}^{19} = \mathbb{P}H^0(\mathscr{O}_{\mathbb{P}^3}(3))$ is the space of cubic surfaces. For a fixed line $L \subset \mathbb{P}^3$, $h^0(\mathscr{I}_L(3)) = 16$, so there is $\mathbb{P}^{15}$ subspace of cubics containing L. We conclude that $\dim \Sigma = 19$. The fiber of $\pi : \Sigma \to \mathbb{P}^{19}$ over a cubic surface X consists of lines contained in X. Therefore, we expect finitely many lines contained in a general cubic surface, and the number of these lines is given by the degree of π.

Let X be a smooth cubic surface in $\mathbb{P}^3$ defined by a polynomial F. Recall that S^* is the dual of the rank 2 tautological bundle on $\mathbb{G}(1,3)$. Note that $\operatorname{Sym}^3 S^*$ is a rank four vector bundle whose fiber over $[L]$ is $H^0(L, \mathscr{O}(3))$, the space of cubic forms on a line L in $\mathbb{P}^3$. The upshot is that F can be regarded as a section of $\operatorname{Sym}^3 S^*$, i.e. over $[L]$ it specifies the cubic form $F|_L \in H^0(L, \mathscr{O}(3))$. Moreover, L is contained in X if and only if $F|_L = 0$. In other words, the locus of lines lying on X as a cycle in $\mathbb{G}(1,3)$ has class $c_4(\operatorname{Sym}^3 S^*)$. We know that $c(S^*) = 1 + \sigma_1 + \sigma_{1,1}$. It follows that

$$c_4(\operatorname{Sym}^3 S^*) = 9\deg(\sigma_{1,1}(2\sigma_1^2 + \sigma_{1,1})) = 27\deg(\sigma_{1,1}^2) = 27.$$

Modulo the multiplicity check, it implies that a smooth cubic surface contains 27 lines.

3.3. **Schubert cycles.** We define Schubert cycles for a general Grassmannian $G(k, V)$, where V is an n-dimensional vector space. Choose a complete flat $\mathcal{V}$ in V, i.e. a sequence of subspaces

$$0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

with $\dim V_i = i$.

If $\Lambda \subset V$ is a general k-plane, then $V_j \cap \Lambda = 0$ for $j \leq n-k$ and $\dim V_{n-k+i} \cap \Lambda = i$ for $1 \leq i \leq k$. Let $a = (a_1, \ldots, a_k)$ be a sequence of integers with

$$n - k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0.$$

Define the *Schubert cycle* $\Sigma_a(\mathcal{V})$ to be the closed subset

$$\Sigma_a(\mathcal{V}) = \{\Lambda \in G(k, V) \mid \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i, \ 1 \leq i \leq k\}.$$

In other words, $\Sigma_a(\mathcal{V})$ consists of Λ such that $\dim(V_j \cap V) \geq i$ occurs for a value j that is a_i steps earlier than expected. Alternatively, consider the sequence

$$0 \subset (V_1 \cap \Lambda) \subset \cdots \subset (V_n \cap \Lambda) = \Lambda.$$

Two consecutive subspaces are either equal or their dimensions differ by one. Since Λ is of k-dimension, the dimension jump phenomenon occurs exactly k times. Then $\Sigma_a(\mathcal{V})$ consists of Λ for which the ith jump in the above sequence occurs at least a_i steps early.

Remark 3.13. In the notation Σ_a, we often drop those a_i that are equal to 0.

Example 3.14. The cycle $\Sigma_1 = \{\Lambda \mid \Lambda \cap V_{n-k} \neq 0\}$. In terms of $\mathbb{G}(1,3)$, it is the locus of lines meeting a fixed line. The cycle $\Sigma_{1,1} = \{\Lambda \mid \Lambda \cap V_{n-k} \neq 0,\ \dim(\Lambda \cap V_{n-k+1}) > 1\}$. In terms of $\mathbb{G}(1,3)$, it is the locus of lines contained in a fixed plane H and meeting a fixed line L, i.e. lines contained in H and passing through $p = L \cap H$.

It is easy to see that the codimension of $\Sigma_a(\mathcal{V})$ is $|a| = \sum_{i=1}^{k} a_i$ in $G(k,V)$. Moreover, if $(a_1,\ldots,a_k) \leq (b_1,\ldots,b_k)$, i.e. if $a_i \leq b_i$ for all i, then $\Sigma_b \subset \Sigma_a$. Let

$$\widetilde{\Sigma}_a = \Sigma_a \backslash (\cup_{\substack{b \geq a \\ b \neq a}} \Sigma_b).$$

Then $\widetilde{\Sigma}_a$ is affine, hence the Σ_a provide an affine stratification of $G(k,V)$. It follows that the Chow group of $G(k,V)$ is (freely) generated by the Schubert cycle classes.

Calculating (effectively) the intersection products between Schubert cycles is called *Schubert calculus*, which is a central topic in the study of the Grassmannians. Below we present one sample calculation.

Proposition 3.15. *Let $\mathcal{V}$ and $\mathcal{W}$ be two general flags in V. If $|a|+|b| = \dim G(k,V) = k(n-k)$, then $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ intersect transversely in a unique point if $a_i + b_{k+1-i} = n-k$ for all i, and are disjoint otherwise. As a result,* $\deg(\sigma_a\sigma_b) = 1$ *if $a_i + b_{k+1-i} = n-k$ for all i and* $\deg(\sigma_a\sigma_b) = 0$ *otherwise.*

Proof. By Proposition 3.2, we have

$$\begin{aligned} \deg(\sigma_a\sigma_b) &= \#(\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})) \\ &= \{\Lambda \mid \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i,\ \dim(W_{n-k+i-b_i} \cap \Lambda) \geq i,\ \forall i\} \end{aligned}$$

If $\dim(V_{n-k+i-a_i} \cap \Lambda) \geq i$ and $\dim(W_{n-k+(k+1-i)-b_{k+1-i}} \cap \Lambda) \geq k-i+1$, since $\dim \Lambda = k$, it follows that

$$V_{n-k+i-a_i} \cap W_{n-i+1-b_{k+1-i}} (\cap \Lambda) \neq 0.$$

Since $\mathcal{V}$ and $\mathcal{W}$ are general, we thus conclude that

$$(n-k+i-a_i) + (n-i+1-b_{k+1-i}) \geq n+1,$$

i.e.

$$a_i + b_{k+1-i} \leq n-k.$$

Since

$$\sum_{i=1}^{k} (a_i + b_{k+1-i}) = |a| + |b| = k(n-k),$$

we see that $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ will be disjoint unless $a_i + b_{k+1-i} = n-k$ for all i. In this case,

$$\Gamma_i = V_{n-k+i-a_i} \cap W_{n-i+1-b_{k+1-i}} = V_{n-k+i-a_i} \cap W_{k-i+1+a_i}$$

is 1-dimensional for all i. It follows that $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$ if and only if $\Gamma_i \subset \Lambda$ for each $i = 1,\ldots,k$. Thus Λ is the span of the Γ_i. $\square$

It thus makes sense to define $a^* = (n-k-a_k, \dots, n-k-a_1)$ as the *dual index* of $a = (a_1, \dots, a_k)$. The above proposition says that $\deg(\sigma_a \sigma_b) = 1$ if $b = a^*$ and 0 otherwise for all $|a| + |b| = \dim G$.

Corollary 3.16. *The Schubert cycle classes form a free basis for $A(G)$. In particular, they form dual bases for the degree map $A^m(G) \times A^{\dim G - m}(G) \to \mathbb{Z}$.*

Finally, we can determine the Chern classes of the tautological and quotient bundles on $G(k,V)$ in terms of Schubert cycle classes. Recall the exact sequence

$$0 \to S \to V \to Q \to 0,$$

where S is the rank k tautological bundle and Q is the rank $n-k$ quotient bundle.

Proposition 3.17. *The Chern classes of S and Q are given by*

$$c(S) = 1 - \sigma_1 + \sigma_{1,1} + \cdots + (-1)^k \sigma_{1,\dots,1},$$
$$c(Q) = 1 + \sigma_1 + \cdots + \sigma_{n-k}.$$

Proof. Let $\phi_1, \dots, \phi_k$ be k general linear maps from V to $\mathbb{C}$. Let $\Phi_i|_{[\Lambda]} = \phi_i|_\Lambda$ for $[\Lambda] \in G$. Then Φ_i is a section of S^*. The degeneracy locus D_m where $\Phi_1, \dots, \Phi_{k-m+1}$ fail to be linearly independent has class $c_m(S^*)$. Let $K = \ker(\phi_1) \cap \cdots \cap \ker(\phi_{k-m+1})$ and $\dim K = n-k+m-1$. Thus D_m consists of Λ such that Λ fails to intersect K properly, i.e.

$$\dim(\Lambda \cap K) \geq m = (\dim \Lambda + \dim K - n) + 1,$$

which implies that $\dim(\Lambda \cap W) \geq (\dim \Lambda + \dim W - n) + 1$ for any subspace W containing K. Therefore, we have $c_m(S^*) = [D_m] = \sigma_{1^m}$ and

$$c(S^*) = 1 + \sigma_1 + \sigma_{1,1} + \cdots + \sigma_{1,\dots,1}.$$

Next, take $n-k$ general vectors $v_1, \dots, v_{n-k} \in V$. Then $v_i + \Lambda \in V/\Lambda$ gives a global section ν_i of Q for each i. The degeneracy locus E_m where $\nu_1, \dots, \nu_{n-k-m+1}$ fail to be linearly independent has class $c_m(Q)$, i.e. E_m consists of Λ such that $\Lambda \cap W \neq 0$, where W is the span of $\nu_1, \dots, \nu_{n-k-m+1}$ with $\dim W = n-k-m+1$. Thus $c_m(Q) = [E_m] = \sigma_m$ and

$$c(Q) = 1 + \sigma_1 + \cdots + \sigma_{n-k}.$$

□

Remark 3.18. By $c(S)c(Q) = c(V) = 1$, we get interesting relations between the Schubert cycles σ_m and σ_{1^l}. Moreover, since $T_{\mathbb{G}} = S^* \otimes Q$, we obtain that $c_1(T_{\mathbb{G}}) = n\sigma_1$ and $K_{\mathbb{G}} = -n\sigma_1$.

4. Fano scheme

In this section we study k-planes contained in a degree d hypersurface in $\mathbb{P}^n$. First, we define the *universal Fano scheme* as

$$\Phi(n,d,k) = \{(X, L) \mid X \text{ is a hypersurface of degree } d \text{ in } \mathbb{P}^n \text{ and } L \subset X \text{ is a } k\text{-plane}\}.$$

The scheme structure of Φ can be given as a subscheme in $\mathbb{G}(k,n) \times \mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$, as follows. Let

$$\left(s_0, s_1, \dots, s_k, s_{k+1} = \sum_{i=0}^{k} a_{i,k+1} s_i, \dots, s_n = \sum_{i=0}^{k} a_{i,n+1} s_i \right)$$

parameterize k-planes around $[\Lambda] \in \mathbb{G}(k,n)$ with the $a_{i,j}$ as local coordinates of $\mathbb{G}(k,n)$. Let F be a hypersurface defined by a homogeneous degree d polynomial

$$f = \sum_{|\delta|=d} z_\delta x^\delta,$$

where the x_i are coordinates of $\mathbb{P}^n$ and z_δ are coordinates of $\mathbb{P}^N$. Replacing x_i by s_i, i.e. restricting f to the corresponding k-plane, we obtain a form $f(s_0, \ldots, s_k)$ of degree d, whose coefficients are bi-homogeneous polynomials in the $a_{i,j}$ and in the z_δ of bi-degree $(d, 1)$, respectively. Note that $\Lambda \subset F$ if and only if $f(s_0, \ldots, s_k)$ is the zero form. Thus we define $\Phi \subset \mathbb{G}(k,n) \times \mathbb{P}^N$ locally as the subscheme cut out by those bi-homogeneous polynomials.

For a given hypersurface X of degree d, define the *Fano scheme* $F_k(X) \subset \mathbb{G}(k,n)$ as the fiber of Φ over $[X] \in \mathbb{P}^N$. Set-theoretically $F_k(X)$ parameterizes k-planes contained in X. If Y is an arbitrary subvariety in $\mathbb{P}^n$, we can define $F_k(Y)$ as the intersection of $F_k(X)$, where the X are hypersurfaces whose defining equations generate the ideal of Y.

4.1. **Dimension of the Fano scheme.**

Proposition 4.1. *Let $N = \binom{n+d}{n} - 1$. The universal Fano scheme $\Phi(n,d,k)$ is a smooth irreducible variety of dimension*

$$\dim \Phi(n,d,k) = N + (k+1)(n-k) - \binom{k+d}{k}.$$

Proof. The fiber of Φ over $[\Lambda] \in \mathbb{G}(k,n)$ is isomorphic to $\mathbb{P}H^0(\mathcal{I}_\Lambda(d)) = \mathbb{P}^{N-\binom{k+d}{k}}$. □

Corollary 4.2. *Let X be a hypersurface of degree d in $\mathbb{P}^n$.*

(1) The dimension of any (nonempty) component of $F_k(X)$ is at least

$$\phi(n,d,k) = (k+1)(n-k) - \binom{k+d}{k}.$$

(2) If $\phi(n,d,k) < 0$, a general X contains no k-planes.

(3) If $\phi(n,d,k) \geq 0$ and if $F_k(X)$ is nonempty for a general X, then $F_k(X)$ is nonempty for every X, and every component of $F_k(X)$ for a general X has dimension exactly $\phi(n,d,k)$.

Proof. For (1), the fiber of Φ over $[X] \in \mathbb{P}^N$ is $F_k(X)$, which is cut out by N equations.

For (2), it follows from $\dim \Phi = \dim \mathbb{P}^N + \dim F_k(X)$ for a general X.

For (3), if $F_k(X)$ is nonempty for a general X, then $\Phi \to \mathbb{P}^N$ is dominant, and hence surjective. Then the claim follows from (1) combining the formula used in (2). □

Remark 4.3. When $\phi(n,d,k) \geq 0$, the above result does not rule out the possibility that a general hypersurface of degree d in $\mathbb{P}^n$ contains no k-planes. However, later on we will show that except in some cases where $k > 1$ and $d = 2$, a general hypersurface of degree d in $\mathbb{P}^n$ does contain k-planes.

Example 4.4. Consider $n = 3$, $d = 2$ and $k = 1$, i.e. lines on quadric surfaces Q in $\mathbb{P}^3$. In this case $\phi(3,2,1) = 1$. We know that $\dim F_1(Q) = 1$ when Q is a smooth quadric or even a quadric cone. On the other hand, if Q is a double plane, then $\dim F_1(Q) = 2$.

Example 4.5. Consider $n = 3$, $d \geq 4$ and $k = 1$, i.e. lines on degree d surfaces in $\mathbb{P}^3$. Since $\phi(3, d, 1) = 3 - d < 0$, a general degree d surface in $\mathbb{P}^3$ contains no lines. Since $N - \dim \Phi = d - 3$, the locus of degree d surfaces containing a line in $\mathbb{P}^N$ has codimension at least $d - 3$. Later on we will show that a general degree d surface containing a line contains only one. Thus this locus has codimension exactly $d - 3$ in $\mathbb{P}^N$.

Of course for a special hypersurface X, $\dim F_k(X)$ may be strictly bigger than the expected dimension $\phi(n, d, k)$. But we can give an upper bound on its possible dimension.

Proposition 4.6. *If $X \subset \mathbb{P}^n$ is an m-dimensional subvariety, then*

$$\dim F_k(X) \leq (k + 1)(m - k) = \dim \mathbb{G}(k, m).$$

The equality holds if and only if X is an m-plane.

Proof. Without loss of generality, assume that X is non-degenerate. Thus $n + 1$ general points in X are linearly independent. Let $U \subset X^{k+1}$ be the open subset parameterizing $(k+1)$-tuples of linearly independent points. Consider the incidence correspondence

$$\Gamma = \{(p_0, \ldots, p_k, \Lambda) \mid p_i \in \Lambda \text{ for all } i\} \subset U \times F_k(X).$$

The fiber of Γ over $(p_0, \ldots, p_k) \in U$ is a single point corresponding to the k-plane spanned by the p_i. Thus $\dim \Gamma \leq \dim U = m(k+1)$. The projection $\Gamma \to F_k(X)$ is surjective with fibers all isomorphic to $(\mathbb{P}^k)^{k+1} \backslash \Delta$, which has dimension $k(k + 1)$. We conclude that $\dim F_k(X) \leq m(k + 1) - k(k + 1) = (k + 1)(m - k)$.

The equality holds if and only if $\Gamma \to U$ is dominant, i.e. every $(k + 1)$-tuple of points in X spans a linear subspace contained in X. I claim that X is also a linear subspace in $\mathbb{P}^n$. If $\deg X = d$, let $Y = X \cap H$ for a general hyperplane H and $\deg Y = d$. Then every $(k + 1)$-tuple of points in Y spans a linear subspace contained in Y. By induction on $\dim X$, it follows that $d = 1$. □

4.2. **Chern classes and the Fano scheme.** Let X be a hypersurface of degree d in $\mathbb{P}^n$. Set $m = \binom{k+d}{k}$.

Proposition 4.7. *When $F_k(X)$ has the expected dimension $\phi(n, d, k) = (k+1)(n-k) - m$, its cycle class equals*

$$[F_k(X)] = c_m(\mathrm{Sym}^d(S^*)),$$

where S^* is the dual of the tautological bundle S on $\mathbb{G}(k, n)$.

Proof. Note that $S^*|_\Lambda = \mathrm{Hom}(\Lambda, V/\Lambda)$. Thus $\mathrm{Sym}^d(S^*)|_\Lambda$ is the space of degree d homogenous polynomials on $\Lambda \cong \mathbb{P}^k$. Let G be the defining equation of X. Then $G|_\Lambda$ is a d homogenous polynomial on Λ. It implies that G gives rise to a section of $\mathrm{Sym}^d(S^*)$. The locus when the section is zero consists of Λ such that $\Lambda \subset X$. Therefore, we have

$$[F_k(X)] = c_m(\mathrm{Sym}^d(S^*)),$$

where $m = \binom{k+d}{k}$ is the rank of $\mathrm{Sym}^d(S^*)$. □

Example 4.8. Consider lines on a quintic threefold in $\mathbb{P}^4$. We have $m = 6 = \dim \mathbb{G}(1,4)$. Thus we expect finitely many lines contained in a general quintic threefold. Using the Chern class $c(S^*) = 1 + \sigma_1 + \sigma_{1,1} \in A(\mathbb{G}(1,4))$, this expected number is

$$\deg c_6(\mathrm{Sym}^5(S^*)) = 2875.$$

4.3. **Tangent space of the Fano scheme.** Recall that the Hilbert scheme is the universal flat family of subschemes that have the same Hilbert polynomials. If Y is a subscheme of X, the Hilbert scheme of X has tangent space at $[Y]$ given by $H^0(N_{Y/X})$, where $N_{Y/X}$ is the normal sheaf of Y in X. If X is smooth and Y is a local complete intersection, then $N_{Y/X}$ is a vector bundle of rank equal to the codimension of Y in X.

The Fano scheme $F_k(X)$ is a special case of the Hilbert scheme parameterizing k-planes in X. Thus we have to following corollary.

