Harmonic spinors on axisymmetric 3-manifolds with Melvin ends

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Abstract. We prove the existence of harmonic spinor fields in axisymmetric Riemannian 3-manifolds having nonnegative scalar curvature and asymptotic to the usual constant time hypersurface of Melvin's magnetic universe. Such a spinor can be used in the proof of the uniqueness of the magnetized Schwarzschild solution. We also consider some rigidity type results for some cylindrical geometry.

Key words. Dirac equation, Cylindrical Ends, Magnetic Universe

1 Introduction

It appears that there are not many works on the Dirac equation on an asymptotically cylindrical Riemannian manifold, although the Laplacian and similar elliptic differential operators on complete manifolds with warped cylindrical ends have been studied (Lockhart and McOwen [2], Ma and McOwen [1] and references cited therein). On the other hand harmonic spinors are important tools for proving rigidity and uniqueness results in differential geometry and in the study of time-symmetric black-hole solutions of Einstein equation. Many of the physical problems in potential theory originate under the assumption of isolated bodies surrounded by empty space and under this setting asymptotically flat assumption naturally arises. Strictly speaking in the physical world existence of a magnetic field is more natural than empty space. Speaking very roughly the solenoidal nature of the magnetic field relates to lack of asymptotic spherical symmetry. Also one of the reasons for assuming asymptotical flatness is that it can provide finite energy and hence stable solutions. Magnetic fields provide an example of an infinite energy solution stable under radial perturbations. Thus in the Melvin magnetic universe (MMU) solution of Einstein-Maxwell equations diverging coaxial tubes of magnetic lines of force are held together by gravitational attraction in such a way that under a radial perturbation they neither collapse nor explode but settle down to the original solution. MMU is not a finite energy solution using a physically reasonable definition of energy. Its “constant time” Riemannian 3-fold is not an asymptotically flat 3-manifold. We want to solve the Dirac equation on a Riemannian 3-manifold which is in some sense asymptotically Melvin. An asymptotically flat Riemannian 3-manifold has a concept of mass when the decay to flatness is reasonably rapid (Bartnik [7]). This concept is of differential geometric origin though it has been discovered in the study of physics and it corresponds to the energy of an appropriate spacetime.

Thorne [8, 9] has defined a concept of energy (C-energy) for a finite region of a cylindrical spacetime. Radinschi and Yang [10] considered another concept of such (quasi-local) energy for MMU. Although MMU has infinite energy comparing it with the magnetized Schwarzschild solution an asymptotically Melvin (defined rigorously later) Riemannian 3-manifold can be assigned a mass-like parameter using the decay coefficient of the metric. It is the coefficient of a term of faster decay. While in the absence field equations giving more information on the Ricci curvature we could not prove the positivity of this decay coefficient assuming only the nonnegativity of the scalar curvature, Weitzenbök-Lichnerowictz identity applied to the harmonic spinor of appropriate decay gives a positive mass type theorem involving the decay coefficient of the term of slower deacy. Unfortunately nonnegativity of this latter parameter has to be assumed from the beginning to avoid singularity in the metric near the axis at large distances. This parameter gives the magnetic field in the physical spacetime. Our formula gives this asymptotic parameter in terms of the harmonic spinor weighted scalar curvature integral and the integral for the norm of the covariant derivative of the spinor. Finally in the appendix paper we also include a rigidity theorem for a cylinder which follows from the part of the positive mass theorem of Schoen and Yau [11] giving rigidity of $\mathbb{R}^3$.

In this paper we have not considered higher dimensions and all types of cylindrical ends. We also did not look for optimal generalizations of the parameters of the weighted Sobolev spaces involved. These generalizations and related boundary value problems will be studied elsewhere by one of the authors (QW). Here we want to show the steps involved in our proofs by considering not too technical problems and we keep the article readable by multi-disciplinary researchers. One application of the existence of a harmonic spinor we prove here occurs in the proof of the uniqueness of the magnetized Schwarzschild solution in [6] where this existence is assumed without proof. In fact existence of such harmonic spinors may be useful in the hitherto unsolved problems of extending the black hole uniqueness theorems ([12]) in many other magnetized worlds.