Proposition 4.9. *Let $\Lambda \subset X$ be a k-plane in a smooth subvariety $X \subset \mathbb{P}^n$. Then the tangent space of $F_k(X)$ at $[\Lambda]$ is isomorphic to $H^0(N_{\Lambda/X})$. The dimension of $F_k(X)$ at $[\Lambda]$ is at most $h^0(N_{\Lambda/X})$. In particular, $F_k(X)$ is smooth at $[\Lambda]$ if and only if equality holds.*

Let us list some properties of normal bundle (sheaf) that we will use later. In general, N_Y/X is defined as a sheaf by

$$\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{I}_Y/\mathscr{I}_Y^2, \mathscr{O}_Y).$$

If $Y \subset X \subset W$ are three smooth schemes, then we have an exact sequence

$$0 \to N_{Y/X} \to N_{Y/W} \to N_{X/W}|_Y \to 0.$$

If Y is a (Cartier) divisor on X, then

$$N_{Y/X} = \mathscr{O}_X(Y).$$

More generally, if Y has codimension e in X and if Y is the zero locus of a section of a vector bundle E of rank e on X, then

$$N_{Y/X} = E|_Y.$$

In particular, if Y is a complete intersection of X with hypersurfaces on $\mathbb{P}^n$ of degrees d_i, then

$$N_{Y/X} = \bigoplus_i \mathscr{O}_X(d_i).$$

Corollary 4.10. *Let X be a smooth surface of degree $d \geq 3$ in $\mathbb{P}^3$. If $F_1(X) \neq \varnothing$, then $F_1(X)$ is smooth and 0-dimensional. In particular, X contains at most finitely many lines, and for $d = 3$, X contains exactly 27 distinct lines.*

Proof. Let $L \subset X$ be a line. Since $K_X = (d-4)H|_S$, where H is the hyperplane class of $\mathbb{P}^3$, by adjunction we have

$$-2 = L(L + K_X) = L^2 + d - 4,$$

$$L^2 = 2 - d < 0.$$

Therefore, $h^0(N_{L/X}) = h^0(\mathscr{O}_L(2-d)) = 0$. Hence $F_1(X)$ is reduced of dimension at most 0 at $[L]$. ∎

Let us study in detail the normal bundle of a k-plane Λ in a degree d hypersurface $X \subset \mathbb{P}^n$. Suppose $I_\Lambda = (x_{k+1}, \ldots, x_n)$, and $I_X = (g) \subset I_\Lambda$. We can write g uniquely as

$$g = \sum_{i=k+1}^{n} x_i g_i(x_0, \ldots, x_k) + h,$$

where $h \in I_\Lambda^2$. The derivatives $\partial g/\partial x_i = g_i$ restricted to Λ for $i = k+1, \ldots, n$ are forms of degree $d-1$. We have the left exact sequence

$$0 \to N_{\Lambda/X} \to N_{\Lambda/\mathbb{P}^n} \xrightarrow{\alpha} N_{X/\mathbb{P}^n}|_\Lambda,$$

i.e.

$$0 \to N_{\Lambda/X} \to \mathscr{O}_\Lambda^{n-k}(1) \xrightarrow{\alpha} \mathscr{O}_\Lambda(d),$$

which is also right exact if X is smooth. Using the interpretation $N_{Z/W} = (T_W|_Z)/T_Z$ for $Z \subset W$, the sheaf map α is given by

$$\alpha = (g_{k+1}, \ldots, g_n) : \mathscr{O}_\Lambda^{n-k}(1) \to \mathscr{O}_\Lambda(d).$$

Proposition 4.11. *With the above notation,*

(1) X is smooth along Λ if and only if α is a surjective sheaf map.

(2) $[\Lambda]$ is a smooth point of $F_k(X)$ and $\dim F_k(X)|_\Lambda = \phi(n,d,k) = (k+1)(n-k) - \binom{k+d}{k}$ if and only if α is surjective on global sections.

(3) $[\Lambda]$ is an isolated reduced point of $F_k(X)$ if and only if α is injective on global sections.

Proof. For (1), since $\Lambda \subset X$, all the derivatives $\partial g/\partial x_i$ are zero along Λ for $i = 0, \ldots, k$. Thus X is smooth at $p \in \Lambda$ if and only if at least one of the normal derivatives $\partial g/\partial x_i$ for $i = k+1, \ldots, n$ is nonzero at p, i.e. α is a surjective sheaf map.

For (2), we have seen that $\phi(n,d,k) = h^0(\mathscr{O}_\Lambda^{n-k}(1)) - h^0(\mathscr{O}_\Lambda(d))$ is a lower bound for the local dimension of $F_k(X)$. Since $H^0(N_{\Lambda/X}) = T_{[\Lambda]}F_k(X)$, the claim follows right away.

For (3), $[\Lambda]$ is an isolated reduced point of $F_k(X)$ if and only if $h^0(N_{\Lambda/X}) = 0$, i.e. if and only if α is injective on global sections. □

Remark 4.12. We can do a little better in (1). By Bertini, a general member of a linear system on $\mathbb{P}^n$ is smooth outside the base locus. It follows that a general X containing Λ with a given induced surjective map α (i.e. only h varies in g) is not only smooth along Λ but also smooth everywhere.

Corollary 4.13. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d > 1$. If $\Lambda \subset X$ is a k-plane on X and if X is smooth along Λ, then*

$$k \leq \frac{n-1}{2}.$$

Proof. If $k > (n-1)/2$, then $\dim \Lambda = k \geq n-k$, so the $n-k$ forms $g_{k+1}, \ldots, g_n$ of positive degree must have a common zero $p \in \Lambda$, contradicting that X is smooth at p. □

As mentioned in Remark 4.3, the following result says that except for some cases in $k > 1$ and $d = 2$, $F_k(X)$ has the expected dimension for a general hypersurface X of degree d.

Theorem 4.14. *Let* $\phi(n,d,k) = (k+1)(n-k) - \binom{d+k}{k}$.

(1) In the case $k = 1$ *or* $d \geq 3$*, if* $\phi \geq 0$*, then every hypersurface of degree* d *contains* k*-planes, and a general hypersurface* $X \subset \mathbb{P}^n$ *of degree* d *has* $\dim F_k(X) = \phi$.

(2) If $\phi \leq 0$ *and* X *is a general hypersurface of degree* d *containing a given* k*-plane* Λ*, then* $[\Lambda] \in F_k(X)$ *is an isolated reduced point.*

Sketch of proof. For (1), the first part follows from the second part. By Proposition 4.11, the second part follows if α is surjective on global sections, i.e. we need to show that under the given assumption, a general $(n-k)$-dimensional vector space of forms of degree $d-1$ generates an ideal containing all degree d forms on Λ. For (2), we have to show that a general $(n-k)$-dimensional vector space of forms of degree $d-1$ generates an ideal on Λ without relations in degree 1 (no linear syzygies).

Combining the two statements, it says that if $g_{k+1}, \ldots, g_n$ are $n-k$ general forms of degree $d-1$ in $k+1$ variables, then the degree d component of the ideal $(g_{k+1}, \ldots, g_n)$ has dimension equal to $\min\{(k+1)(n-k), \binom{k+d}{k}\}$, which is a special case of Fröberg's conjecture. □

4.4. **Fano scheme of lines.** The case of lines is particularly interesting, because every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles, proved by Grothendieck.

Let $L \subset X$ be a line in a hypersurface $X \subset \mathbb{P}^n$ of degree d. Choose coordinates so that $I_L = (x_2, \ldots, x_n)$. Write the equation of X in the form

$$g = \sum_{i=2}^{n} x_i g_i(x_0, x_1) + h,$$

where $h \in I_L^2$. Let α be the map

$$\alpha = (g_2, \ldots, g_n) : \mathscr{O}_L^{n-1}(1) \to \mathscr{O}_L(d).$$

The expected dimension of $F_1(X)$ is

$$\phi = 2n - 3 - d.$$

Suppose $N_{L/X} = \oplus_{i=1}^{n-2} \mathscr{O}_{\mathbb{P}^1}(e_i)$ with $e_i \in \mathbb{Z}$.

Proposition 4.15. *Suppose that* $n \geq 3$ *and* $d \geq 1$*. There exists a smooth degree* d *hypersurface* $X \subset \mathbb{P}^n$ *and a line* $L \subset X$ *such that* $N_{L/X} = \oplus_{i=1}^{n-2} \mathscr{O}_{\mathbb{P}^1}(e_i)$ *if and only if* $e_i \leq 1$ *for all* i *and* $\sum_{i=1}^{n-2} e_i = n - 1 - d$.

Proof. If $N_{L/X} = \oplus_{i=1}^{n-1} \mathscr{O}_{\mathbb{P}^1}(e_i)$, then by the inclusion $N_{L/X} \to N_{L/\mathbb{P}^n} = \mathscr{O}_{\mathbb{P}^1}^{n-1}(1)$, we thus conclude that $e_i \leq 1$ for all i, since there is no nontrivial map $\mathscr{O}_{\mathbb{P}^1}(m) \to \mathscr{O}_{\mathbb{P}^1}(l)$ for $m > l$. Moreover, by the exact sequence

$$0 \to N_{L/X} \xrightarrow{\beta} \mathscr{O}_{\mathbb{P}^1}^{n-1}(1) \xrightarrow{\alpha} \mathscr{O}_{\mathbb{P}^1}(d) \to 0,$$

we thus have

$$\sum_{i=1}^{n-2} e_i = \deg c_1(N_{L/X}) = n - 1 - d.$$

Conversely, let β be an $(n-1) \times (n-2)$ matrix, whose two main diagonals have entries $x_0^{1-e^1}, \ldots, x_0^{1-e^{n-2}}$ and $x_1^{1-e_1}, \ldots, x_1^{1-e_{n-2}}$, respectively. Then α can be given by the $(n-2)\times(n-2)$ minors of β. The top and the bottom $(n-2)\times(n-2)$ minors of β are x_0^{d-1} and x_1^{d-1}, which do not have a common zero on L, hence α

is a surjective sheaf map. By Remark 4.12, a general degree d hypersurface X containing L with the induced surjective sheaf map α is smooth. □

Corollary 4.16. *Suppose that* $\dim \phi(n,d,1) \geq 0$, *i.e.* $d \leq 2n-3$.

(1) There exists a smooth degree d hypersurface $X \subset \mathbb{P}^n$ containing a line L such that $F_1(X)$ is smooth of dimension $2n-3-d$ at $[L]$.

(2) Every hypersurface of degree d in $\mathbb{P}^n$ contains a line.

(3) If $d \leq 3$, then $F_1(X)$ is smooth of dimension $2n-3-d$ for any smooth hypersurface X of degree d.

(4) If $n \geq 4$ and $d \geq 4$, then there exists a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d such that $F_1(X)$ is either singular or of dimension $> 2n-3-d$.

Proof. If $d \leq 2n-3$, we can find $e_1, \dots, e_{n-2}$ such that $-1 \leq e_i \leq 1$ and $\sum_i e_i = n-1-d$. By Proposition 4.15, there exists a smooth degree d hypersurface X containing L. Moreover, with this choice of the e_i we have $h^0(\mathscr{O}_{\mathbb{P}^1}(e_i)) = e_i + 1$ and hence $h^0(N_{L/X}) = (\sum_i e_i) + (n-2) = 2n-3-d = \phi \geq 0$. Since $\dim F_1(X) \geq 2n-3-d$ everywhere, the claim (1) follows.

For (2), the universal Fano scheme $\Phi(n,d,1)$ is irreducible and has dimension $N+\phi$. By (1), there exists $(X,L) \in \Phi$ such that the fiber of Φ over $[X] \in \mathbb{P}^N$ has dimension equal to ϕ. It follows that the projection $\Phi \to \mathbb{P}^N$ is surjective.

For (3), since $e_i \leq 1$ and $\sum_{i=1}^{n-2} e_i = n-1-d \geq n-4$, we have $e_i \geq -1$ for all i, for otherwise their sum would be at most $-2+(n-3) < n-4$. Then $h^0(\mathscr{O}_{\mathbb{P}^1}(e_i)) = e_i+1$ and hence $h^0(N_{L/X}) = (\sum_i e_i) + (n-2) = 2n-3-d = \phi$ for any $L \subset X$.

For (4), if $n \geq 4$ and $d \geq 4$, then we may choose $e_1 = \cdots = e_{n-3} = 1$ and $e_{n-2} = 2-d \leq -2$. Then $h^0(N_{L/X}) = 2(n-3) > 2n-3-d$, which implies that either $\dim F_1(X) > 2n-3-d$ at $[L]$ or $F_1(X)$ is singular at $[L]$. □

For $d, k \ll n$, a more general result says that $F_k(X)$ has the expected dimension for *every* smooth hypersurface X of degree d in $\mathbb{P}^n$. For $k=1$, we can make a precise (conjectural) upper bound for d in terms of n.

Conjecture 4.17 (The Debarre-de Jong Conjecture)**.** *If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d \leq n$, then* $\dim F_1(X) = 2n-3-d$.

Remark 4.18. The conjecture does not hold if $d > n$. For example, let X be a general hypersurface of degree $d > 2m+1$ in $\mathbb{P}^{n=2m+1}$ that contains an m-plane Λ. One can show that X is smooth, by showing that X is smooth along Λ plus Bertini. On the other hand, $\dim F_1(X) \geq \dim \mathbb{G}(1,m) = 2m-2 > 2n-3-d$. The condition $d \leq n$ is equivalent to the fact that X is a Fano variety, i.e. the anti-canonical bundle $\omega_X^* = \mathscr{O}_X(n+1-d)$ is ample. The conjecture has been proved for $d \leq 6$.

5. Singular hypersurfaces

Let F and G be two general forms of degree d on $\mathbb{P}^2$. In the pencil (1-dimensional linear series) $|sF+tG|$ where $[s,t] = \mathbb{P}^1$, how many members correspond to singular degree d plane curves? In this section we will develop some tools to answer questions of this type.

5.1. **Universal singular point.** Let $\mathbb{P}^N$ be the parameter space of degree d hypersurfaces in $\mathbb{P}^n$, where $N = \binom{n+d}{d} - 1$. Define the *universal singular point* as

$$\Sigma = \{(Y, p) \mid p \in Y_{\text{sing}}\} \subset \mathbb{P}^N \times \mathbb{P}^n.$$

If we write a degree d form on $\mathbb{P}^n$ as $F = \sum_{|I|=d} a_I x^I$, where the a_I are the coordinates of $\mathbb{P}^N$ and $x_0, \ldots, x_n$ are coordinates of $\mathbb{P}^n$, then Σ is cut out by

$$F(x) = 0, \quad \frac{\partial F}{\partial x_i} = 0 \text{ for } i = 0, \ldots, n.$$

Since $dF(x) = \sum_{i=0}^n x_i \frac{\partial F}{\partial x_i}$, the last $n+1$ equations determine the first one. Thus Σ is a complete intersection of $n+1$ hypersurfaces of bidegree $(1, d-1)$ in $\mathbb{P}^N \times \mathbb{P}^n$.

Denote by $\mathcal{D}$ the image of Σ in $\mathbb{P}^N$, called the *discriminant*, parameterizing singular hypersurfaces of degree d in $\mathbb{P}^n$.

Proposition 5.1. *Suppose $d \geq 2$. Then Σ is smooth, irreducible of dimension $N-1$. Moreover, a general singular hypersurface of degree d has a unique singularity. In particular, $\Sigma \to \mathcal{D}$ is birational and $\mathcal{D}$ is an irreducible hypersurface in $\mathbb{P}^N$.*

Proof. Set $p = [1, 0, \ldots, 0] \in \mathbb{P}^n$ and let $x_1, \ldots, x_n$ be local coordinates at p. The fiber Σ_p of Σ over $p \in \mathbb{P}^n$ consists of hypersurfaces whose defining equations belong to $(x_1, \ldots, x_n)^2$, i.e., the coefficients of $x_0^n, x_0^{n-1}x_1, \ldots, x_0^{n-1}x_n$ are all 0. Therefore, $\Sigma_p \cong \mathbb{P}^{N-n-1}$, hence Σ is smooth, irreducible of dimension $N - n - 1 + n = N - 1$.

By Bertini's theorem, a general member in the linear series of hypersurfaces singular at p is smooth away from p, hence a general singular hypersurface possesses only one singularity. Thus $\Sigma \to \mathcal{D}$ is birational and $\dim \mathcal{D} = \dim \Sigma = N - 1$. □

Remark 5.2. A hypersurface Y singular at $p = [1, 0, \ldots, 0]$ has equation

$$f_2 + \cdots + f_n,$$

where f_i is a form of degree i in $x_1, \ldots, x_n$. If f_2 is of full rank as a quadratic form, the singularity type of Y at p is called an *ordinary double point*. It looks like the vertex of the quadric cone defined by f_2, which is a cone over a smooth quadric hypersurface in $\mathbb{P}^{n-1}$.

5.2. **Bundles of principal parts.** Let L be a vector bundle on a smooth variety X. We want to define a bundle $\mathcal{P}^m(L)$ on X such that its fiber over p is canonically identified with the vector space

$$V_p^m(L) = \frac{\{\text{germs of sections of } L \text{ at } p\}}{\{\text{germs of sections of } L \text{ with vanishing order} \geq m+1 \text{ at } p\}},$$

where we view a section locally as a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}^k$, where $n = \dim X$ and $k = \operatorname{rank} L$. Two sections define the same element in $V_p^m(L)$ if locally they are the same modulo terms of vanishing order $\geq m+1$ at p.

In general, $\mathcal{P}^m(L)$ can be defined as follows. Consider $X \times X$ and let π_i be the projections to X, $i = 1, 2$. Let $\Delta \subset X \times X$ be the diagonal. Set

$$\mathcal{P}^m(L) = \pi_{2*}\left(\pi_1^* L \otimes \mathscr{O}_{X \times X} / \mathscr{I}_\Delta^{m+1}\right).$$

Theorem 5.3. *In the above setting,*

(a) $\mathcal{P}^m(L)$ is a vector bundle and the fiber of $\mathcal{P}^m(L)$ at p is canonically identified with

$$V_p^m(L) = H^0(L \otimes \mathscr{O}_{X,p} / \mathfrak{m}_{X,p}^{m+1}).$$

(b) $\mathcal{P}^0(L) = L$ *and for* $m \geq 1$ *there is an exact sequence*

$$0 \to L \otimes \mathrm{Sym}^m(\Omega_X) \to \mathcal{P}^m(L) \to \mathcal{P}^{m-1}(L) \to 0,$$

where Ω_X *is the cotangent bundle of* X*, i.e. the sheaf of differentials on* X*.*

Sketch of proof. Part (a) can be reduced to a local verification on an affine open subset of X. For (b), note that $\Delta \cong X$, hence $\mathcal{P}^0(L) = L$ follows from definition. In addition, $N_{\Delta/X\times X} = T_\Delta$, hence $\Omega_\Delta = \mathscr{I}_\Delta/\mathscr{I}_\Delta^2$. Finally, the exact sequence follows from

$$\begin{aligned} V_p^m(L)/V_p^{m-1}(L) &= L|_p \otimes (\mathfrak{m}_{X,p}^m/\mathfrak{m}_{X,p}^{m+1}) \\ &= L|_p \otimes \mathrm{Sym}^m(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2) \\ &= L|_p \otimes (\mathrm{Sym}^m \Omega_X)|_p. \end{aligned}$$

□

The above exact sequence can help us calculate the Chern class of $\mathcal{P}^m(L)$.

Example 5.4. Let $L = \mathscr{O}_{\mathbb{P}^n}(d)$. Then we have $\mathcal{P}^0(L) = L = \mathscr{O}_{\mathbb{P}^n}(d)$. Moreover, we have

$$0 \to \Omega_{\mathbb{P}^n}(d) \to \mathcal{P}^1(L) \to \mathscr{O}_{\mathbb{P}^n}(d) \to 0.$$

Thus we obtain that

$$c(\mathcal{P}^1(L)) = (1 + dh)c(\Omega_{\mathbb{P}^n}(d)),$$

where h is the hyperplane class of $\mathbb{P}^n$. By the twisted Euler sequence

$$0 \to \Omega_{\mathbb{P}^n}(d) \to \mathscr{O}_{\mathbb{P}^n}(d-1)^{\oplus(n+1)} \to \mathscr{O}_{\mathbb{P}^n}(d) \to 0,$$

we have

$$c(\Omega_{\mathbb{P}^n}(d)) = \frac{(1 + (d-1)h)^{n+1}}{1 + dh}.$$

It follows that

$$c(\mathcal{P}^1(L)) = (1 + (d-1)h)^{n+1}.$$

In particular,

$$\deg c_n(\mathcal{P}^1(L)) = (n+1)(d-1)^n.$$

In general, one can show that

$$c(\mathcal{P}^m(L)) = (1 + (d-m)h)^{\binom{n+m}{m}}.$$

Given a global section F of L, the germ of F at p gives rise to a section τ_F of $\mathcal{P}^m(L)$. Therefore, F has vanishing order $\geq m+1$ at p if and only if τ_F is vanishing at p.

Consider the following application. Let F and G be two general forms of degree d on $\mathbb{P}^n$. They span a pencil of hypersurfaces $C_t = V(t_0F + t_1G)$ for $t = [t_0, t_1] \in \mathbb{P}^1$. The number of singular C_t in the pencil is given by the number of points p where τ_F and τ_G are linearly dependent in the fiber of $\mathcal{P}^1(\mathscr{O}_{\mathbb{P}^n}(d))$ over p. Since $\mathrm{rank}\, \mathcal{P}^1(\mathscr{O}_{\mathbb{P}^n}(d)) = n+1$, this number equals

$$\deg c_n(\mathcal{P}^1(\mathscr{O}_{\mathbb{P}^n}(d)) = (n+1)(d-1)^n.$$

Rephrasing differently, we have

Proposition 5.5. *The discriminant hypersurface* $\mathcal{D} \subset \mathbb{P}^N$ *parameterizing singular degree* d *hypersurfaces in* $\mathbb{P}^n$ *has degree* $(n+1)(d-1)^n$*.*

Example 5.6. A general pencil of plane degree d curves contains $3(d-1)^2$ singular members. We may also solve this number by a topological Riemann-Hurwitz formula.

Alternatively, $V(F)$ is singular at p if and only if $\partial F/\partial X_i = 0$ at p for $i = 0, \dots, n$. Consider the $n+1$ hypersurfaces Z_i in $\mathbb{P}^1 \times \mathbb{P}^n$ cut out by

$$H_i(t; X) = \frac{\partial(t_0 F + t_1 G)}{\partial X_i}.$$

Since H_i is of bidegree $(1, d-1)$, Z_i has class

$$l + (d-1)h \in A^1(\mathbb{P}^1 \times \mathbb{P}^n).$$

Therefore, the number of common zeros of H_i is

$$\deg(l + (d-1)h)^{n+1} = (n+1)(d-1)^n.$$

Next, consider plane curves with a triple point. If C is a curve of degree d defined by $F = 0$, we say that F has a triple point at p if locally it has expansion

$$f_3 + f_4 + \cdots$$

at the origin p, where f_i is of degree i. Equivalently, $F \in I^3_{p/\mathbb{P}^2}$. Similarly, one can construct the universal triple point. Since the coefficients of $1, x, y, x^2, xy, y^2$ are zero in F at p and p moves in $\mathbb{P}^2$, the locus T of plane degree d curves with a triple point is irreducible of dimension $N - 6 + 2 = N - 4$ in $\mathbb{P}^N$, where $N = \binom{d+2}{2} - 1$. We can ask for the degree of T.

Take five general degree d forms on $\mathbb{P}^2$: $F_0, \dots, F_4$. The degree of T equals the number of curves in the 4-dimensional linear series $\sum t_i F_i$ that have a triple point. View each F_i as a section τ_i of the rank 6 bundle $\mathcal{P}^2(\mathscr{O}_{\mathbb{P}^2}(d))$. The curve C_t cut out by $\sum t_i F_i$ has a triple point at p precisely when $\tau_0, \dots, \tau_4$ fail to be linearly independent at p. Thus

$$\deg T = \deg c_2(\mathcal{P}^2(\mathscr{O}_{\mathbb{P}^2}(d))) = 15(d-2)^2.$$

5.3. **Topological Riemann-Hurwitz formula.** Denote by $\chi_{\text{top}}(\cdot)$ the topological Euler characteristic. For instance, $\chi(S) = 1 - 2g + 1 = 2 - 2g$ for a smooth genus g Riemann surface S.

Let X be a smooth projective variety and $Y \subset X$ a subvariety. Triangulate X in such a way that Y forms a subcomplex. Thus we obtain that

$$\chi_{\text{top}}(X) = \chi_{\text{top}}(Y) + \chi_{\text{top}}(X \backslash Y).$$

Theorem 5.7 (Topological Riemann-Hurwitz Formula)**.** *Let $f : X \to B$ be a morphism from a smooth projective variety X to a smooth projective curve B. Let $\eta \in B$ be a general point. Then*

$$\chi_{\text{top}}(X) = \chi_{\text{top}}(B)\chi_{\text{top}}(X_\eta) + \sum_{b \in B} (\chi_{\text{top}}(X_b) - \chi_{\text{top}}(X_\eta)).$$

Proof. Suppose $b_1, \dots, b_k \in B$ are special points where the fiber X_{b_i} is singular. Let $Y = \cup_i X_{b_i} \subset X$. Then we have

$$\begin{aligned}
\chi_{\text{top}}(X) &= \chi_{\text{top}}(X\backslash Y) + \chi_{\text{top}}(Y) \\
&= \chi_{\text{top}}(X_\eta)(\chi_{\text{top}}(B) - k) + \sum_{i=1}^{k} \chi_{\text{top}}(X_{b_i}) \\
&= \chi_{\text{top}}(X_\eta)(\chi_{\text{top}}(B)) + \sum_{i=1}^{k} (\chi_{\text{top}}(X_{b_i}) - \chi_{\text{top}}(X_\eta)).
\end{aligned}$$

□

Suppose $\pi : X \to B$ is a branched cover between two curves of genus h and genus g. Let $b_1, \dots, b_k \in B$ be the branched points, where $\pi^*(b_i) = \sum_j m_{i,j} p_{i,j}$ for $p_{i,j} \in X$, so each $p_{i,j}$ has ramification order $m_{i,j} - 1$. Then we have

$$2 - 2h = d(2 - 2g) - \sum_{i=1}^{k} \sum_{j} (m_{i,j} - 1),$$

$$2h - 2 = d(2g - 2) + \sum_{i=1}^{k} \sum_{j} (m_{i,j} - 1),$$

which recovers the classical Riemann-Hurwitz formula of branched covers.

Now we can calculate the number of singular curves in a general pencil of degree d plane curves using the topological Riemann-Hurwitz formula. Let S be the blowup of $\mathbb{P}^2$ at d^2 base points of the pencil. Then S is a fibration over $\mathbb{P}^1$ with fibers given by curves in the pencil. A smooth degree d plane curve C_η has genus $g = (d-1)(d-2)/2$. If C is irreducible one-nodal of geometric genus $g-1$, its topological Euler characteristic drops by 1 compared to its normalization, since it loses one point. Thus

$$\chi(C) - \chi(C_\eta) = (2 - 2(g-1) - 1) - (2 - 2g) = 1.$$

By the topological Riemann-Hurwitz formula,

$$\chi_{\text{top}}(S) = 2 \cdot (2 - 2g) + \# \text{ of singular members.}$$

On the other hand,

$$\chi_{\text{top}}(S) = \chi_{\text{top}}(\mathbb{P}^2) + d^2 = d^2 + 3.$$

We thus obtain that the number of singular members in the pencil equals $3(d-1)^2$, which recovers the result in Example 5.6.

What happens if C has a singularity other than a node? Say, if C has a cusp and no other singularities, then it is homeomorphic to its normalization, hence

$$\chi_{\text{top}}(C) = 2 - 2(g-1),$$

$$\chi_{\text{top}}(C) - \chi_{\text{top}}(C_\eta) = 2,$$

i.e. the fiber C counts with multiplicity 2 in the above.

If C has a tacnode, it loses one point compared to its normalization, hence

$$\chi_{\text{top}}(C) = 2 - 2(g-2) - 1 = 2 - 2g + 3,$$

$$\chi_{\text{top}}(C) - \chi_{\text{top}}(C_\eta) = 3,$$

i.e. the fiber C counts with multiplicity 3.

In general, this tells us information about the discriminant locus $\mathcal{D} \subset \mathbb{P}^N$.

Proposition 5.8. *Let C be a plane curve of degree d with isolated singularities. Then*

$$\mathrm{mult}_{[C]}(\mathcal{D}) = \chi_{\mathrm{top}}(C) - \chi_{\mathrm{top}}(C_\eta).$$

Proof. Let D be a general plane curve of degree d. Consider the pencil B spanned by C and D. Then

$$B \cdot \mathcal{D} = \deg \mathcal{D} = 3(d-1)^2.$$

On the other hand, by the topological Riemann-Hurwitz formula, the number of singular elements of B other than C is

$$3(d-1)^2 - (\chi_{\mathrm{top}}(C) - \chi_{\mathrm{top}}(C_\eta)).$$

It follows that the multiplicity of $B \cap \mathcal{D}$ at $[C]$ equals $\chi_{\mathrm{top}}(C) - \chi_{\mathrm{top}}(C_\eta)$. □

Remark 5.9. A plane curve with a cusp and no other singularities is a double point of $\mathcal{D}$. A plane curve with a tacnode and no other singularities is a triple point of $\mathcal{D}$. A plane curve with a node and no other singularities is necessarily a smooth point of $\mathcal{D}$.

6. Stable maps

As the Hilbert scheme, the Kontsevich moduli space of stable maps provides an alternative compactification of the parameter space of smooth curves in an ambient space, on which one can carry out intersection calculations to derive enumerative results.

6.1. **The five conic problem.** The space of conics is $\mathbb{P}^5$. Fix a conic C in $\mathbb{P}^2$. The locus of conics tangent to C is a hypersurface $T \subset \mathbb{P}^5$. The degree of T is equal to the number of conics in a pencil that are tangent to C. Since two conics intersect at four points, we conclude that $\deg T$ equals the number of ramification points of a degree 4 branched cover $\mathbb{P}^1 \to \mathbb{P}^1$, which is 6 by Riemann-Hurwitz.

Let $C_1, \ldots, C_5$ be five general conics in $\mathbb{P}^2$. Let $T_i \subset \mathbb{P}^5$ be the locus of conics tangent to C_i. Then one would expect that there are $6^5 = 7776$ conics tangent to the five conics. Well, this is unfortunately incorrect, for the T_i do not intersect transversely. Indeed they all contain the locus D of double lines, which are tangent to every conic! One approach is to apply the *excess intersection formula*, which we will discuss later. Alternatively, we will describe another compactification of the space of smooth conics to avoid the excess intersection.

For a smooth conic C, let $C^* \subset \mathbb{P}^{2*}$ parameterize tangent lines of C. Then C^* is also of degree 2 and smooth, called the *dual conic* of C. Moreover, $C^{**} = C$. Set

$$U = \{(C, C^*) \in \mathbb{P}^5 \times \mathbb{P}^{5*} \mid C \text{ is a smooth conic and } C^* \text{ is its dual}\}.$$

Define the *variety of complete conics* as the closure of U:

$$X = \overline{U} \subset \mathbb{P}^5 \times \mathbb{P}^{5*}.$$

Since U is an open dense subset of X, $\dim X = \dim U = 5$. Later on we will see that X can be identified with the moduli space of stable degree 2 and genus 0 maps to $\mathbb{P}^2$, which is also isomorphic to the blowup of $\mathbb{P}^5$ along the locus of double lines.

Let us analyze the dual conic when C is singular. Consider a family of conics $C_t : y^2 = x^2 - t$ parameterized by t. For $t \neq 0$, C_t is a smooth conic. The limit C_0 is a union of two lines meeting at $p = (0,0)$. It is easy to see that tangent lines of C_t approach to lines passing through p as their limits. Conversely, every line passing though p appears as the limit of some tangent lines of C_t as $t \to 0$. It implies that C_0^* set-theoretically equals $p^* \subset \mathbb{P}^{2*}$ parameterizing lines passing through p. Since it has degree 2, it should be the double line $2p^*$.

If C_0 is a double line $2L$ for $L \subset \mathbb{P}^2$, we can think of it as the limit of maps $f_t : \mathbb{P}^1 \to \mathbb{P}^2$ such that the generic map f_t is an embedding and the limit f_0 is a double cover of L, hence f_0 has two branch points $p, q \in L$. If a line H tangent to the image of f_t, it implies that $f_t^{-1}H$ consists of a single (nonreduced) point in $\mathbb{P}^1$. Therefore, in this case $C_0^* = p^* + q^*$ parameterizes lines passing through p or q. If p coincides with q, then $C_0^* = 2p^*$. Here we implicitly used the idea of the Kontsevich space of stable maps.

In summary, there are four types of complete conics $(C, C^*) \in X$:
(1) C and C^* are both smooth and dual to each other.
(2) $C = L_1 \cup_p L_2$ and $C^* = 2p^*$.
(3) $C = 2L$ and $C^* = p^* \cup_{L^*} q^*$, where $p \neq q \in L$.
(4) $C = 2L$ and $C^* = 2p^*$, where $p \in L$.

Next, let us describe the Chow ring of X. Let $\alpha, \beta \in A^1(X)$ be the pullback to X of the hyperplane classes of $\mathbb{P}^5$ and $\mathbb{P}^{5*}$. They can be represented by divisors

$$A_p = \{(C, C^* \mid p \in C\},$$
$$B_L = \{(C, C^* \mid L \in C^*\},$$

respectively, where $p \in \mathbb{P}^2$ and $L \in \mathbb{P}^{2*}$.

Lemma 6.1. *The group $A^1(X) \otimes \mathbb{Q}$ is generated by α and β.*

Proof. First we claim that the rank of $A^1(X)$ over $\mathbb{Q}$ is at most two. Note that U is the complement of the discriminant hypersurface in $\mathbb{P}^5$, hence $A^1(U)$ is a torsion group. The complement $X \backslash U$ has two irreducible components D_2 and D_3, given by complete conics of type (2) and (3), respectively. It is easy to see that D_2 and D_3 are divisors in X, thus proving the claim.

Next we show that α and β are independent in $A^1(X)$. Define two test curves $\gamma, \eta \in A_1(X)$ where γ is the class of (C, C^*) such that C contains four general points in $\mathbb{P}^2$ and η is the class of (C, C^*) such that C^* contains four general lines in $\mathbb{P}^{2*}$. It is easy to see that

$$\alpha\gamma = 1, \quad \beta\eta = 1,$$
$$\alpha\eta = 2, \quad \beta\gamma = 2,$$

where the second line follows from the number of conics in a pencil that are tangent to a fixed line. Since the intersection matrix is nonsingular, it follows that α and β generate $A^1(X) \otimes \mathbb{Q}$. □

Now let us determine the divisor class of the locus of conics in X that are tangent to a fixed conic.

Lemma 6.2. *Let $Z \subset X$ be the locus of conics $(C, C^*$ such that C is tangent to a given conic Q or equivalently C^* is tangent to Q^*. Then Z has divisor class*

$$\zeta = 2\alpha + 2\beta \in A^1(X).$$

Proof. Assume that $\zeta = a\alpha + b\beta$ for $a, b \in \mathbb{Q}$. Using the test curves γ and η, we have

$$6 = a + 2b,$$
$$6 = 2a + b,$$

we thus obtain that $a = b = 2$. □

Finally we can calculate the number of conics that are tangent to five given conics.

Theorem 6.3. *There are* 3264 *plane conics tangent to five general conics.*

Proof. It amounts to calculate the degree of

$$\zeta^5 = 32(\alpha + \beta)^5.$$

We need to evaluate the degree of $\alpha^{5-i}\beta^i$ for $0 \leq i \leq 5$. By symmetry between α and β, it suffices to do the cases $i = 0, 1, 2$.

$i = 0$: α^5 is represented by conics containing 5 general points, hence it has degree equal to 1.

$i = 1$: $\alpha^4\beta$ is represented by (C, C^*) where C contains 4 general points and C^* contains a general line, i.e. C is also tangent to a given line. We have seen this number equal to 2.

$i = 2$: Note that the locus of conics tangent to a line is a quadric hypersurface in $\mathbb{P}^5$. Moreover, $\alpha^3\beta$ is represented by conics containing 3 general points and tangent to a general line. This give a degree 2 curve Q in $\mathbb{P}^5$ of conics. Not all conics in Q are tangent to another general line, say, by using degenerate conics in Q. Therefore, $\deg(\alpha^3\beta^2) = 2 \cdot 2 = 4$ and there is no excess intersection in this case.

Putting everything together, we obtain that

$$\deg(\alpha + \beta)^5 = 1 + 5 \cdot 2 + 10 \cdot 4 + 10 \cdot 4 + 5 \cdot 2 + 1 = 102.$$

Consequently $\deg \zeta^5 = 32 \cdot 102 = 3264$. □

6.2. **Stable maps.** The space of complete conics is a special example of the *Kontsevich moduli space of stable maps* $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ parameterizing maps

$$f : C \to \mathbb{P}^r,$$

where C is a connected curve of arithmetic genus g with at worst nodal singularities and n ordered marked points in its smooth locus, $f_*[C] = dl \in A_1(\mathbb{P}^r)$ with l a line class of $\mathbb{P}^r$, and the map f has a finite automorphism group. The last stability condition is equivalent to that any genus 0 component of C contracted by f possesses at least three special points (nodes or marked points).

Example 6.4. The moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ is the space of complete conics. The complete conics of the four types correspond to:

(1) f maps a $\mathbb{P}^1$ to a smooth conic.
(2) f maps a union of two $\mathbb{P}^1$ to a union of two lines in $\mathbb{P}^2$.
(3) $f : \mathbb{P}^1 \to L$ is a double cover of a line $L \subset \mathbb{P}^2$.
(4) f maps a union of two $\mathbb{P}^1$ to the same line in $\mathbb{P}^2$.

The Kontsevich space has nice properties when $g = 0$. For instance, $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is $\mathbb{Q}$-factorial, irreducible of dimension $(d + 1)(r + 1) + n - 4$. Its rational Picard group is generated by the boundary divisors and the locus of maps whose images intersect a fixed codimension-2 subspace in $\mathbb{P}^r$. Here let us use it to solve the following enumerative problem.