2 Preliminaries

We consider a complete Riemannian 3-manifold $(\Sigma, \hat{g})$ having the metric of the form

$$\hat{g} = \bar{g} + Xd\phi^2$$

where the function X and the 2-metric $\bar{g}$ are independent of the coordinate ϕ. We say that Σ has ends if outside a compact subset K it is a finite union of disjoint sets U_i each diffeomorphic to $\mathbb{R}^3 \setminus B$, B being a closed ball. Here i counts the number of ends. Let b be a nonnegative constant. We say that $\hat{g}$ is asymptotically Melvin with parameter b if at any end $U = U_i$ w.r.t. the Euclidean spherical coordinates $\{r, \theta, \phi\}$ in $\mathbb{R}^3 \setminus B$,

$$\bar{g} = (1 + v_1) F^2 \left(dr^2 + r^2 d\theta^2\right) \quad (1)$$

$$X = (1 + v_2) F^{-2} r^2 \sin^2 \theta \quad (2)$$

where

$$F = 1 + br^2 \sin^2 \theta$$

and $v_1, v_2 \in W^{2,p}_{-\tau+1}$ for large enough q and $\tau > 1/2$. For the existence of harmonic spinors the restrictions on p and τ are not optimal. We assume $p \geq 4$ so that geodesic equation at each point of the tangent bundle has a unique solution. We shall define the weighted Sobolev spaces later. They say for $i = 1, 2$

$$v_i = O\left(r^{-1}\right), \partial v_i = O\left(r^{-2}\right), \ \partial^2 v_i \in L^p_{-\tau-2}\left(R^3 \setminus B\right)$$

where the last conditions imply that in some sense $\partial^2 v_i = o\left(r^{-2}\right)$. In some applications we can take out the $O(r^{-1})$ term from v_1 and then we assume

$$\bar{g} = (1 + v_3) F^2 \left((1 - 2M/r)^{-1} dr^2 + r^2 d\theta^2\right) \quad (3)$$

where M is a constant and $v_3 \in W^{2,p}_{-\tau}$.

$W^{2,p}_{\delta} \equiv W^{2,p}_{\hat{g},\delta}$ is the space of measurable $\mathbb{C}^2$ valued functions u in L^p_{loc} such that

$$\int_{\Sigma,\hat{g}} \left| \frac{\partial^{|l|}}{\partial x^{l_1} \partial x^{l_2} \partial x^{l_3}} u \right|^p \left(\sqrt{1+r^2} \right)^{-\delta p + |l| p - 3} \tag{4}$$

are finite for $|l| = 0, 1, 2$. Here $l = \{l_1, l_2, l_3\}$ is the multi-index. $\dfrac{\partial^{|l|}}{\partial x^{l_1} \partial x^{l_2} \partial x^{l_3}}$ means an l-th weak partial derivative. Powers of $\sqrt{1+r^2}$ is included to specify the decay rate as $r \to \infty$. Since r is not defined outside U we interpret this factor to be a positive function in the subset $\Sigma \setminus U$.

By Melvin's 3-metric we mean the following metric.

$$g_{\text{MMU}} = \left(1 + (1/4) B^2 r^2 \sin^2\theta\right)^2 \left(dr^2 + r^2 d\theta^2\right) + \left(1 + (1/4) B^2 r^2 \sin^2\theta\right)^{-2} r^2 \sin^2\theta d\phi^2$$

It is the 3-metric induced on the "constant time" hypersurface of Melvin's magnetic universe. Since we shall work mainly with $SU(2)$ spinors in 3-dimension we give the definition of weighted Sobolev spaces for $\mathbb{C}^2$-valued functions. When we shall denote the spaces for the real-valued functions by the same notation, it will be clear from the context what we mean. In general for $u = (u^1, u^2) \in \mathbb{C}^2$ we have $|u| = u^1 \bar{u}^1 + u^2 \bar{u}^2$. The integral in Eq. (4) has the volume form of $\hat{g}$. Thus we define the $W^{2,p}_{\delta}$ norm for functions on $(\Sigma, \hat{g})$ by

$$||u||_{\hat{g},2,p,\delta} = \sum_{|l|=0}^{|l|=3} \left(\int \left| \frac{\partial^{|l|}}{\partial x^{l_1} \partial x^{l_2} \partial x^{l_3}} u \right|^p \left(\sqrt{1+r^2} \right)^{-\delta p + |l| p - 3} \sqrt{\det \hat{g}} \right)^{\frac{1}{p}} \tag{5}$$

$||u||_{\hat{g},p,\delta}$ will denote the weighted Lebesgue norm $L^p_{\hat{g},\delta}$.