Question 6.5. *How many degree d rational plane curves pass through $3d-1$ general points in $\mathbb{P}^2$?*

Since the space of degree d rational curves has dimension $3d-1$, we expect a finite number for the above question. Denote this number by N_d. We know that $N_1 = 1$, $N_2 = 1$, $N_3 = 12$ (the number of rational nodal curves in a pencil of plane cubics).

For general d, let us work on $M_d = \overline{\mathcal{M}}_{0,4}(\mathbb{P}^2, d)$, where the marked points are p_1, p_2, p_3, p_4. If we fix the cross-ratio

$$\phi = \frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)},$$

then we get a divisor D_ϕ in M_d whose divisor class is independent of ϕ. This follows from the fact that D_ϕ is the fiber of the forgetful morphism $M_d \to \overline{\mathcal{M}}_{0,4}$ over ϕ, where $\overline{\mathcal{M}}_{0,4}$ is the moduli space of stable genus zero curves with four marked points. In particular, D_0 parameterizes stable maps such that C has two components containing p_1, p_2 and p_3, p_4, respectively. Similarly D_∞ parameterizes stable maps such that C has two components containing p_1, p_3 and p_2, p_4, respectively.

Fix two general lines $L, M \subset \mathbb{P}^2$ intersecting at a point p, as well as $3d-2$ general points $q, r, z_1, \ldots, z_{3d-4} \in \mathbb{P}^2$. Let $Z = \{z_1, \ldots, z_{3d-4}\}$. Consider the locus $B \subset M_d$ of stable maps $f : (C, p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ such that $f(p_1) = q$, $f(p_2) = r$, $f(p_3) \in L$, $f(p_4) \in M$ and $Z \in f(C)$. Note that $f(C)$ is a rational curve of degree d containing $3d-2$ general points. It follows that B is a curve in M_d.

Let us calculate the intersection number of B with D_0. Note that if $f(C)$ passes through p, then $f(p_3) = f(p_4)$, so in the domain curve we need to blow it up and thus the map belongs to D_0. In this case $f(C)$ contains p, q, r, Z. Hence such maps contribute N_d to $B \cdot D_0$. If $f(C)$ does not go through p, the other possibility is that $C = C_1 \cup C_2$ with $d_1 + d_2 = d$ such that $f : C_i \to \mathbb{P}^2$ is of degree d_i, $f(C_1)$ contains q, r, Z_1 and $f(C_2)$ contains Z_2, where $Z_1 \subset Z$ such that $|Z_1| = 3d_1 - 3$ and $Z_2 = Z \backslash Z_1$ with $|Z_2| = 3d_2 - 1$. Since $f(C_{d_1})$ and $f(C_{d_2})$ intersect at $d_1 d_2$ nodes, the domain curve C has $d_1 d_2$ choices with a given image $f(C)$. Since p_3 is an intersection point of L with C_{d_2}, it has d_2 choices, so does p_4. Therefore, such maps contribute $\sum_{d_1+d_2=d} \binom{3d-4}{3d_1-3} d_1 d_2^3 N_{d_1} N_{d_2}$ to $B \cdot D_0$. Altogether we obtain that

$$B \cdot D_0 = N_d + \sum_{d_1+d_2=d} \binom{3d-4}{3d_1-3} d_1 d_2^3 N_{d_1} N_{d_2}.$$

Next, we calculate the intersection number of B with D_∞. The only possibility is that $C = C_1 \cup C_2$ with $d_1 + d_2 = d$ such that $f : C_i \to \mathbb{P}^2$ is of degree d_i, $f(C_1)$ contains q, Z_1 and $f(C_2)$ contains r, Z_2, where $Z_1 \cup Z_2 = Z$ and $|Z_i| = 3d_i - 2$. Such maps without marked points have $d_1 d_2 N_{d_1} N_{d_2}$ choices. Moreover, p_3 and p_4 have d_1 and d_2 many choices, respectively. We thus conclude that

$$B \cdot D_\infty = \sum_{d_1+d_2=d} \binom{3d-4}{3d_1-2} d_1^2 d_2^2 N_{d_1} N_{d_2}.$$

By the relation $B \cdot D_0 = B \cdot D_\infty$, we obtain the following recursive formula to count N_d.

Theorem 6.6. *The number N_d of plane rational degree d curves containing $3d-1$ general points satisfies the relation*

$$N_d = \sum_{d_1+d_2=d} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1 d_2^3 \binom{3d-4}{3d_1-3} \right) N_{d_1} N_{d_2}.$$

Using the initial values $N_1 = N_2 = 1$, one can calculate that $N_3 = 12$, $N_4 = 620$ ...

7. Projective bundles

7.1. **Projective bundles.** A *projective bundle* over X is a map $\pi : Y \to X$ such that there is an open covering $\{U_i\}$ with $\pi^{-1}(U_i) \cong U_i \times \mathbb{P}^r$.

Projective bundles can be made from vector bundles (locally free sheaves) on X. Let $\mathcal{E}$ be a vector bundle of rank $r+1$ on X. Then $\mathbb{P}\mathcal{E} := \operatorname{Proj}\operatorname{Sym}\mathcal{E}^* \to X$ is a projective bundle with fiber $\mathbb{P}^r$, which can be locally verified. Geometrically a point in $\mathbb{P}\mathcal{E}_x$ parameterizes a 1-dimensional subspace in $\mathcal{E}_x$. From this we can define the tautological bundle $\mathscr{O}_{\mathbb{P}\mathcal{E}}(-1)$ and let $\mathscr{O}_{\mathbb{P}\mathcal{E}}(1)$ be its dual as the universal bundle.

Remark 7.1. In many cases the notation $\mathbb{P}\mathcal{E}$ is also used for the parameterization of hyperplanes in $\mathcal{E}_x$. It has the advantage that $\pi_*\mathscr{O}_{\mathbb{P}\mathcal{E}}(1) = \mathbb{P}\mathcal{E}$. Sometimes we will switch between the two notions for convenience.

Projective spaces are universal spaces of one-dimensional subspaces in an ambient vector space. More precisely, let V be a vector space of dimension $r+1$. If $\mathcal{L} \hookrightarrow B \times V$ is a family of one-dimensional subspaces in V parameterized by B, i.e. $\mathcal{L}$ is a line bundle on B such that the fiber over $b \in B$ is a one-dimensional subspace $L_b \subset V$, then there exists a map $\phi : B \to \mathbb{P}^r$ such that $\mathcal{L} = \phi^*\mathscr{O}_{\mathbb{P}^r}(-1)$, where $\mathscr{O}_{\mathbb{P}^r}(-1)$ is the universal object over (the moduli space) $\mathbb{P}^r$. We can generalize this property to projectivized vector bundles.

Proposition 7.2. *Let $\pi : \mathbb{P}\mathcal{E} \to X$ be a projectivized vector bundle. Commutative diagrams as follows*

$$\begin{array}{ccc} Y & \xrightarrow{\ \phi\ } & \mathbb{P}\mathcal{E} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

are in one-to-one correspondence with line bundle inclusions $\mathcal{L} \hookrightarrow p^*\mathcal{E}$ on Y.

Proof. We have the natural inclusion $\mathscr{O}_{\mathbb{P}\mathcal{E}}(-1) \hookrightarrow \pi^*\mathcal{E}$. Given a commutative diagram as above, set $\mathcal{L} = \phi^*\mathscr{O}_{\mathbb{P}\mathcal{E}}(-1)$. Note that $p^*\mathcal{E} = \phi^*(\pi^*\mathcal{E})$. The desired inclusion follows right away.

Conversely, given $\mathcal{L} \hookrightarrow p^*\mathcal{E}$, we may cover X by open sets U on which $\mathcal{L}$ and $\mathcal{E}$ are trivial, hence get a unique map over U by the universal property of projective spaces. These maps glue together to give a map over X. □

Let $\mathscr{O}_{\mathbb{P}\mathcal{E}}(m)$ be the mth tensor power of $\mathscr{O}_{\mathbb{P}\mathcal{E}}(1)$. For a sheaf $\mathcal{F}$ on $\mathbb{P}\mathcal{E}$, we also write $\mathcal{F}(m) = \mathcal{F} \otimes \mathscr{O}_{\mathbb{P}\mathcal{E}}(m)$.

Proposition 7.3. *If $\pi : \mathbb{P}\mathcal{E} \to X$ is a projectivized vector bundle, then for $m \geq 0$, $\pi_*\mathscr{O}_{\mathbb{P}\mathcal{E}}(m) = \operatorname{Sym}^m \mathcal{E}^*$ and $\pi_*\mathscr{O}_{\mathbb{P}\mathcal{E}}(m) = 0$ for $i > 0$. In particular for $m = 1$, $\mathscr{O}_{\mathbb{P}\mathcal{E}}(1)$ determines $\mathcal{E}^*$ (and hence $\mathcal{E}$).*

Proof. Suppose that $\mathcal{E}$ has rank $r+1$. Let $U \subset X$ be an open affine subset such that $\mathcal{E}|_U = \mathscr{O}_U^{r+1}$. Then we have a canonical isomorphism $H^0(\mathscr{O}_{\mathbb{P}\mathcal{E}}(m)|_{\pi^{-1}(U)}) \cong H^0(\pi^*\operatorname{Sym}^m \mathcal{E}^*|_U)$. Moreover, $H^i(\mathscr{O}_{\mathbb{P}\mathcal{E}}(m)|_{\pi^{-1}(U)}) = 0$ for $i > 0$, which follows from $H^i(\mathscr{O}_{\mathbb{P}^r}(m)) = 0$ for $m \geq 0$ and $i > 0$. Thus the claim follows from the definition of direct images. □

We also have a relative version of the Euler sequence that computes the relative tangent bundle and dualizing line bundle class of $\mathbb{P}\mathcal{E}$ over X.

Proposition 7.4. *The relative dualizing line bundle of* $\mathbb{P}\mathcal{E}$ *over* X *is* $\wedge^{r+1}\pi^*\mathcal{E}^*(-r-1)$.

Proof. We have the relative Euler sequence

$$0 \to \mathscr{O}_{\mathbb{P}\mathcal{E}}(-1) \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is the quotient bundle. Twisting by $\mathscr{O}_{\mathbb{P}\mathcal{E}}(1)$, we obtain that

$$0 \to \mathscr{O}_{\mathbb{P}\mathcal{E}} \to \pi^*\mathcal{E}(1) \to T_{\mathbb{P}\mathcal{E}/X} \to 0,$$

where $T_{\mathbb{P}\mathcal{E}/X}$ is the relative tangent bundle of $\mathbb{P}\mathcal{E}$ over X. The claim follows by taking the dual of $c_1(T_{\mathbb{P}\mathcal{E}/X})$. $\square$

Suppose that $\mathcal{E}^*$ has a section σ. It induces a section $\tilde{\sigma} \in H^0(\mathscr{O}_{\mathbb{P}\mathcal{E}}(1))$ given by $\tilde{\sigma}(x,[L]) = \sigma(x)|_L \in \mathrm{Hom}(L,\mathbb{C})$ for $x \in X$ and $[L] \in \mathbb{P}\mathcal{E}_x$. The section $\tilde{\sigma}$ vanishes at $(x,[L])$ if and only if L is contained in $\ker\sigma(x)$. In other words, given $\sigma(x) : \mathcal{E}_x \to \mathbb{C}$, if it is not identically zero, i.e. if σ is not vanishing at x, then the divisor $(\tilde{\sigma})$ meets the fiber $\mathbb{P}\mathcal{E}_x$ at the hyperplane $\ker\sigma(x)$. If σ is vanishing at x, then $(\tilde{\sigma})$ contains the entire fiber $\mathbb{P}\mathcal{E}_x$.

Below we show that every projective bundle is the projectivization of a vector bundle.

Proposition 7.5. *Let* $\pi : Y \to X$ *be a smooth morphism whose (scheme-theoretic) fibers are all isomorphic to* $\mathbb{P}^r$. *Then the following are equivalent:*

(a) $Y = \mathbb{P}\mathcal{E}$ *is the projectivization of a vector bundle* $\mathcal{E}$ *on* X.

(b) $\pi : Y \to X$ *is a projective bundle, i.e. there exists an open covering of* X *that locally trivializes* Y.

(c) There exists a line bundle L *on* Y *such that* $L|_{Y_x} \cong \mathscr{O}_{\mathbb{P}^r}(1)$.

(d) There exists a Cartier divisor $D \subset Y$ *such that* $D \cap Y_x$ *for a general fiber* $Y_x \cong \mathbb{P}^r$ *is a hyperplane.*

Proof. Clearly (a) implies (b) and (c). Using local trivialization and gluing $\mathscr{O}(1)$ globally, (b) implies (c). Moreover, (c) and (d) are equivalent. Given D in (d), the line bundle $L = \mathscr{O}_Y(D)$ does what we want in (c). Conversely, given L in (c), tensoring it with the pullback of a very ample line bundle from X, we may assume that L has a nonzero section, whose zero locus thus provides the divisor D in (d).

If suffices to show that (c) implies (a). Let L be the line bundle given in (c). Since $h^0(L|_{Y_x}) = h^0(\mathscr{O}_{\mathbb{P}^r}(1)) = r+1$ and $h^1(L|_{Y_x}) = h^1(\mathscr{O}_{\mathbb{P}^r}(1)) = 0$, the direct image $\mathcal{E}^* = \pi_*L$ is a vector bundle of rank $r+1$ on X, whose fibers are isomorphic to $H^0(\mathscr{O}_{\mathbb{P}^r}(1))$. Let $\rho : \mathbb{P}\mathcal{E} \to X$ be the natural map.

We need to find an isomorphism $\alpha : Y \to \mathbb{P}\mathcal{E}$. such that the follow diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & \mathbb{P}\mathcal{E} \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle\rho} \\ & & X \end{array}$$

By the universal property of $\mathbb{P}\mathcal{E}$, such α can be defined by giving a line subbundle of $\pi^*\mathcal{E}$, or equivalently from the dual viewpoint, a line bundle that is the homomorphism image of $\pi^*\mathcal{E}^* = \pi^*\pi_*L$. By definition, there is a natural onto map

$\pi^*\pi_*L \to L$, locally induced by $U \times H^0(\mathscr{O}_{\mathbb{P}^r}(1)) \to \mathscr{O}_U(1)$. Let $\alpha : Y \to \mathbb{P}\mathcal{E}$ be the corresponding morphism. It is an isomorphism because restricted to each fiber (or trivialization) of $\pi : \mathbb{P}^r \to \mathbb{P}^r$, it is given by the complete linear series $|\mathscr{O}_{\mathbb{P}^r}(1)|$. □

Next, we show that two projective bundles are isomorphic if and only if their associated vector bundles differ by the pullback of a line bundle from the base.

Proposition 7.6. *Two projective bundles $\pi : \mathbb{P}\mathcal{E} \to X$ and $\pi' : \mathbb{P}\mathcal{E}' \to X$ are isomorphic if and only if there is a line bundle L on X such that $\mathcal{E} = L \otimes \mathcal{E}'$. In this case $\mathscr{O}_{\mathbb{P}\mathcal{E}}(-1)$ corresponds to $\pi^*L \otimes \phi^*\mathscr{O}_{\mathbb{P}\mathcal{E}'}(-1)$ under the isomorphism $\phi : \mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{E}'$.*

Proof. Suppose that $\mathcal{E} = L \otimes \mathcal{E}'$. We have

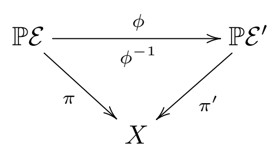
$$\mathscr{O}_{\mathbb{P}\mathcal{E}}(-1) \hookrightarrow \pi^*\mathcal{E} = \pi^*L \otimes \pi^*\mathcal{E}'.$$

Tensoring it with π^*L^*, we obtain that

$$\pi^*L^* \otimes \mathscr{O}_{\mathbb{P}\mathcal{E}}(-1) \hookrightarrow \pi^*\mathcal{E}',$$

which gives rise to a morphism $\phi : \mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{E}'$ such that $\phi^*\mathscr{O}_{\mathbb{P}\mathcal{E}'}(-1) = \pi^*L^* \otimes \mathscr{O}_{\mathbb{P}\mathcal{E}}(-1)$. The inverse map can be defined similarly. The composite $\mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{E}' \to \mathbb{P}\mathcal{E}$ is the identity map because it corresponds to the tautological line bundle $\mathscr{O}_{\mathbb{P}\mathcal{E}}(-1) \hookrightarrow \pi^*\mathcal{E}$.

Conversely, suppose that $\phi : \mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{E}'$ is an isomorphism satisfying the commutative diagram



Then $\phi^*\mathscr{O}_{\mathbb{P}\mathcal{E}'}(1)$ and $\mathscr{O}_{\mathbb{P}\mathcal{E}}(1)$ have the same restriction $\mathscr{O}_{\mathbb{P}^r}(1)$ to every fiber of π. Since π is flat, there exists a line bundle L on X such that $\mathscr{O}_{\mathbb{P}\mathcal{E}}(1) = \phi^*\mathscr{O}_{\mathbb{P}\mathcal{E}'}(1) \otimes \pi^*L^*$. Thus we have

$$\begin{aligned}
\mathcal{E}^* &= \pi_*(\mathscr{O}_{\mathbb{P}\mathcal{E}}(1)) \\
&= \pi_*(\phi^*\mathscr{O}_{\mathbb{P}\mathcal{E}'}(1) \otimes \pi^*L^*) \\
&= L^* \otimes \pi_*\phi^*\mathscr{O}_{\mathbb{P}\mathcal{E}'}(1) \text{ (projection formula)} \\
&= L^* \otimes \pi_*\phi_*^{-1}\mathscr{O}_{\mathbb{P}\mathcal{E}'}(1) \\
&= L^* \otimes \pi'_*\mathscr{O}_{\mathbb{P}\mathcal{E}'}(1) \\
&= L^* \otimes \mathcal{E}'^*.
\end{aligned}$$

It follows that $\mathcal{E} = L \otimes \mathcal{E}'$ and $\mathscr{O}_{\mathbb{P}\mathcal{E}}(-1) = \pi^*L \otimes \phi^*\mathscr{O}_{\mathbb{P}\mathcal{E}'}(-1)$. □

7.2. **Ample vector bundle.** In this section we use $\mathbb{P}\mathcal{E}$ to parameterize hyperplanes in each fiber $\mathcal{E}_x$ for $x \in X$. The universal line bundle $\mathscr{O}_{\mathbb{P}\mathcal{E}}(1)$ has fiber $\mathcal{E}_x/H$ at $(x, [H])$, where $[H] \in \mathbb{P}\mathcal{E}_x$. If $\mathscr{O}_{\mathbb{P}\mathcal{E}}(1)$ is very ample, we say that $\mathcal{E}$ is a *very ample* vector bundle.

Proposition 7.7. *If $\mathcal{E}$ is very ample, then $\mathcal{E}$ is generated by global sections.*

Proof. Let $L = \mathscr{O}_{\mathbb{P}\mathcal{E}}(1)$ be very ample. A global section $\sigma \in H^0\mathcal{E}$ specifies $\sigma(x) = v \in \mathcal{E}_x$. It induces a section $\tilde{\sigma} \in H^0L$ by $\tilde{\sigma}(x, [H]) = v/H \in \mathcal{E}_x/H$. Conversely, a section $\tilde{\sigma} \in H^0L$ restricted to a section of $H^0(\mathscr{O}_{\mathbb{P}^r}(1))$ on $\mathbb{P}\mathcal{E}_x$, which is of the form v/H at each $[H] \in \mathbb{P}\mathcal{E}_x$ for the same $v \in \mathcal{E}_x$. Therefore, we obtain a natural identification $H^0\mathcal{E} = H^0L$. If σ vanishes at x, then $\tilde{\sigma}$ vanishes entirely along $\mathbb{P}\mathcal{E}_x$.