We also need to consider Sobolev norm $||u||_{2,p,\delta}$ relative to the asymptotically flat metric

$$g = (1 + v_1)[(1 - 2M/r)^{-1} dr^2 + r^2 d\theta^2] + (1 + v_2) r^2 \sin^2\theta d^2\phi \tag{6}$$

Since the 3-measure of $\hat{g}$ in U is

$$\sqrt{\det \hat{g}} d\theta d\phi dr = F r^2 |\sin\theta| d\theta d\phi dr, \quad \text{and } 1 \le F \le C_1(1 + r^2) \tag{7}$$

functions in $W^{2,p}_{\delta}$ with finite $||\cdot||_{\hat{g},2,p,\delta}$ norm will be in $W^{2,p}_{\delta+2/p}$ with finite $||\cdot||_{2,p,\delta+2/p}$ norm for the asymptotically flat metric. These two norms are however not equivalent because $\sin\theta = 0$ kills the r^2 growth in F. Now to show the existence of a harmonic spinor suitable for proving the positive mass theorem for an asymptotically flat Riemannian 3-manifold Bartnik showed (Proposition 6.1 in [7], minding a typo) that the Dirac operator is an isomorphism from $W^{2,p}_{-\delta}$ onto $W^{1,p}_{-\delta-1}$ where $\delta \in (0, 2)$. So for an asymptotically Melvin manifold with the metric independent of ϕ we seek harmonic spinors in the Sobolev spaces

$$\mathbb{W}^{2,p}_{-\delta} = \{\xi \in W^{2,p}_{-\delta} \mid \delta \in (2/p, 2), \partial_\phi \xi = 0\} \tag{8}$$

We now state the main result. It is proved in the next section.

Theorem 2.1. *Suppose the scalar curvature $R_{\hat{g}} \geq 0$ and $(\Sigma, \hat{g})$ is asymptotically Melvin with parameter b having finite number of asymptotic ends. Let $\epsilon \in (2/p, 2)$. Then the Dirac operator $D_{\hat{g}} : \mathbb{W}^{2,p}_{-\epsilon} \longrightarrow \mathbb{W}^{1,p}_{-\epsilon-1-2/p}$ is an isomorphism.*

3 Existence of a harmonic spinor

We prove Theorem 2.1 using the method of Bartnik [7] and the fact that Fredholm property is the same for two bounded linear operators sufficiently close.

Proof of Theorem 2.1. First we show that the Dirac operator of $\hat{g}$ is a small perturbation in some appropriate sense of a "weighted" Dirac operator of the asymptotically flat metric g given in Eq. (6) so that arguments as in Theorem 1.10 of Bartnik [7] prove the Fredholm property of the first operator in case we can show that the later operator is Fredholm.

For simplicity let us first assume that we have only one end U. Let $\{e^i\}$ be an orthonormal frame field of 1-forms relative to g. We choose $e^3 = (1+\nu_2)^{1/2} r \sin\theta d\phi$ in U. Then $\{\hat{e}^i\}$ where $\hat{e}^i = Fe^i$ for $i = 1, 2$, and $\hat{e}^3 = F^{-1}e^3$ will be an orthonormal frame field of 1-forms relative to $\hat{g}$. We note that $\hat{e}_i = F^{-1}e_i$ for $i = 1, 2$, and $\hat{e}_3 = Fe_3$. By direct calculation, we see

$$D_{\hat{g}} = \hat{e}^i \cdot \hat{\nabla}_{\hat{e}_i} = \sum_{i=1,2} F^{-1}e^i \cdot \nabla_{e_i} + Fe^3 \cdot \partial_{e_3} + O(r^{-1}). \tag{9}$$

We denote the operator $\sum_{i=1,2} F^{-1}e^i \cdot \nabla_{e_i} + Fe^3 \cdot \partial_{e_3}$ by P. Here P is the perturbation of $D_{\hat{g}}$ which we can handle. In fact $P = F^{-1}D_g + O(r^{-1}) : \mathbb{W}^{2,p}_{-\epsilon} \longrightarrow \mathbb{W}^{1,p}_{-\epsilon-1-2/p}$