The sections of L give an embedding of $\mathbb{P}\mathcal{E}$. It implies that the sections of L restricted to a fiber $\mathbb{P}\mathcal{E}_x = \mathbb{P}^r$ are the complete linear series $|\mathscr{O}_{\mathbb{P}^r}(1)|$. The above identification implies that $H^0\mathcal{E}/H^0(I_{x/X}\mathcal{E})$ has dimension at least $r+1$. On the other hand, $H^0\mathcal{E}/H^0(I_{x/X}\mathcal{E}) \subset H^0(\mathcal{E}_x)$ is $(r+1)$-dimensional. Therefore, we conclude that $H^0\mathcal{E}/H^0(I_{x/X}\mathcal{E}) = H^0(\mathcal{E}_x)$, hence the sections of $\mathcal{E}$ generate $\mathcal{E}$ at each point $x \in X$. □

7.3. **Chow ring of a projective bundle.** First, consider the trivial vector bundle $Y = X \times \mathbb{P}^r$. The Künneth formula holds for the Chow ring of the product of a variety with a projective space (or in general, varieties with affine stratifications). Therefore, we have

$$\begin{aligned} A(Y) &= A(X) \otimes A(\mathbb{P}^r) \\ &= A(X) \otimes \mathbb{Z}[\zeta]/(\zeta^{r+1}) \\ &= A(X)[\zeta]/(\zeta^{r+1}), \end{aligned}$$

where ζ is the pullback of the hyperplane class from $\mathbb{P}^r$ to Y. In particular,

$$A(Y) = \bigoplus_{i=0}^{r} \zeta^i A(X)$$

as groups. Here for $\alpha \in A(Y)$ and $\beta \in A(X)$, $\alpha\beta$ means $\alpha(\pi^*\beta) \in A(Y)$, where $\pi : Y \to X$.

For a general projective bundle, we have the following description of its Chow ring.

Theorem 7.8. *Let $\mathcal{E}$ be a vector bundle of rank $r+1$ on a smooth variety X. Let $\zeta = c_1(\mathscr{O}_{\mathbb{P}\mathcal{E}}(1)) \in A^1(\mathbb{P}\mathcal{E})$. Let $\pi : \mathbb{P}\mathcal{E} \to X$ be the projection. The map $\pi^* : A(X) \to A(\mathbb{P}\mathcal{E})$ is injective, and via this map*

$$A(\mathbb{P}\mathcal{E}) \cong A(X)[\zeta]/\big(\zeta^{r+1} + c_1(\mathcal{E})\zeta^r + \cdots + c_{r+1}(\mathcal{E})\big).$$

In particular, as groups we have

$$A(\mathbb{P}\mathcal{E}) \cong \bigoplus_{i=0}^{r} \zeta^i A(X).$$

We isolate the proof into several parts.

Lemma 7.9. *In the above setting,*

$$\zeta^{r+1} + c_1(\mathcal{E})\zeta^r + \cdots + c_{r+1}(\mathcal{E}) = 0 \in A(\mathbb{P}\mathcal{E}).$$

Proof. Let $S = \mathscr{O}_{\mathbb{P}\mathcal{E}}(-1)$ and $c(S) = 1 - \zeta$. By the relative Euler sequence

$$0 \to S \to \pi^*\mathcal{E} \to Q \to 0,$$

we have

$$\begin{aligned} c(Q) &= (\pi^* c(\mathcal{E})) \cdot c(S^*) \\ &= (1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots) \cdot (1 + \zeta + \zeta^2 + \cdots). \end{aligned}$$

Since the rank of Q is r, the claim follows from $c_{r+1}(Q) = 0$. □

Lemma 7.10. *For $\alpha \in A(X)$, we have*

$$\pi_*(\zeta^i \alpha) = 0, \ i < r,$$

$$\pi_*(\zeta^r \alpha) = \alpha.$$

Proof. By the projection formula,

$$\pi_*(\zeta^i \alpha) = \pi_*(\zeta^i)\alpha.$$

If $i < r$, ζ^i has positive dimension restricted to each fiber of π, hence $\pi_*(\zeta^i) = 0$ by dimension considerations. Similarly, $\pi_*(\zeta^r)$ is a multiply of $[X]$. Since its restriction to each fiber is a point class, we conclude that $\pi_*(\zeta^r) = [X]$, hence $\pi_*(\zeta^r)\alpha = \alpha$. □

For $Z \subset \mathbb{P}\mathcal{E}$ a k-dimensional subvariety, let $W \subset X$ be the image of Z under π. We say that Z has *footprint l* if $\dim W = l$, i.e. a general fiber of $\pi : Z \to W$ has dimension $k - l$.

Lemma 7.11. *If $Z \subset \mathbb{P}\mathcal{E}$ is a subvariety of dimension k and footprint l, $k \geq l$, then*

$$Z \sim Z' + \sum n_i B_i \in A_k(\mathbb{P}\mathcal{E})$$

such that
(*a*) *$[Z'] = \zeta^{r-k+l}\alpha$ for some $\alpha \in A_l(X)$;*
(*b*) *each B_i has footprint strictly less than l.*

Proof. If we tensor $\mathcal{E}$ with π^*L for a very ample line bundle L on X, it does not change $\mathbb{P}\mathcal{E}$ and ζ is replaced by $\zeta + c_1(\pi^*L)$, which does not affect the desired conclusion (the extra terms will have footprint smaller than l). Hence we assume that $\mathcal{E}$ is very ample and in particular, generated by its global sections.

Let $\tau_0, \ldots, \tau_r$ be a general collection of global sections of $\mathcal{E}$. Let $U \subset X$ be the locus of $x \in X$ such that $\tau_0(x), \ldots, \tau_r(x)$ generate $\mathcal{E}_x$ and such that the zero locus $(\tau_0(x) = \cdots = \tau_{k-l}(x) = 0) \subset \mathbb{P}\mathcal{E}_x$ is disjoint from $Z_x = Z \cap \mathbb{P}\mathcal{E}_x$. Note that these are open conditions. Moreover, for a general point $x \in \pi(Z) \subset X$, Z_x is $(k-l)$-dimensional, hence U contains an open subset of $W = \pi(Z)$.

Since $\mathcal{E}_U$ is trivialized by $\tau_0, \ldots, \tau_r$, we have $\mathbb{P}\mathcal{E}_U \cong U \times \mathbb{P}^r$. In terms of this basis, consider the one-parameter group of automorphisms A_t of $\mathbb{P}\mathcal{E}_U$ given by $\operatorname{diag}(I_{k-l+1}, t \cdot I_{r-k+l})$. Let Z_t be the closure of the image of $A_t(Z \cap \mathbb{P}\mathcal{E}_U)$ for $t \neq 0$. In particular, $Z_1 = Z$.

We would like to understand the limit Z_0. Over $U \subset X$, Z is pushed to a multiple of the zero locus $\tau_{k-l+1} = \cdots = \tau_r = 0$. Let Z' be the unique component of Z_0 dominating $W = \pi(Z)$. In other words, Z' is the intersection of the zero locus $\tau_{k-l+1} = \cdots = \tau_r = 0$ with $\pi^{-1}(W \cap U)$. By the general choice of the sections τ_i and by the assumption that $\mathcal{E}$ is very ample, the zero locus $\tau_{k-l+1} = \cdots = \tau_r = 0$ intersect $\pi^{-1}W$ generically transversely. We thus conclude that

$$[Z'] = m[W] \cdot \zeta^{r-k+l}$$

for some $m \in \mathbb{Z}$.

If Z_0 has a component B that does not dominate W under π, then it maps to the complement of $W \cap U$ in W, hence it has footprint smaller than l. □

Now we can finish the proof of Theorem 7.8.

Proof of Theorem 7.8. Let $\psi : A(\mathbb{P}\mathcal{E}) \to \oplus_{i=0}^r A(x)\zeta^i$ be the map

$$\beta \to \sum_i \pi_*(\zeta^{r-i}\beta)\zeta^i.$$

Conversely, let $\phi : \oplus_{i=0}^r A(x) \to A(\mathbb{P}\mathcal{E})$ be the map

$$\phi(\alpha_0, \dots, \alpha_r) = \sum_i \zeta^i \alpha_i.$$

By Lemma 7.10, we have

$$\begin{aligned} \psi\phi(\alpha_0, \dots, \alpha_r) &= \sum_{i,j=0}^r \pi_*(\zeta^{r-i+j}\alpha_j)\zeta^i \\ &= \sum_{i=0}^r \Big(\sum_{j \geq i} \pi_*(\zeta^{r-i+j})\alpha_j \Big)\zeta^i. \end{aligned}$$

We thus conclude that $\psi\phi$ is upper triangular with 1 on the diagonal, which implies that ϕ is injective.

On the other hand, Lemma 7.11 implies that the subgroups $\zeta^i A(X)$ generate $A(\mathbb{P}\mathcal{E})$ additively. Therefore, ϕ is a group isomorphism.

Finally, from the proof we see that ζ^{r+1} can be represented by linear combinations of $\zeta^i\alpha_i$ for $0 \leq i \leq r$ and $\alpha_i \in A(X)$. In other words, ζ satisfies a monic polynomial f defined over $A(X)$. This polynomial has been found in Lemma 7.9. □

7.4. **Projectivization of a subbundle.** Let $\mathcal{F} \subset \mathcal{E}$ be a subbundle of a vector bundle $\pi : \mathcal{E} \to X$. Then $\mathbb{P}\mathcal{F}$ is naturally a subscheme of $\mathbb{P}\mathcal{E}$. Suppose that the ranks of $\mathcal{E}$ and $\mathcal{F}$ are $r+1$ and $s+1$, respectively.

There is another description for $\mathbb{P}\mathcal{F}$. Consider the composite map

$$\phi : \mathscr{O}_{\mathbb{P}\mathcal{E}}(-1) \to \pi^*\mathcal{E} \to \pi^*(\mathcal{E}/\mathcal{F}).$$

Then $p \in \mathbb{P}\mathcal{F}$ if and only if ϕ vanishes at p. Note that ϕ can be regarded as a global section of the rank $r-s$ bundle

$$\mathrm{Hom}(\mathscr{O}_{\mathbb{P}\mathcal{E}}(-1), \pi^*(\mathcal{E}/\mathcal{F})) \cong \mathscr{O}_{\mathbb{P}\mathcal{E}}(1) \otimes \pi^*(\mathcal{E}/\mathcal{F}).$$

Therefore, the vanishing locus of ϕ is $\mathbb{P}\mathcal{F}$. As a byproduct, we can calculate the cycle class of $\mathbb{P}\mathcal{F}$ in $A(\mathbb{P}\mathcal{E})$.

Proposition 7.12. *In the above setting, we have*

$$\begin{aligned} [\mathbb{P}\mathcal{F}] &= c_{r-s}(\mathscr{O}_{\mathbb{P}\mathcal{E}}(1) \otimes \pi^*(\mathcal{E}/\mathcal{F})) \\ &= \zeta^{r-s} + \gamma_1\zeta^{r-s-1} + \cdots + \gamma_{r-s} \in A^{r-s}(\mathbb{P}\mathcal{E}), \end{aligned}$$

where $\zeta = c_1(\mathscr{O}_{\mathbb{P}\mathcal{E}}(1))$ *and* $\gamma_i = c_i(\mathcal{E}/\mathcal{F})$.

Proof. The first equality follows from the degeneracy locus definition of Chern classes. For the second, note that

$$c(\mathscr{O}_{\mathbb{P}\mathcal{E}}(1) \otimes \pi^*(\mathcal{E}/\mathcal{F})) = (1 + a_1 + \zeta) \cdots (1 + a_{r-s} + \zeta),$$

where $a_1, \dots, a_{r-s}$ are the Chern roots of $\pi^*(\mathcal{E}/\mathcal{F})$. Therefore,

$$\begin{aligned} c_{r-s}(\mathscr{O}_{\mathbb{P}\mathcal{E}}(1) \otimes \pi^*(\mathcal{E}/\mathcal{F})) &= (a_1 + \zeta) \cdots (a_{r-s} + \zeta) \\ &= \zeta^{r-s} + \gamma_1\zeta^{r-s-1} + \cdots + \gamma_{r-s} \in A^{r-s}(\mathbb{P}\mathcal{E}). \end{aligned}$$

□

There is an important correspondence between line subbundles of $\mathcal{E}$ and sections of $\pi : \mathcal{E} \to X$. We say that a morphism $\sigma : X \to \mathbb{P}\mathcal{E}$ is a *section*, if the composite $\pi\sigma : X \to X$ is the identity map. We will also refer to the image of σ as a section.

Proposition 7.13. *If $\mathcal{L} \subset \mathcal{E}$ is a line subbundle, then $\mathbb{P}\mathcal{L} \subset \mathbb{P}\mathcal{E}$ is a section of $\mathbb{P}\mathcal{E}$ over X. Conversely, every section has this form. In other words, giving a section is the same as specifying a point of $\mathbb{P}\mathcal{E}$, i.e. specifying a 1-dimensional subspace of each fiber of $\mathcal{E}$.*

Proof. The claim is obvious, because determining $\sigma(x) = [L] \in \mathbb{P}\mathcal{E}_x$ amounts to specifying a 1-dimensional subspace $L \subset \mathcal{E}_x$. A more conceptual argument is the following. By the universal property of projective bundles, giving $\sigma : X \to \mathbb{P}\mathcal{E}$ satisfying the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \sigma\ } & \mathbb{P}\mathcal{E} \\ & {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

is equivalent to giving a line bundle inclusion $\mathcal{L} \subset \mathcal{E}$. □

7.5. **Ruled surfaces.** Let $\mathcal{E}$ be a rank 2 vector bundle over a smooth curve X. Then the fibers of $\mathbb{P}\mathcal{E}$ over X are isomorphic to $\mathbb{P}^1$. We call such $\mathbb{P}\mathcal{E}$ a *ruled surface*, see [Hartshorne, Chap V2].

Proposition 7.14. *A ruled surface can contain at most one irreducible curve of negative self-intersection.*

Proof. The fiber class has zero self-intersection. Let C_1, C_2 be two irreducible curves on $\mathbb{P}\mathcal{E}$ such that $[C_1]^2, [C_2]^2 < 0$. Note that $\pi : C_i \to X$ is a finite map, i.e. C_i is only a multi-section in general. Below we show that from this setting it gives rise to a ruled surface with two *sections* of negative self-intersection.

Let $\alpha : C_1' \to X$ be the normalization of C_1. Then $\mathbb{P}(\alpha^*\mathcal{E})$ is a ruled surface over C_1'. We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\alpha^*\mathcal{E}) & \xrightarrow{\ \beta\ } & \mathbb{P}\mathcal{E} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ C_1' & \xrightarrow[\ \alpha\ ]{} & X \end{array}$$

Then $\beta^{-1}(C_1)$ gives a section Σ_1 over C_1' such that $[\Sigma_1]^2 < 0$ (draw a picture to verify it). Under a finite morphism f of degree m, $\deg[f^*C]^2 = m \deg[C]^2$. Therefore, we can repeat the process with $\beta^{-1}(C_2)$ to arrive at a ruled surface, still denoted by $\pi : \mathbb{P}\mathcal{E} \to X$, such that it has two sections Σ_1, Σ_2 of negative self-intersection. By Proposition 7.12, we may assume that $\Sigma_i = \mathbb{P}\mathcal{L}_i \subset \mathbb{P}\mathcal{E}$ for line subbundles $\mathcal{L}_i$ for $i = 1, 2$.

By Theorem 7.8, we have

$$A(\mathbb{P}\mathcal{E}) = A(X)[\zeta]/(\zeta^2 + c_1(\mathcal{E})\zeta),$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$. In particular, by Lemma 7.10

$$\deg c_1(\mathcal{E})\zeta = \deg \pi_* c_1(\mathcal{E})\zeta = \deg c_1(\mathcal{E}).$$

It follows that $\deg \zeta^2 = -\deg c_1(\mathcal{E})$. By Proposition 7.12,

$$[\Sigma_i] = \zeta + c_1(\mathcal{E}/\mathcal{L}_i) = \zeta + c_1(\mathcal{E}) - c_1(\mathcal{L}_i).$$

Thus we obtain that

$$\begin{aligned} 0 &> \deg[\Sigma_i]^2 \\ &= \deg(\zeta + c_1(\mathcal{E}) - c_1(\mathcal{L}_i))^2 \\ &= \deg \zeta^2 + 2\deg c_1(\mathcal{E}) - 2\deg \mathcal{L}_i \\ &= \deg c_1(\mathcal{E}) - 2\deg \mathcal{L}_i. \end{aligned}$$

It implies that $2\deg \mathcal{L}_i > \deg c_1(\mathcal{E})$.

Since $\Sigma_1 \neq \Sigma_2$, we have an exact sequence

$$0 \to \mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{E} \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ has finite support at the locus of points x where $\mathcal{L}_1|_x$ and $\mathcal{L}_2|_x$ fail to be linearly independent in $\mathcal{E}_x$. We conclude that $\deg c_1(\mathcal{E}) \geq \deg \mathcal{L}_1 + \deg \mathcal{L}_2 > \deg c_1(\mathcal{E})$, leading to a contradiction. □

Remark 7.15. It is possible to construct a (non-ruled) surface S such that S contains infinitely many irreducible curves of negative self-intersection. For example, take S to be $\mathbb{P}^2$ blown up at (at least) 9 points.

7.6. **The zero section of a vector bundle.** For a vector bundle $\pi : \mathcal{E} \to X$ of rank r, define $\overline{\mathcal{E}} = \mathbb{P}(\mathcal{E} \oplus \mathscr{O}_X)$. In terms of local coordinates, $\mathcal{E}$ can be regarded as the complement of $\mathbb{P}\mathcal{E}$ in $\overline{\mathcal{E}}$, where $\mathbb{P}\mathcal{E}$ restricted to each fiber is a hyperplane by setting the last coordinate to be zero. In other words, $\overline{\mathcal{E}}$ is a natural compactification of $\mathcal{E}$ by adding a hyperplane at the infinity in each fiber.

Since $c(\mathcal{E} \oplus \mathscr{O}_X) = c(\mathcal{E})$, we have

$$A(\overline{\mathcal{E}}) = A(X)[\zeta]/\big(\zeta^{r+1} + c_1(\mathcal{E})\zeta^r + \cdots + c_r(\mathcal{E})\zeta\big),$$

where $\zeta = c_1(\mathscr{O}_{\overline{\mathcal{E}}}(1))$.

Applying Proposition 7.12 to $\mathcal{E} \hookrightarrow \mathcal{E} \oplus \mathscr{O}_X$, we obtain that

$$[\mathbb{P}\mathcal{E}] = \zeta + c_1(\pi^* \mathscr{O}_X) = \zeta.$$

Let $\mathbb{P}\mathscr{O}_X \subset \overline{\mathcal{E}}$ be the locus where all the coordinates of $\mathcal{E}$ vanish. It is just the zero section of $\mathcal{E}$. Denote this section by Σ_0. Applying Proposition 7.12 to $\mathscr{O}_X \hookrightarrow \mathcal{E} \oplus \mathscr{O}_X$, we obtain that

$$[\Sigma_0] = \zeta^r + c_1(\mathcal{E})\zeta^{r-1} + \cdots + c_r(\mathcal{E}).$$

More generally, if τ is a section of $\pi : \mathcal{E} \to X$, then $(\tau, 1)$ is a nowhere vanishing section of $\mathcal{E} \oplus \mathscr{O}_X$. Hence it gives rise to a section Σ_τ of $\overline{\mathcal{E}}$. Using the family $\Sigma_{t\tau}$ as $t \to 0$, one sees that $[\Sigma_0] = [\Sigma_\tau]$, where Σ_0 can be regarded as the section associated to $(\underline{0}, 1)$.

Proposition 7.16. *If τ vanishes along a codimension r subscheme $V(\tau)$ in X, then $\pi_*([\Sigma_0]^2) = c_r(\mathcal{E}) \in A^r(X)$.*

Proof. By the degeneracy locus definition of Chern classes, we have

$$\pi_*([\Sigma_0]^2) = \pi_*([\Sigma_0][\Sigma_\tau]) = [V(\tau)] = c_r(\mathcal{E}).$$

□

The above equality indeed holds in general without the assumption that τ vanishes in a codimension r.

Proposition 7.17. *Let $\iota : X \to \overline{\mathcal{E}}$ be the map whose image is the section Σ_0. We have*

$$\pi_*([\Sigma_0]^2) = c_r(\mathcal{E}).$$

Proof. We know that

$$[\Sigma_0] = \zeta^r + c_1(\mathcal{E})\zeta^{r-1} + \cdots + c_r(\mathcal{E}).$$

Since Σ_0 is disjoint with $\mathbb{P}\mathcal{E}$ and $[\mathbb{P}\mathcal{E}] = \zeta$, we obtain that $[\Sigma_0]\zeta = 0$. Therefore,

$$\begin{aligned} [\Sigma_0]^2 &= [\Sigma_0]\Big(\zeta^r + c_1(\mathcal{E})\zeta^{r-1} + \cdots + c_r(\mathcal{E})\Big) \\ &= [\Sigma_0]c_r(\mathcal{E}). \end{aligned}$$

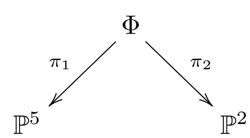
It follows that

$$\pi_*([\Sigma_0]^2) = (\pi_*\Sigma_0)c_r(\mathcal{E}) = c_r(\mathcal{E}).$$

□

7.7. **Brauer-Severi varieties.** A *Brauer-Severi variety* is a variety Y with a morphism $\pi : Y \to X$ such that all the fibers of π are isomorphic to $\mathbb{P}^r$ for a fixed r. Projective bundles are Brauer-Severi varieties, but the converse is false, illustrated in the following example.

Let $\Phi = \{(C, p) \in \mathbb{P}^5 \times \mathbb{P}^2\}$ be the universal conic. Consider the two projections:



Via π_2, Φ is a $\mathbb{P}^4$-bundle over $\mathbb{P}^2$, hence it is smooth. Indeed, it is the projectivization of a rank 5 subbundle $\mathcal{E} \hookrightarrow \mathrm{Sym}^2 V$, where $V = \mathbb{P}^2 \times H^0(\mathscr{O}_{\mathbb{P}^2}(1))$, $\mathcal{E}_p$ is the subspace of quadratic forms vanishing at p. In particular, the universal line bundle $\mathscr{O}_{\mathbb{P}\mathcal{E}}(1)$ can be identified with $\pi_1^*(\mathscr{O}_{\mathbb{P}^5}(1))$. By Theorem 7.8, $A^1(\Phi)$ is generated by the pullbacks of the hyperplane classes from $\mathbb{P}^5$ and $\mathbb{P}^2$. Note that they restrict to classes of degrees 2 and 0 to any fiber of π_1. Thus the intersection of the fiber of π_1 with any divisor on Φ has even degree.

Let $X \subset \mathbb{P}^5$ be the open locus of smooth conics. Let $\pi : \Phi_X \to X$ be the restriction of π_1 over X. Every fiber of π is isomorphic to $\mathbb{P}^1$. However, we claim that Φ_X is *not* a projective bundle over X. If there exists a Zariski open set $U \subset X$ such that $U \times \mathbb{P}^1 \cong \Phi_U = \pi^{-1}(U)$, take a section of π_U and consider its closure in Φ. Since Φ is smooth, it gives rise to a Cartier divisor, which intersects a general fiber of π at a reduced point, leading to a contradiction.

As a result, there do not exist rational functions $X(a,b,c,d,e,f)$, $Y(a,b,c,d,e,f)$ and $Z(a,b,c,d,e,f)$ such that

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ \equiv 0,$$

for otherwise we would obtain a section of π_U for some open subset $U \subset \mathbb{P}^5$.

Remark 7.18. Under the analytic topology, $\pi : \Phi_X \to X$ is a projective bundle. Let $C_0 \in X$ be a smooth conic. Choose two general lines $L, M \subset \mathbb{P}^2$. In a small analytic neighborhood U of $C_0 \in X$, we can solve analytically for a point $q_C \in C \cap L$.

Projecting C to M from q_C, it gives an identification of C with M. We thus obtain a trivialization $\pi^{-1}(U) \cong U \times M \cong U \times \mathbb{P}^1$.

8. Segre class

Let $\mathcal{E}$ be a vector bundle on X. Suppose that $\mathcal{E}$ is globally generated. How many global sections does it take to generate $\mathcal{E}$? Moreover, given a set of global sections of $\mathcal{E}$, what is the class of the locus $p \in X$ where they fail to generate $\mathcal{E}_p$?

Example 8.1. If $\mathcal{E}$ is a line bundle, then it is base point free. The vanishing locus of a section of $\mathcal{E}$ is a codimension-one divisor in X. Hence the common vanishing locus of i general sections has codimension-i in X with class $c_1(E)^i$.

Example 8.2. If the rank of $\mathcal{E}$ is $r > 1$, we need (at least) r general global sections to generate $\mathcal{E}_p$ at a general point $p \in X$. In this case, their common vanishing locus has codimension-one with class $c_{r-r+1}(E) = c_1(E)$.

In general, this sort of questions can be effectively answered by using Segre classes, which play an "orthogonal" role of Chern classes.

Definition 8.3 (Segre class)**.** Let $\mathcal{E}$ be a vector bundle of rank r on X, let $\pi : \mathbb{P}\mathcal{E} \to X$ and $\zeta = c_1(\mathscr{O}_{\mathbb{P}\mathcal{E}}(1))$. Define the i-th *Segre class*

$$s_i(\mathcal{E}) = \pi_*(\zeta^{r-1+i}) \in A^i(X)$$

and the *total Segre class*

$$s(\mathcal{E}) = 1 + s_1(\mathcal{E}) + s_2(\mathcal{E}) + \cdots$$

Proposition 8.4. *The total Segre class and Chern class are reciprocals in the Chow ring of X:*

$$s(\mathcal{E})c(\mathcal{E}) = 1 \in A(X).$$

Proof. Suppose that $\mathcal{E}$ has rank r. Let S be the tautological bundle with $c(S) = 1 - \zeta$. By the Euler sequence

$$0 \to S \to \pi^*\mathcal{E} \to Q \to 0,$$

we have

$$c(Q) = \frac{\pi^* c(\mathcal{E})}{c(S)} = (\pi^* c(\mathcal{E}))(1 + \zeta + \zeta^2 + \cdots).$$

Since Q has rank $r - 1$, which equals the relative dimension of $\mathbb{P}\mathcal{E}$ over X, we know that $\pi_* c_i(Q) = 0$ for $i < r - 1$. The top Chern class $c_{r-1}(Q) = \zeta^{r-1} + \zeta^{r-2}\pi^* c_1(E) + \cdots$, hence $\pi_* c_{r-1}(Q) = \pi_*(\zeta^{r-1})$ is the fundamental class of X. Therefore, we obtain that

$$1 = \pi_* Q = c(\mathcal{E})\pi_*(1 + \zeta + \zeta^2 + \cdots) = c(\mathcal{E})s(\mathcal{E}).$$

□

Corollary 8.5. *We have $s_i(\mathcal{E}^*) = (-1)^i s_i(\mathcal{E})$. Moreover, for an exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$, we have $s(\mathcal{F}) = s(\mathcal{E})s(\mathcal{G})$.*

Proof. It follows from Proposition 8.4 and the corresponding properties of Chern classes. □

Segre classes have an enumerative interpretation.

Proposition 8.6. *Let $\mathcal{E}$ be a vector bundle of rank r on X that is globally generated. Let $X_i \subset X$ be the locus where a given set of $r+i-1$ general global sections of $\mathcal{E}$ fail to generate $\mathcal{E}$. Then every component of X_i has codimension exactly i, and $s_i(\mathcal{E}^*)$ is represented by a positive linear combination of the components of X_i.*

Remark 8.7. We have the following comparison between Chern classes and Segre classes:

- $c_i(\mathcal{E})$ is the class of the locus of fibers where a general bundle map
$$\mathscr{O}_X^{\oplus r+1-i} \to \mathcal{E}$$
fails to be injective.
- $s_i(\mathcal{E})$ is $(-1)^i$ times the class of the locus of fibers where a general bundle map
$$\mathscr{O}_X^{\oplus r-1+i} \to \mathcal{E}$$
fails to be surjective.

Proof. Recall the identification between $H^0(\mathcal{E})$ and $H^0(\mathscr{O}_{\mathbb{P}\mathcal{E}^*}(1))$. A global section $\sigma \in H^0(\mathcal{E})$ specifies $\sigma(x) = v \in \mathcal{E}_x$. It induces a section $\tilde{\sigma} \in H^0(\mathscr{O}_{\mathbb{P}\mathcal{E}^*}(1))$ by $\tilde{\sigma}(x,[H]) = v/H \in \mathcal{E}_x/H = \mathscr{O}_{\mathbb{P}\mathcal{E}^*}(1)|_{x,[H]}$. Conversely, a section $\tilde{\sigma} \in H^0(\mathscr{O}_{\mathbb{P}\mathcal{E}^*}(1))$ restricted to $\mathbb{P}\mathcal{E}_x^*$ gives a section of $H^0(\mathscr{O}_{\mathbb{P}^{r*}}(1))$, which is of the form v/H at each $[H] \in \mathbb{P}\mathcal{E}_x$ for the same $v \in \mathcal{E}_x$. We thus obtain a natural identification between $H^0(\mathcal{E})$ and $H^0(\mathscr{O}_{\mathbb{P}\mathcal{E}^*}(1))$. Note that σ vanishes at x if and only if $\tilde{\sigma}$ vanishes entirely along $\mathbb{P}\mathcal{E}_x^*$.

Therefore, a collection of sections σ_k of $\mathcal{E}$ generates $\mathcal{E}_x$ if and only if the corresponding sections $\tilde{\sigma}_k$ generate $\mathscr{O}_{\mathbb{P}\mathcal{E}^*}(1)$ at $(x,[H])$ for every $[H] \in \mathcal{E}_x^*$. Define $Y_i \subset \mathbb{P}\mathcal{E}^*$ to be the locus where $\mathscr{O}_{\mathbb{P}\mathcal{E}^*}(1)$ is not generated by the $\tilde{\sigma}_k$. Then Y_i is the intersection of the $r+i-1$ divisors $D_k = \{y \in \mathbb{P}\mathcal{E}^* \mid \tilde{\sigma}_k(y) = 0\}$ and $X_i = \pi(Y_i)$.

Since $\mathcal{E}$ is generated by global sections, so is $\pi^*\mathcal{E}$ and $\mathscr{O}_{\mathbb{P}\mathcal{E}^*}(1)$. Since the $r+i-1$ sections are general, we conclude that every component of Y_i has codimension exactly $r+i-1$, i.e. Y_i has dimension equal to $\dim X - i$. Consequently X_i as the image of Y_i has dimension at most $n-i$, where $n = \dim X$. On the other hand, by the above remark X_i has a determinantal structure – it is defined locally by the vanishing of the $r \times r$ minors of an $r \times (r+i-1)$ matrix. Hence the codimension of X_i in X is at most i (unless it is empty). Combining the two bounds, it implies that $\dim X_i = n-i$. Since a component of X_i is the image of some component of Y_i, the same argument implies that every component of X_i has dimension equal to $n-i$. It also implies that $\pi : Y_i \to X_i$ is a generically finite map, hence $\pi_*[Y_i]$ is a positive linear combination of the components of X_i. □

Example 8.8. By the Euler sequence, we know that $T_{\mathbb{P}^n}$ is globally generated. Take $2n-1$ general sections of $T_{\mathbb{P}^n}$. At how many points of $\mathbb{P}^n$ do they fail to generate $T_{\mathbb{P}^n}$? In this setting $n+i-1 = 2n-1$ and $i = n$, hence the answer is given by $(-1)^n \cdot s_n(T_{\mathbb{P}^n})$. Let h be the hyperplane class of $\mathbb{P}^n$. We have
$$s(T_{\mathbb{P}^n}) = \frac{1}{c(T_{\mathbb{P}^n})} = \frac{1}{(1+h)^{n+1}} = 1 - (n+1)h + \binom{n+2}{2}h^2 + \cdots,$$
hence $(-1)^n \cdot s_n(T_{\mathbb{P}^n}) = \binom{2n}{n}$.

Example 8.9. Let $B \subset \mathbb{G}(k,n)$ be a subvariety of dimension m. Let
$$X = \bigcup_{b \in B} \Lambda_b \subset \mathbb{P}^n$$

be the union of the k-planes parameterized by B. More precisely, consider the universal k-planes

$$\Phi = \{(\Lambda, p) \mid p \in \Lambda\} \subset B \times \mathbb{P}^n.$$

Then X is the image of the projection $\eta : \Phi \to \mathbb{P}^n$. Since $\dim \Phi = m + k$, we conclude that $\dim X \leq m + k$.

Assume that $\dim X = m+k$. Then η is generically finite. Suppose that $\deg \eta = d$. It means that for a general point $p \in X$, there are d planes in B containing p. Take a general $(n-m-k)$-plane Γ in $\mathbb{P}^n$. Then $\deg(X)$ is the number of times X meets Γ. Since $\deg(\sigma_m \cdot [B])$ is the number of k-planes in B that meet Γ, we conclude that

$$\deg(X) = \frac{1}{d} \deg(\sigma_m \cdot [B]).$$

We can also interpret the above by a Segre class. Let $\mathcal{E}$ be the universal bundle of rank $k+1$ over B. Let L be a linear form on $\mathbb{P}^n$. Then L defines a section of $\mathcal{E}^*$ by restriction, hence a section σ_L of $\mathscr{O}_{\mathbb{P}\mathcal{E}}(1)$. Let $H = \ker(L) \subset \mathbb{P}^n$ a hyperplane. Then $\eta^{-1}H$ is the zero locus of σ_L, hence $[\eta^{-1}H] = c_1(\mathscr{O}_{\mathbb{P}\mathcal{E}}(1)) = \zeta$. It follows that

$$d \deg(X) = \deg \zeta^{m+k} = \deg s_m(\mathcal{E}).$$

9. Secant varieties

9.1. **Secant varieties in general.** Let $X \subset \mathbb{P}^r$ be a nondegenerate subvariety of dimension n. For any $m \leq r$, let $\mathrm{Sec}_m(X) \subset \mathbb{P}^r$ be the union of $(m-1)$-planes spanned by all possible collections of m points $p_1, \ldots, p_m \in X$. We call $\mathrm{Sec}_m(X)$ the *m-th secant variety of X*. More precisely, let $X^{(m)}$ be the m-th symmetric product of X. Consider the rational map $\tau : X^{(m)} \dashrightarrow \mathbb{G}(m-1, r)$, sending a general collection of $p_1, \ldots, p_m \in X$ to $\Lambda = \overline{p_1 \ldots p_m} \in \mathbb{G}(m-1, r)$. Let $\Sigma_m(X)$ be the closure of the image of τ. Then

$$\mathrm{Sec}_m(X) = \bigcup_{\Lambda \in \Sigma_m(X)} \Lambda \subset \mathbb{P}^r.$$

The following lemma is useful and we skip its proof.

Lemma 9.1 (Uniform Position Lemma)**.** *Let $X \subset \mathbb{P}^r$ be a nondegenerate subvariety of dimension n. Let $\Gamma \subset \mathbb{P}^r$ be a general $(r-n)$-plane. Then the points of $\Gamma \cap X$ are in linear general position, i.e. any $r-n+1$ of them span Γ.*

Proposition 9.2. *Let $X \subset \mathbb{P}^r$ be a nondegenerate subvariety of dimension n. If $m \leq r-n$, then $\tau : X^{(m)} \dashrightarrow \mathbb{G}(m-1, r)$ is birational and $\dim \Sigma_m(X) = \dim X^{(m)} = mn$.*

Proof. It suffices to show that if $p_1, \ldots, p_m \in X$ are general points, then $\Lambda = \overline{p_1 \ldots p_m}$ does not contain any other points of X.

Let $U \subset X^{(m)}$ be the open subset of m-tuples of distinct, linearly independent points. Consider the incidence correspondence

$$\Psi = \{(p_1 + \cdots + p_m, \Gamma) \in U \times \mathbb{G}(r-n, r) \mid p_1, \ldots, p_m \in \Gamma\}.$$

Via the projection $\Psi \to U$, we see that Ψ is irreducible. By the Uniform Position Lemma, the other projection $\Psi \to \mathbb{G}(r-n, r)$ is dominant. It follows that a general $\Gamma \in \mathbb{G}(r-n, r)$ containing m general points $p_1, \ldots, p_m \in X$ is a general $(r-n)$-plane, hence by the Uniform Position Lemma again $\overline{p_1 \ldots p_m}$ does not contain any other points of X. □

Let $\Phi = \{(\Lambda, p) \in \mathbb{G}(m-1, r) \times \mathbb{P}^r \mid p \in \Lambda\}$ be the universal $(m-1)$-plane. It admits two projections

$$\begin{array}{ccc} & \Phi & \\ {}^{\eta}\swarrow & & \searrow^{\pi} \\ \mathbb{P}^r & & \mathbb{G}(m-1, r) \end{array}$$

Let $\Phi_m(X) = \pi^{-1}(\Sigma_m(X))$, hence $\mathrm{Sec}_m(X) = \eta(\Phi_m(X))$. We call $\Phi_m(X)$ the *m-th abstract secant variety of X*. Since $\Sigma_m(X)$ is birational to $X^{(m)}$, it is irreducible, and hence $\Phi_m(X)$ is irreducible of dimension $mn + m - 1$. It follows that $\dim \mathrm{Sec}_m(X) \leq mn+m-1$, with equality holding if a general point of $\mathrm{Sec}_m(X)$ lies on only finitely many m-secant $(m-1)$-planes to X. Therefore, it makes sense to call $\min(mn+m-1, r)$ the *expected dimension* of $\mathrm{Sec}_m(X)$. In addition, we say that X is *m-defective* if $\dim \mathrm{Sec}_m(X) < \min(mn + m - 1, r)$ for some m.