since spinors in $\mathbb{W}$ spaces are independent of ϕ. Now since g is asymptotically flat $D_g : \mathbb{W}^{2,p}_{-\epsilon} \longrightarrow \mathbb{W}^{1,q}_{-\epsilon-1}$ is a Fredholm operator by Proposition 1.6 in [7] and F^{-1} is bounded. So P is a bounded linear operator. To check the Fredholm property of P, we first show that $F^{-1}D_g$ is Fredholm. First we note that $\ker F^{-1}D_g = \{\xi \in \mathbb{W}^{2,p}_{-\epsilon} : F^{-1}D_g\xi = 0\}$ is finite dimensional because $\ker D_g$ is finite dimensional. Second we note that range of $F^{-1}D_g$, $F^{-1}D_g\left(\mathbb{W}^{2,p}_{-\epsilon}\right)$ is closed in $\mathbb{W}^{1,p}_{-\epsilon-1}$. This is because the range of $F^{-1}D_g$ is the set of spinors in the range of D_g multiplied by the bounded function F^{-1}, and the range of D_g is closed in $\mathbb{W}^{1,p}_{-\epsilon-1}$. Now the range of $F^{-1}D_g$ is a subset of $\mathbb{W}^{1,p}_{-\epsilon-1-2/p}$. So by the compact inclusion of $\mathbb{W}^{1,p}_{-\epsilon-1-2/p}$ into $\mathbb{W}^{1,p}_{-\epsilon-1}$ it is closed in $\mathbb{W}^{1,p}_{-\epsilon-1-2/p}$.

Now the operator P is a perturbation of $F^{-1}D_g$ in the operator norm

$$||P - F^1 D_g||_{\text{op}} = \sup\left\{||(P - F^{-1}D_g)u||_{1,p,-\epsilon-1-2/p} : u \in \mathbb{W}^{2,p}_{-\epsilon}, ||u||_{2,p,-\epsilon} = 1\right\}$$

because $||P - F^1 D_g||_{\text{op},R} = o(1)$ as $R \to \infty$. Thus by the arguments of Theorem 1.10 of Bartnik [7], P is also Fredholm. Since $D_{\hat{g}}$ is a perturbation of P similar arguments shows that

$$D_{\hat{g}} : \mathbb{W}^{2,p}_{-\epsilon} \longrightarrow \mathbb{W}^{1,q}_{-\epsilon-1-2/p} \tag{10}$$

is Fredholm. As the scalar curvature of $\hat{g}$ is nonnegative, the kernel of $D_{\hat{g}}$ and its adjoint would be trivial. Thus (10) is an isomorphism. To generalize the result for finite number of ends we define the Sobolev spaces with r extended in Σ so that it matches at all ends with the asymptotic radial coordinates (for details see p.230 in Parker and Taubes [5]). □

To get a suitable harmonic spinor we need the transformation formulas stated in the following lemma.

Lemma 3.1. *If $\hat{g} = \zeta^2 g$, then $D_{\hat{g}}(\zeta^{-1}\xi) = \zeta^{-2}D_g\xi$. If the spinor satisfies $\partial_\phi\xi = 0$ and $\hat{g} = \bar{g} + fd\phi^2, g = \bar{g} + qfd\phi^2$, then $D_{\hat{g}}(q^{-\frac{3}{8}}\xi) = q^{-\frac{3}{8}}D_g\xi$.*

In the above lemma the first formula is the well-known conformal transformation formula. A derivation of the second formula can be found in [6].

Let ξ_0 be a spinor *constant near infinity* relative to the asymptotically flat metric g. In particular all partial derivatives of ξ_0 vanishes in U and we extend ξ_0 outside by keeping $\partial_\phi \xi_0 = 0$. We define $g_1 = F^2 g$. Taking $q = F^{-4}$ in Lemma 3.1 we get in U

$$D_{\hat{g}}(q^{-\frac{3}{8}} F^{-1}\xi_0) = q^{-\frac{3}{8}} D_{g_1}(F^{-1}\xi_0) = q^{-\frac{3}{8}} F^{-2} D_g \xi_0. \tag{11}$$

If we choose $\Theta_0 = q^{-\frac{3}{8}} F^{-1}\xi_0 = F^{\frac{1}{2}}\xi_0$ then $D_{\hat{g}}\Theta_0 \in \mathbb{W}^{1,p}_{-\epsilon-1}$ for $2/p < \epsilon < 2$. We get a unique spinor $\hat{\Theta} \in \mathbb{W}^{1,p}_{-\epsilon}$ satisfying $D_{\hat{g}}\hat{\Theta} = \Theta_0$. So we get a nontrivial spinor $\Theta = \hat{\Theta} - \Theta_0$ harmonic relative to $\hat{g}$.

Remark 3.1. In the case of multiple ends one usually chooses the asymptotically constant spinor to be zero at all but one end. We then take Θ to be the average of the harmonic spinors obtained.

4 Positive mass type theorems

Theorem 4.1. *Suppose $(\Sigma, \hat{g})$ is complete having a single end and the scalar curvature $R_{\hat{g}} \geq 0$. Let Θ be the harmonic spinor as stated before Remark 3.1. Then*

$$b = \frac{3}{8\pi} \lim_{r\to\infty} \left(r^{-3} \int_{\hat{g},\mathcal{B}_r} \left[R_{\hat{g}} \|\Theta\|^2 + 4\|\nabla_{\hat{g}}\Theta\|^2 \right] \right) \tag{12}$$

In case $b = 0$, $(\Sigma, \hat{g})$ is flat. In particular $M = 0$ if we use the asymptotic condition Eq. (3).

Proof. Let $n_{\hat{g}}$ be the unit normal form relative to $\hat{g}$ on $\partial\mathcal{B}_r$ where $\mathcal{B}_r$ is a ball of large radius r in asymptotic region pointing in the direction of increasing r. By integrating the Weitzenbök-Lichnerowicz identity namely

$$2\Delta_{\hat{g}}\|\Theta\|^2 = R_{\hat{g}}\|\Theta\|^2 + 4\|\nabla_{\hat{g}}\Theta\|^2 \tag{13}$$

we get

$$\int_{\hat{g},\partial\mathcal{B}_r} \frac{\partial\|\Theta\|^2}{\partial n_{\hat{g}}} \geq 0 \tag{14}$$

As before we assume $q/2 < \epsilon < 2$. $\Theta + F^{1/2}\xi_0 \in W^{1,q}_{\hat{g},-\epsilon-1}$ where ξ_0 was defined before Eq. (11).

$$\begin{aligned} \|\Theta\|^2 &= \|\hat{\Theta}\|^2 - \left\langle \hat{\Theta}, \Theta_0 \right\rangle - \left\langle \Theta_0, \hat{\Theta} \right\rangle + \|\Theta_0\|^2 \\ &= \|\hat{\Theta}\|^2 - F^{1/2}\left\langle \hat{\Theta}, \xi_0 \right\rangle - F^{1/2}\left\langle \xi_0, \hat{\Theta} \right\rangle + F \end{aligned}$$

since for the asymptotically constant spinor $||\xi_0|| = 1$. Now $\hat{\Theta} \in W^{2,q}_{\hat{g},-\epsilon}$, and $n_{\hat{g}} = n_{\hat{g},r}dr = \sqrt{\bar{g}_{rr}}dr$. So

$$\begin{aligned}\frac{\partial||\Theta||^2}{\partial n_{\hat{g}}} &= \hat{g}^{jk}(||\Theta||^2)_j n_{\hat{g},k} = \sqrt{\bar{g}^{rr}}\frac{\partial||\Theta||^2}{\partial r} = \sqrt{\bar{g}^{rr}}\frac{\partial F}{\partial r} + O(r^{-2}) \\ &= (1/2)F^{-1}br\sin^2\theta + O(r^{-2}).\end{aligned}$$

Thus

$$\begin{aligned}\int\limits_{\hat{g},\partial\mathcal{B}_r}\frac{\partial||\Theta||^2}{\partial n_{\hat{g}}} &= 2\pi\int\limits_0^\pi \left((1/2)F^{-1}br\sin^2\theta + O(r^{-2})\right)Fr^2\sin\theta d\theta \\ &= 2\pi\int\limits_0^\pi \left((1/2)br^3\sin^3\theta + O(r^2)\right)d\theta = \frac{4}{3}\pi br^3 + O(r^2)\end{aligned}$$

Hence from Eq. (14) we get Eq. (12). If $b = 0$, then Θ is a constant spinor. Since we can start with three linearly independent constant spinors at infinity, this means $\hat{g}$ is flat. □