Proposition 9.3. *The Veronese embedding of $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ given by*

$$[x_0, x_1, x_2] \mapsto [x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2]$$

is 2-defective.

Proof. Let X be the Veronese image of $\mathbb{P}^2$ in $\mathbb{P}^5$. Consider the symmetric 3×3 matrix

$$M = \begin{pmatrix} z_{0,0} & z_{0,1} & z_{0,2} \\ z_{0,1} & z_{1,1} & z_{1,2} \\ z_{0,2} & z_{1,2} & z_{2,2} \end{pmatrix}.$$

If $z_{i,j} = x_i x_j$ for all $0 \leq i, j \leq 2$, then the determinants of all the 2×2 minors of M are zero. It implies that the rank of M at any point $p \in X \subset \mathbb{P}^5$ is at most one. Hence the rank of M at any point in the linear span $\overline{pq}$ for $p, q \in X$ is at most two, i.e. the determinant of M is zero at any point contained in $\mathrm{Sec}_2(X)$. Therefore, $\mathrm{Sec}_2(X)$ is contained in the vanishing locus of the cubic form $\det M$, hence $\dim \mathrm{Sec}_2(X) \leq 4 < 2 \times 2 + 1 = 5$. □

9.2. **Secant varieties of rational normal curves.** Recall that a rational normal curve in $\mathbb{P}^d$ is the embedding of $\mathbb{P}^1$ via the complete linear system $\mathscr{O}_{\mathbb{P}^1}(d)$.

Lemma 9.4. *Let $C \subset \mathbb{P}^d$ be a rational normal curve. If $D \subset C$ is a divisor of degree $m \leq d+1$, then D is not contained in any linear subspace of $\mathbb{P}^d$ of dimension $< m - 1$. In other words, finite sets of points on C are as independent as possible.*

Proof. If D is contained in a linear subspace of dimension $n < m - 1$, then adding $d - n - 1$ points to D, we obtain a divisor $D' \subset C$ of degree $d + m - n - 1$ such that D' is contained in a linear subspace D of dimension $n + (d - n - 1) = d - 1$, i.e. H is a hyperplane. Since C is nondegenerate, it implies that

$$d = \deg(C) = \deg(C \cap H) \geq \deg(D') > d,$$

leading to a contradiction. □

Proposition 9.5. *Let $C \subset \mathbb{P}^d$ be a rational normal curve. Then for $2m - 1 \leq d$, the map $\eta : \Phi_m(C) \to \mathbb{P}^d$ is birational onto the image $\mathrm{Sec}_m(C)$ and is one-to-one over the complement of $\mathrm{Sec}_{m-1}(C)$.*

Proof. A point of $\Phi_m(C)$ can be identified with $(\overline{D}, p)$, where D is a divisor of degree m and $p \in \overline{D}$. Suppose that there exist two points $(\overline{D}, p)$ and $(\overline{D'}, p)$ in $\Phi_m(C)$, both mapping to $p \in \mathrm{Sec}_m(C)$. Let $k = \dim(\overline{D} \cap \overline{D'})$, then $0 \leq k \leq m-2$. Since the span of D and D' has dimension $2m-2-k$, the divisor D union D' (as a subscheme of C) has degree at most $2m-1-k$ by the preceding lemma. It follows that the divisor $D \cap D'$ has degree at least $k+1$. Hence $\overline{D} \cap \overline{D'}$ contains a $(k+1)$-secant k-plane of C, i.e. $p \in \mathrm{Sec}_{k+1}(C) \subset \mathrm{Sec}_{m-1}(C)$. □

Remark 9.6. When $2m-1 = d$, the result implies that a general point of $\mathbb{P}^d$ lies in a unique $(m+1)$-secant m-plane of a rational normal curve.

Next, we study the degrees of secant varieties of a rational normal curve $C \subset \mathbb{P}^d$. Some special cases can be worked out by elementary methods.

Example 9.7. Since $S_1(C) = C$, $\deg S_1(C) = d$. The variety $S_2(C)$ has dimension 3, hence its degree is the number of points in its intersection with a general $(d-3)$-plane Λ. Project C from Λ to a $\mathbb{P}^2 \subset \mathbb{P}^d$. This number corresponds to the number of nodes in the image of C, which is $\binom{d-1}{2}$ by the genus formula. Hence $\deg \mathrm{Sec}_2(C) = \binom{d-1}{2}$. Finally, if d is odd and $2m-1 = d$, then $S_m(C) = \mathbb{P}^d$ hence $\deg S_m(C) = 1$.

In general, identify the space X of degree m divisors in $\mathbb{P}^1$ (or equivalently, the space of m-secant $(m-1)$-planes of C) with $\mathrm{Sym}^m(\mathbb{P}^1) \cong \mathbb{P}^m$. Then $\pi : \Phi_m(C) \to \mathbb{P}^m$ sending $(\Lambda = \overline{D}, p \in \Lambda)$ to $[D] \in \mathbb{P}^m$ is $\mathbb{P}\mathcal{E}$ restricted to $X \cong \mathbb{P}^m \subset G(m, d+1)$, where $\mathcal{E}$ is the tautological bundle of $G(m, d+1)$. Thus the question transforms to a calculation of Segre classes of this bundle.

Theorem 9.8. *Let $C \subset \mathbb{P}^d$ be a rational normal curve. For $2m-1 \leq d$,*

$$\deg \mathrm{Sec}_m(C) = \binom{d-m+1}{m}.$$

Moreover, suppose that $m \leq d$ and let ζ denote the hyperplane class of $X \cong \mathbb{P}^m \subset G(m, d+1)$. Then

$$s(\mathcal{E}) = (1+\zeta)^{d-m+1},$$
$$c(\mathcal{E}) = \sum_i (-1)^i \binom{d-m+i}{i} \zeta^i.$$

Proof. By Example 8.9, $\deg \mathrm{Sec}_m(C) = s_m(\mathcal{E})$, where $\mathcal{E}$ is the universal bundle. Moreover, $s(\mathcal{E})c(\mathcal{E}) = 1$, hence it suffices to prove the last identity about the Chern classes of $\mathcal{E}$, i.e. to prove that $c_i(\mathcal{E}^*) = \binom{d-m+i}{i}$.

The ambient space $\mathbb{P}^d$ can be identified with $\mathbb{P}H^0(\mathscr{O}_{\mathbb{P}^1}(d))$, which is the space of d-forms on $\mathbb{P}^1$. Then the fiber of $\mathcal{E}$ over $\overline{D}$ can be identified with $H^0(\mathscr{O}_D(d))$.

Let $F_1, \ldots, F_{m-i+1}$ be a collection of $m-i+1$ general d-forms. Each F_j induces a section σ_j of $\mathcal{E}$ by restriction to D. Then the locus $V \subset X \cong \mathbb{P}^m$ where $\sigma_1, \ldots, \sigma_{m-i+1}$ are dependent consists of degree m divisors D of C such that some nontrivial linear combination of $F_1, \ldots, F_{m-i+1}$ vanishes on D. If this locus has codimension i (and reduced), then the associated cycle represents $c_i(\mathcal{E})$. Since $c_i(\mathcal{E}) = \alpha\zeta^i \in A^i(\mathbb{P}^m)$ for some α, we can determine α by restricting $c_i(\mathcal{E})$ to an i-plane $\Lambda \subset \mathbb{P}^m$ and taking the intersection degree.

Take $p_1, \ldots, p_{m-i} \in C$ general points and consider

$$\Lambda = \{D \in \mathbb{P}^m \mid p_1, \ldots, p_{m-i} \in D\}.$$

Then $\Lambda \cap V$ consists of degree m divisors D such that $p_1, \ldots, p_{m-i} \in D$ and some nontrivial linear combination $\sum_{j=1}^{m-i+1} a_j F_j$ vanishes on D. It implies that $\sum_{j=1}^{m-i+1} a_j F_j$ vanishes at $p_1, \ldots, p_{m-i}$, hence up to scalar multiple only one linear combination $\sum_{j=1}^{m-i+1} a_j F_j$ vanishes at those points. Since $p_1, \ldots, p_{m-i}$ are general, the divisor of F is a sum of distinct points

$$(F) = p_1 + \cdots + p_{m-i} + q_1 + \cdots + q_{d-m+i}.$$

It follows that $D \in \Lambda \cap V$ is of the form

$$D = p_1 + \cdots + p_{m-i} + q_{\alpha_1} + \cdots + q_{\alpha_i},$$

where $\{\alpha_1, \ldots, \alpha_i\} \subset \{1, \ldots, d-m+i\}$. There are $\binom{d-m+i}{i}$ choices for such D, thus proving the theorem. □

Theorem 9.8 has the following interesting application. If we realize $\mathbb{P}^d$ as the projective space of forms of degree d on $\mathbb{P}^1$, then the curve C of pure d-th powers of linear forms is a rational normal curve in $\mathbb{P}^d$. Indeed, write $[X, Y]$ for the coordinates of $\mathbb{P}^1$. A general d-form on $\mathbb{P}^1$ is $a_0 X^n + a_1 X^{n-1} Y + \cdots + a_n Y^n$. If it is $(sX + tY)^d$, then the coefficients are $s^d, ds^{d-1}t, \binom{d}{2} s^{d-2} t^2, \ldots, t^d$. Hence C is the image $[s, t] \mapsto [s^d, ds^{d-1}t, \binom{d}{2} s^{d-2} t^2, \ldots, t^d]$.

In this setting, a point $p \in \mathbb{P}^d$ lies on the plane spanned by distinct points $q_1, \ldots, q_m \in C$ if and only if the homogeneous coordinates of p can be expressed as a linear combination of the homogeneous coordinates of $q_1, \ldots, q_m$. In other words, a form of degree d is a linear combination of m d-th powers of linear forms if and only if the corresponding point in $\mathbb{P}^d$ lies in the union of the m-secant $(m-1)$-planes to C.

Corollary 9.9. *Suppose that $d \geq 2m-1$. Then the number of linear combinations of $d - 2m + 2$ general forms of degree d on $\mathbb{P}^1$ that can be expressed as the sum of m pure d-th powers is*

$$\deg \operatorname{Sec}_m(C) = \binom{d-m+1}{m}.$$

Example 9.10. Let f and g be general polynomials of degree $d = 2m$ in one variable. Then the number of linear combinations of f and g that are expressible as a sum of m pure d-th powers of linear forms is $m+1$. For instance, take $m = 1$, $f = a_2 x^2 + a_1 x + a_0$ and $g = b_2 x^2 + b_1 x + b_0$. Then $\lambda f + \mu g$ is a square of a one form if and only if $(\lambda a_1 + \mu b_1)^2 - 4(\lambda a_2 + \mu b_2)(\lambda a_0 + \mu b_0) = 0$, which has degree 2.

9.3. **Special secant planes.** Let $C \subset \mathbb{P}^d$ be a rational normal curve. Consider the secant plane map

$$\tau : \operatorname{Sym}^m C \cong \mathbb{P}^m \to \mathbb{G}(m-1, d)$$

and denote the image by $\Sigma_m(C)$. Let S be the tautological bundle of $\mathbb{G}(m-1, d)$ and Q the quotient bundle. In Section 3, we have seen that

$$c(S^*) = 1 + \sigma_1 + \sigma_{1,1} + \cdots,$$

$$s(S) = c(Q) = 1 + \sigma_1 + \sigma_2 + \cdots.$$

In particular, $c_i(S^*) = \sigma_{1^i}$ and $s_i(S) = \sigma_i$. As a corollary of Theorem 9.8, we obtain that

$$\tau^* \sigma_{1^i} = \binom{d-m+i}{i} \zeta^i \in A^i(\mathbb{P}^m), \tag{1}$$

$$\tau^*\sigma_i = \binom{d-m+1}{i}\zeta^i \in A^i(\mathbb{P}^m). \tag{2}$$

Let us apply these formulas to study special secant planes of general rational curves.

Proposition 9.11. *Let $C \subset \mathbb{P}^4$ be a general rational curve of degree $d > 4$. Then C has $\binom{d-2}{3}$ trisecant lines.*

Proof. By assumption, C is the projection of a rational normal curve R in $\mathbb{P}^d$ from a general $(d-5)$-plane Λ. Let $\Sigma_3(\Lambda)$ be the Schubert cycle in $\mathbb{G}(2,d)$ of 2-planes that meet Λ. Then the number of trisecant lines of C equals the degree of $\Sigma_3(\Lambda)\cap\Sigma_3(R)$, i.e. the degree of $\tau^*\sigma_3$ on $\mathbb{P}^3$. By (2), this degree equals $\binom{d-3+1}{3} = \binom{d-2}{3}$. □

Proposition 9.12. *Let $C \subset \mathbb{P}^3$ be a general rational curve of degree $d > 3$. Then the degree of the surface S swept out by trisecant lines of C is $2\binom{d-1}{3}$.*

Proof. By assumption, C is the projection of a rational normal curve R in $\mathbb{P}^d$ from a general $(d-4)$-plane Λ. Let $L \subset \mathbb{P}^3$ be a general line and let Γ be the the preimage of L in $\mathbb{P}^d$ under the projection, which is a $(d-2)$-plane. The points of intersection of L with S correspond to 3-secant 2-planes P of R that meet Λ and intersect Γ in a line (spanned by the intersection of P with Λ and the intersection of the corresponding trisecant line with L). We thus obtain that

$$\deg S = \deg \tau^*(\sigma_{2,1}).$$

We may express

$$\sigma_{2,1} = \sigma_1\sigma_2 - \sigma_3.$$

Hence by (2) it follows that

$$\deg \tau^*(\sigma_{2,1}) = \binom{d-3+1}{1}\binom{d-3+1}{2} - \binom{d-3+1}{3} = 2\binom{d-1}{3}.$$

□

10. Porteous formula

Let $\phi : \mathcal{E} \to \mathcal{F}$ be a map between two vector bundles of rank e and f on X. Let $M_k(\phi) \subset X$ be the locus where ϕ_x has rank $\leq k$. We call such $M_k(\phi)$ *degeneracy loci* of ϕ. Choosing local sections of $\mathcal{E}$ and $\mathcal{F}$, $M_k(\phi)$ can be given a scheme structure by the vanishing of the $(k+1)\times(k+1)$ minors of a matrix representation of ϕ.

The following formula tells us how to calculate the class of the degeneracy loci. We leave the proof to the reader as a reading assignment.

Theorem 10.1 (Porteous formula)**.** *In the above setting, let c_i be the ith piece of $c(\mathcal{F})/c(\mathcal{E}) \in A(X)$. If the scheme $M_k(\phi) \subset X$ has codimension $(e-k)(f-k)$, then its class is given by*

$$[M_k(\phi)] = \det \begin{pmatrix} c_{f-k} & c_{f-k+1} & \cdots & c_{e+f-2k-1} \\ c_{f-k-1} & c_{f-k} & \cdots & c_{e+f-2k-2} \\ \vdots & \vdots & & \vdots \\ c_{f-e+1} & c_{f-e+2} & \cdots & c_{f-k} \end{pmatrix}.$$

Let us consider some applications of Porteous formula.

Theorem 10.2. *Let A be an $e \times f$ matrix of general linear forms on $\mathbb{P}^r$. Let $M_k \subset \mathbb{P}^r$ be the scheme defined by the $(k+1) \times (k+1)$ minors of A. Then the degree of M_k is*

$$\deg(M_k) = \prod_{i=0}^{e-k-1} \frac{i!(f+i)!}{(k+i)!(f-k+i)!}.$$

Proof. The matrix A induces a bundle map

$$\mathcal{E} = \mathscr{O}_{\mathbb{P}^r}^{\oplus e} \to \mathcal{F} = \mathscr{O}_{\mathbb{P}^r}(1)^{\oplus f}.$$

Let ζ be the hyperplane class of $\mathbb{P}^r$. Then

$$c(\mathcal{E}) = 1, \quad c(\mathcal{F}) = (1+\zeta)^f = \sum_{j=0}^{f} \binom{f}{j} \zeta^j.$$

By Porteous formula, the class of M_k is given by

$$[M_k] = \det \begin{pmatrix} \binom{f}{f-k}\zeta^{f-k} & \cdots & \binom{f}{e+f-2k-1}\zeta^{e+f-2k-1} \\ \vdots & & \vdots \\ \binom{f}{f-e+1}\zeta^{f-e+1} & \cdots & \binom{f}{f-k}\zeta^{f-k} \end{pmatrix}.$$

Simplifying the determinant, we thus obtain the desired formula. □

Let $\pi : S \to \mathbb{P}^3$ be a general map from a smooth surface S to $\mathbb{P}^3$. We say that $p \in S$ is a *pinch point* if $d\pi : T_S \to \pi^* T_{\mathbb{P}^3}$ fails to be injective at p.

Proposition 10.3. *Let $c_i = c_i(T_S^*)$ and ζ (the pullback of) the hyperplane class of $\mathbb{P}^3$. Then the number of pinch points of S is $6\zeta^2 + 4\zeta c_1 + c_1^2 - c_2$.*

Proof. We have $c(\pi^* T_{\mathbb{P}^3}) = (1+\zeta)^4 = 1 + 4\zeta + 6\zeta^2$. Therefore,

$$\begin{aligned} \frac{c(\pi^* T_{\mathbb{P}^3})}{c(T_S)} &= \frac{1+4\zeta+6\zeta^2}{1 - c_1 + c_2} \\ &= (1+4\zeta+6\zeta^2)(1 + c_1 + (c_2 - c_1^2)). \end{aligned}$$

By Porteous formula, the number of pinch points of S is given by the degree 2 piece of the above, i.e.

$$\deg[M_1(d\pi)] = 6\zeta^2 + 4\zeta c_1 + c_1^2 - c_2.$$

□

11. Excess intersection

11.1. **Excess intersection and normal bundles.** For two curves C, D intersecting transversally on a surface, the degree of $[C] \cdot [D]$ equals the number of their intersection points. What happens if C and D fail to intersect properly, or if they possess a common component? Well, we can still interpret it by the degree of the normal bundle N_D restricted to C.

Let us consider another example. Suppose that S, T and U are three surfaces in $\mathbb{P}^3$ of degree s, t and u, respectively, whose intersection consists of a smooth curve C of degree d and genus g as well as a 0-dimensional scheme Γ. What is the degree of Γ?

Suppose that

$$S \cap T = C + D, \quad S \cap U = C + E.$$

By the adjunction formula, on S we have

$$2g-2 = C\cdot C + C\cdot K_S,$$

hence

$$C\cdot C = 2g-2-(s-4)d.$$

Let H be the hyperplane class of $\mathbb{P}^3$. We know that

$$D \sim tH - C, \quad E \sim uH - C \in A^1(S).$$

It follows that

$$D\cdot E = stu - d(s+t+u-4) + 2g - 2.$$

Proposition 11.1. *In the above setting, the degree of* Γ *is*

$$\begin{aligned}\deg(\Gamma) &= stu - d(s+t+u-4) + 2g - 2\\ &= stu - \deg(N_{S/\mathbb{P}^3}|_C) - \deg(N_{T/\mathbb{P}^3}|_C) - \deg(N_{U/\mathbb{P}^3}|_C) + \deg(N_{C/\mathbb{P}^3}).\end{aligned}$$

Proof. All we need is to calculate the degrees of the normal bundles. We have

$$\deg(N_{S/\mathbb{P}^3}|_C) = \deg(\mathscr{O}_{\mathbb{P}^3}(S)|_C) = ds.$$

By the Euler sequence

$$0 \to T_C \to T_{\mathbb{P}^3}|_C \to N_{C/\mathbb{P}^3} \to 0,$$

we have

$$\begin{aligned}\deg(N_{C/\mathbb{P}^3}) &= \deg T_{\mathbb{P}^3}|_C - \deg T_C\\ &= \deg(\mathscr{O}_{\mathbb{P}^3}(1)^{\oplus 4}|_C) - (2-2g)\\ &= 4d + 2g - 2.\end{aligned}$$

The desired expression follows right away. □

Example 11.2. If S, T and U are three planes containing a common line, then $\deg(\Gamma) = 0$ as predicted by the above result.

Corollary 11.3. *There exist smooth curves* C *in* $\mathbb{P}^3$ *that cannot be expressed as the (scheme-theoretic) intersection of three surfaces.*

Proof. For example, let C be a smooth elliptic quintic, i.e. $d = 5$ and $g = 1$. Then C does not lie in any planes or quadrics. On the other hand, for $s, t, u \geq 3$,

$$stu - d(s+t+u-4) + 2g - 2 = stu - 5(s+t+u-4) > 0.$$

Therefore, any surfaces S, T and U that contain C must contain some nonempty residual scheme of C. □

Next, let $S, T \subset \mathbb{P}^4$ be two surfaces of degree d and e, whose intersection consists of a smooth curve C and a finite set Γ of reduced points. What is the degree of Γ?

We want to deform $S = S_0$ and $T = T_0$ in two families S_λ and T_λ over $\lambda \in \Delta$ such that for $\lambda \neq 0$, S_λ and T_λ intersect transversally at de points $p_1(\lambda), \dots, p_{de}(\lambda)$. The question boils down to: as $\lambda \to 0$, how many of the points $p_i(\lambda)$ approach C and how many approach Γ?

Consider the universal surfaces $\mathcal{S}$ and $\mathcal{T}$:

$$\mathcal{S} = \{(\lambda, p) \in \Delta \times \mathbb{P}^4 \mid p \in S_\lambda\},$$

$$\mathcal{T} = \{(\lambda, p) \in \Delta \times \mathbb{P}^4 \mid p \in T_\lambda\},$$

In the fivefold $\Delta \times \mathbb{P}^4$, suppose that

$$\mathcal{S} \cap \mathcal{T} = \Phi \cup D,$$

where Φ is flat of degree de over Δ, consisting of the intersection $S_\lambda \cap T_\lambda$ for $\lambda \neq 0$ and their limits. On the other hand, $D = \{0\} \times C$ is the positive dimensional component of $S_0 \cap T_0$. We need to figure out the degree of $\Phi \cap D$.

Note that the points of $\Phi \cap D$ are *singular* points of $\mathcal{S} \cap \mathcal{T}$ occurring along D. If $\mathcal{S} \cap \mathcal{T}$ met transversally along D, we would have

$$T_D = T_{\mathcal{S}}|_D \cap T_{\mathcal{T}}|_D \subset T_{\Delta \times \mathbb{P}^4}|_D$$

and hence

$$N_{D/\Delta \times \mathbb{P}^4} = N_{\mathcal{S}/\Delta \times \mathbb{P}^4}|_D \oplus N_{\mathcal{T}/\Delta \times \mathbb{P}^4}|_D.$$

Rephrasing differently, we have a bundle map

$$N_{D/\Delta \times \mathbb{P}^4} \to N_{\mathcal{S}/\Delta \times \mathbb{P}^4}|_D \oplus N_{\mathcal{T}/\Delta \times \mathbb{P}^4}|_D$$

and the locus where this map fails to be surjective is the singular locus of $\mathcal{S} \cap \mathcal{T}$ along D.

Since $D = \{0\} \times C$, we have

$$N_{D/\Delta \times \mathbb{P}^4} = N_{C/\mathbb{P}^4} \oplus \mathscr{O}_C,$$

hence

$$c_1(N_{D/\Delta \times \mathbb{P}^4}) = c_1(N_{C/\mathbb{P}^4}).$$

Since S_0 is the transverse intersection of $\mathcal{S}$ with $\{0\} \times \mathbb{P}^4$, we have

$$N_{\mathcal{S}/\Delta \times \mathbb{P}^4}|_D = N_{S/\mathbb{P}^4}|_C.$$

Similarly,

$$N_{\mathcal{T}/\Delta \times \mathbb{P}^4}|_D = N_{T/\mathbb{P}^4}|_C.$$

By Porteous formula, we conclude that

$$\deg(\Phi \cap D) = c_1(N_{S/\mathbb{P}^4}|_C) + c_1(N_{T/\mathbb{P}^4}|_C) - c_1(N_{C/\mathbb{P}^4}),$$

and hence

$$\deg(\Gamma) = de - c_1(N_{S/\mathbb{P}^4}|_C) - c_1(N_{T/\mathbb{P}^4}|_C) + c_1(N_{C/\mathbb{P}^4}).$$

Example 11.4. Let $S, T \subset \mathbb{P}^4$ be two planes containing a common line. Then Γ is empty. The above formula yields $1 - 2 - 2 + 3 = 0$.

11.2. **Excess intersection formula.** Assume that X is a smooth projective variety of dimension n. Let $S, T \subset X$ be two smooth subvarieties of codimension k and l, respectively. Let $C = S \cap T$ be smooth, with connected components C_α of codimension $k + l - m_\alpha$.

We want to assign to each C_α a cycle class $\gamma_\alpha \in A^{m_\alpha}(C_\alpha)$, which represents the contribution of C_α to $S \cap T$. In other words, denote by $i_\alpha : C_\alpha \to X$ the inclusion. Then

$$\sum_\alpha (i_\alpha)_*(\gamma_\alpha) = [S] \cdot [T] \in A^{k+l}(X).$$

As before, we deform $S = S_0$ and $T = T_0$ in two families $\mathcal{S}$ and $\mathcal{T}$ over Δ. For $\lambda \neq 0 \in \Delta$, suppose that S_λ and T_λ intersect transversally. We want to figure out the limiting position of $S_\lambda \cap T_\lambda$ as $\lambda \to 0$.

In the $(n+1)$-fold $\Delta \times X$, suppose that

$$\mathcal{S} \cap \mathcal{T} = \Phi \cup D$$

where Φ is flat over Δ, consisting of $S_\lambda \cap T_\lambda$ for $\lambda \neq 0$ and their limits, and $D = \{0\} \times C$. Denote by $D_\alpha = \{0\} \times C_\alpha$ the components of D.

The cycle $\Gamma = \Phi \cap (\{0\} \times X)$ has dimension $n - (k+l)$ and

$$[\Gamma] = [S] \cdot [T] \in A^{k+l}(X).$$

Moreover, Γ consists of the union of $\Gamma_\alpha = \Phi \cap D_\alpha$. We need to figure out the class of $\Phi \cap D_\alpha$ for each component D_α.

The key observation is that the points $p \in \Phi \cap D_\alpha$ are the points where $T_p\mathcal{S}$ and $T_p\mathcal{T}$ fail to intersect properly in $T_p D_\alpha$. If they intersected properly, we would have

$$T_{D_\alpha} = T_{\mathcal{S}}|_{D_\alpha} \cap T_{\mathcal{T}}|_{D_\alpha} \subset T_{\Delta \times X}|_{D_\alpha}$$

hence the normal bundle of D_α would have a direct sum decomposition

$$N_{D_\alpha/\Delta \times X} = N_{\mathcal{S}/\Delta \times X}|_{D_\alpha} \oplus N_{\mathcal{T}/\Delta \times X}|_{D_\alpha}.$$

Rephrasing differently, we have the bundle map

$$N_{D_\alpha/\Delta \times X} \to N_{\mathcal{S}/\Delta \times X}|_{D_\alpha} \oplus N_{\mathcal{T}/\Delta \times X}|_{D_\alpha}$$

between two vector bundles of ranks $k + l - m_\alpha + 1$ and $k + l$. The cycle Γ_α is the locus where this map fails to be injective.

We have

$$N_{D_\alpha/\Delta \times X} = N_{C_\alpha/X} \oplus \mathscr{O}_{C_\alpha},$$

$$N_{\mathcal{S}/\Delta \times X}|_{D_\alpha} = N_{S/X}|_{C_\alpha},$$

$$N_{\mathcal{T}/\Delta \times X}|_{D_\alpha} = N_{T/X}|_{C_\alpha}.$$

By Porteous formula, we conclude that

$$\gamma_\alpha = [\Gamma_\alpha] = \left[\frac{c(N_{S/X}|_{C_\alpha}) \cdot c(N_{T/X}|_{C_\alpha})}{c(N_{C_\alpha/X})}\right]_{m_\alpha} \in A^{m_\alpha}(C_\alpha).$$

By the exact sequence

$$0 \to N_{C_\alpha/S} \to N_{C_\alpha/X} \to N_{S/X}|_{C_\alpha} \to 0,$$

we can rewrite the above as

$$\gamma_\alpha = \left[\frac{c(N_{T/X}|_{C_\alpha})}{c(N_{C_\alpha/S})}\right]_{m_\alpha} \in A^{m_\alpha}(C_\alpha).$$

By the exact sequence

$$0 \to N_{C_\alpha/T} \to N_{C_\alpha/X} \to N_{T/X}|_{C_\alpha} \to 0,$$

alternatively we have

$$\gamma_\alpha = \left[\frac{c(N_{C_\alpha/X})}{c(N_{C_\alpha/S})c(N_{C_\alpha/T})}\right]_{m_\alpha} \in A^{m_\alpha}(C_\alpha).$$

In summary, we thus obtain the following excess intersection formula.

Proposition 11.5 (Excess intersection formula)**.** *Let $S, T \subset X$ be two smooth subvarieties of codimension k and l, respectively. Let $C = S \cap T$ be smooth, with* *connected components C_α of codimension $k+l-m_\alpha$. Assuming that there exist deformations of S and T in X that intersect transversally. Then we have*

$$\begin{aligned}
[S]\cdot[T] &= \sum_\alpha (i_\alpha)_*\left(\left[\frac{c(N_{S/X}|_{C_\alpha})\cdot c(N_{T/X}|_{C_\alpha})}{c(N_{C_\alpha/X})}\right]_{m_\alpha}\right)\\
&= \sum_\alpha (i_\alpha)_*\left(\left[\frac{c(N_{T/X}|_{C_\alpha})}{c(N_{C_\alpha/S})}\right]_{m_\alpha}\right)\\
&= \sum_\alpha (i_\alpha)_*\left(\left[\frac{c(N_{C_\alpha/X})}{c(N_{C_\alpha/S})c(N_{C_\alpha/T})}\right]_{m_\alpha}\right) \in A^{k+l}(X).
\end{aligned}$$

One can generalize the formula to the intersection of more than two subvarieties.

Proposition 11.6. *Let X be a smooth projective variety. Let $S_1,\dots,S_l\subset X$ be subvarieties of codimension $k_1,\dots,k_l$, respectively. Let $k=\sum_i k_i$ be the expected codimension of their intersection. Suppose that $S_1\cap\cdots\cap S_l$ is a disjoint union of smooth subvarieties C_α of codimension $k=m_\alpha$. Assuming that $S_1,\dots,S_l$ admit deformations that intersect transversally, then*

$$\begin{aligned}
\prod_{i=1}^{l}[S_i] &= \sum_\alpha (i_\alpha)_*\left(\left[\frac{\prod_i c(N_{S_i/X}|_{C_\alpha})}{c(N_{c_\alpha/X})}\right]_{m_\alpha}\right)\\
&= \sum_\alpha (i_\alpha)_*\left(\left[\frac{c(N_{C_\alpha/X})^{l-1}}{\prod_i c(N_{C_\alpha/S_i})}\right]_{m_\alpha}\right) \in A^{k}(X).
\end{aligned}$$

Let us apply the excess intersection formula to the following example.

Example 11.7. Suppose that $Q_1,\dots,Q_k\subset\mathbb{P}^n$ are k general quadric hypersurfaces containing a common codimension-2 plane $\Lambda\subset\mathbb{P}^n$. Write

$$\bigcap_{i=1}^{k} Q_i = \Lambda\cup Z.$$

What is the degree of Z? We can reduce the question to the case $n=k$ by intersecting with $n-k$ general hyperplanes. It ensures that $\Lambda\cap Z=\emptyset$. Let ζ be the hyperplane class. We have

$$c(N_{Q_i/\mathbb{P}^n}|_\Lambda) = 1+2\zeta,$$

$$c(N_{\Lambda/\mathbb{P}^2}) = (1+\zeta)^2.$$

Thus by the excess intersection formula, the contribution of Λ to the degree of $\prod_i [Q_i]$ is given by

$$\lambda = \deg\left[\frac{(1+2\zeta)^k}{(1+\zeta)^2}\right]_{k-2} \in A^{k-2}(\Lambda)\cong\mathbb{Z}.$$

After simplifying, we obtain that

$$\lambda = 2^k-k-1.$$

It follows that

$$\deg(Z) = 2^k-\lambda = k+1.$$

11.3. **Intersections in a subvariety.** Let X be a smooth projective variety. Suppose that $Z \subset X$ is a smooth subvariety of codimension m. Let $A, B \subset Z$ be subvarieties of codimensions a and b in Z, respectively. Assume that A and B intersect transversally in Z. Then $A \cap B$ is of codimension $a+b$ in Z and hence of codimension $a+b+m$ in X. What can we say about the intersection product $[A] \cdot [B]$ in the Chow rings of Z and of X?

Theorem 11.8. *Let $i : Z \hookrightarrow X$ be an inclusion of smooth projective varieties of codimension m. Let $N = N_{Z/X}$ be the normal bundle of Z in X. Then we have*
(1) *For $\alpha \in A^a(Z)$, $i^*(i_*\alpha) = \alpha \cdot c_m(N) \in A^{a+m}(Z)$;*
(2) *For $\alpha \in A^a(Z)$ and $\beta \in A^b(Z)$, $(i_*\alpha)\cdot(i_*\beta) = i_*(\alpha\cdot\beta\cdot c_m(N)) \in A^{a+b+2m}(X)$.*

Outline of the proof. The tubular neighborhood of Z in X looks like a neighborhood of Z in $\mathbb{P}(N \oplus \mathscr{O}_Z)$. Hence (1) follows in the context of projective bundles. By the projection formula, (2) follows from (1). □

11.4. **Excess intersection in pullbacks.** There is a variation of the excess intersection formula in the context of maps.

Proposition 11.9 (Excess intersection formula for pullbacks)**.** *Let $f : X \to Y$ be a morphism of smooth projective varieties. Let $A \subset Y$ be a smooth subvariety of codimension a. Assume that $f^{-1}(A)$ is smooth. Then we have*

$$f^*[A] = \sum_B i_* \left[\frac{c(f^*N_{A/Y})}{c(N_{B/X})}\right]_{a-\operatorname{codim}(B\subset X)} \in A^a(X),$$

where the sum ranges over all connected components B of $f^{-1}(A)$ and $i : B \hookrightarrow X$ is the inclusion.

Outline of the proof. This follows from the excess intersection formula applied to the graph of f. □

Corollary 11.10. *Let $f : X \to Y$ be a generically finite surjective map of smooth projective varieties. If $q \in Y$ is any point such that $f^{-1}(q)$ is smooth (not necessarily finite), then*

$$\deg(f) = \sum_B \deg s_{\dim B}(N_{B/X}),$$

where the sum ranges over all connected components of $f^{-1}(q)$ and s denotes the Segre class.

Proof. Notice that $N_{q/Y}$ is a trivial vector bundle and $s(E) \cdot c(E) = 1$ for any vector bundle E. The desired formula follows from the excess intersection formula for pullbacks. □

Example 11.11. Let $f : X \to Y$ be the blowdown map, where X is the blowup of a smooth n-dimensional projective variety Y at a point q. The exceptional divisor $E = f^{-1}(q) \cong \mathbb{P}^{n-1}$ and its normal bundle $N_{E/X} \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$. We have

$$s(N_{E/X}) = \frac{1}{1-\zeta} = 1 + \zeta + \cdots + \zeta^{n-1}.$$

Hence $\deg(f) = \deg s_{n-1}(N_{E/X}) = \deg \zeta^{n-1} = 1$.

11.5. **The double point formula.** Let $f : X \to Y$ be a morphism between two smooth projective varieties. Assume that $\dim X = m \leq \dim Y = n$. Further assume that $X \to f(X)$ is birationally onto. We would like to know the class of the closure Φ_0 of the locus of pairs $(p, q) \in X \times X$ such that $p \neq q$ but $f(p) = f(q)$.

Consider the map $f \times f : X \times X \to Y \times Y$. Let $\Delta_Y \subset Y \times Y$ be the diagonal. The preimage $\Phi = (f \times f)^{-1}(\Delta_Y)$ contains the diagonal $\Delta_X \subset X \times X$ of codimension m and Φ_0 of (expected) codimension $2m - n$.

Proposition 11.12 (Double point formula)**.** *In the above setting, we have*

$$[\Phi_0] = (f \times f)^*[\Delta_Y] - \left[\frac{f^*c(T_Y)}{c(T_X)}\right]_{n-m}.$$

Proof. Note that the normal bundle of $\Delta_Y \subset Y \times Y$ is the tangent bundle of $\Delta_Y \cong Y$, and likewise for X. By the excess intersection formula for pullbacks, the contribution of $[\Delta_Y]$ is

$$i_* \left[\frac{f^*c(N_{\Delta_Y/Y \times Y})}{c(N_{\Delta_X/X \times X})}\right]_{m-(2m-n)} = i_* \left[\frac{f^*c(T_Y)}{c(T_X)}\right]_{n-m},$$

where $i : X \cong \Delta_X \hookrightarrow X \times X$ is the inclusion. The desired contribution of $[\Phi_0]$ thus follows. □

Example 11.13. Let $f : C \to \mathbb{P}^2$ be a map of degree d from a smooth curve C of genus g. Suppose that the image $C_0 \subset \mathbb{P}^2$ is a nodal curve. The number of nodes of C_0 is given by one-half of the degree of

$$(f \times f)^*[\Delta_{\mathbb{P}^2}] - \left[\frac{f^*c(T_{\mathbb{P}^2})}{c(T_C)}\right]_1.$$

We have seen that

$$[\Delta_{\mathbb{P}^2}] = \zeta_1^2 + \zeta_1\zeta_2 + \zeta_2^2,$$

where the ζ^i are the pullbacks of the hyperplane class from $\mathbb{P}^2$ via the ith projection for $i = 1, 2$, hence

$$\deg(f \times f)^*[\Delta_{\mathbb{P}^2}] = d^2.$$

Moreover,

$$c_1(T_{\mathbb{P}^2}) = 3\zeta,$$

hence

$$\deg \left[\frac{f^*c(T_{\mathbb{P}^2})}{c(T_C)}\right]_1 = 3d - (2 - 2g) = 2g - 2 + 3d.$$

In sum, the number of nodes of C_0 is

$$\frac{d^2 - (2g - 2 + 3d)}{2} = \binom{d-1}{2} - g = p_a - g,$$

where p_a is the arithmetic genus of C_0. This confirms the fact that the number of nodes in a nodal curve equals the difference of its arithmetic and geometric genera.