Remark 4.1. In this remark we explain the underlying principle how the harmonic spinor found in this paper can be used in the proof of the uniqueness magnetized Schwarzschild solution in [6] where its existence is assumed. Suppose $\zeta^\pm$ and $f^\pm$ are positive functions having decay as follows

$$\zeta^\pm = (1/4)(1 - M/r \pm \sqrt{1-2M/r})^2F^{-2} \quad (15)$$

$$f^\pm = (1/4)(1 - M/r \pm \sqrt{1-2M/r})^2 r^2\sin^2\theta \quad (16)$$

If now $\hat{g}$ has decay as in Eq. (6) then the metric $\eta^+ = \zeta^+\bar{g} + f^+d\phi^2$ is asymptotically flat with mass zero and $\eta^- = \zeta^-\bar{g} + f^-d\phi^2$ compactifies the infinity. If the scalar curvature $R_{\hat{g}} \geq 0$ is such that we also have $R_{\eta^+} \geq 0$ then positive mass theorem says η^+ is flat. η^- is necessary because for this uniqueness problem $\hat{g}$ is not complete but has a smooth totally geodesic surface. Now $(\Sigma, \hat{g})$ has two ends and a totally geodesic surface across which it is symmetric. If Θ be the the average harmonic spinor mentioned in Remark 3.1, then the normal derivative of $||\Theta||^2$ on the totally geodesic surface vanishes. Unfortunately from the field equations we cannot show $R_{\eta^\pm} \geq 0$ directly. But from Θ we get harmonic spinors relative to $\eta^\pm$ and exploiting their Weitzenbök-Lichnerowicz identities we can build two divergence form identities involving the expressions for $R_{\eta^\pm}$. On integration these identities give the uniqueness result by virtue of the properties of Θ on the totally geodesic surface and at infinity.

5 Appendix: Rigidity of a cylinder

We include a rigidity result for a $\mathbb{R} \times S^2$ cylinder that follows from the part of the positive mass theorem of Schoen and Yau [11] giving the rigidity of $\mathbb{R}^3$. First we define a special type of cylindrical end. We use the notation $d\sigma^2 = d\theta^2 + \sin^2\theta d\phi^2$.

Definition 5.1. *A 3-dimensional Riemannian manifold* (Σ, h) *has a cylindrical end of radius a and c-mass m if near the end it is diffeomorhic to* $S^2 \times [0, \infty)$ *and* $h = \left(1 + (2m/a)e^{-l/a}\right)\left[dl^2 + a^2 d\sigma^2\right] + O\left(e^{-4l/a}\right)$ *as* $l \to \infty$.

Later in Remark 5.1 we shall state a curvature condition such that the definition is meaningful. We shall prove the following theorem.

Theorem 5.1. *Suppose the* (Σ, h) *is a complete* 3*-dimensional Riemannian manifold having cylindrical ends of radius a and c-mass m. Suppose the scalar curvature of h,* $R_h \geq 2a^{-2}$ *and f is a positive solution of* $\Delta_h f = (1/4)a^{-2} f$ *on* Σ*. If*

$$\text{as } l \to \infty, \quad f = e^{l/(2a)}\left(1 - m'/(2a)e^{-l/a} + o(e^{-2l/a})\right) \tag{17}$$

then $m \geq m'$*. If* $m = m'$ *and* (Σ, h) *has only two ends then* (Σ, h) *is isometric to* $\left(S^2 \times \mathbb{R}, dl^2 + a^2 d\sigma^2\right)$.

Before proving the theorem we first consider some easy consequences of the hypotheses. In asymptotic region

$$\begin{aligned} f^4 h &= e^{2l/a}\left(1 - (2m'/a)e^{-l/a} + o(e^{-2l/a})\right) \\ &\quad \times \left[\left(1 + (2m/a)e^{-l/a}\right)\left(dl^2 + a^2 d\sigma^2\right) + O(a^{-2}e^{-4l/a})\right] \\ &= e^{2l/a}\left(1 + \frac{2(m - m')}{a}e^{-l/a} + O(e^{-2l/a})\right)\left(dl^2 + a^2 d\sigma^2\right) + O(a^{-2}e^{-2l/a}) \end{aligned}$$

If we put $r = ae^{l/a}$ the asymptotic form of $\eta \equiv f^4 h$ becomes

$$\left(1 + 2(m - m')/r + O(r^{-2})\right)\left(dr^2 + r^2 d\sigma^2\right) + O(r^{-2}) \tag{18}$$

With $r = ae^{l/a}$ Eq. (17) becomes

$$\text{as } r \to \infty, \quad f = \sqrt{r/a}\,(1 - m'/(2r)) + o(r^{-3/2}) \tag{19}$$

We also have $h = (1+2m/r)a^2r^{-2}(dr^2+r^2d\sigma^2)+O(r^{-4})$. Then in case $m = m'$ the metric $\eta = f^4h = (1-4m^2r^{-2}+O(r^{-3}))(dr^2+r^2d\sigma^2)+O(r^{-2})$ is asymptotically flat with mass zero. Still assuming $m = m'$, if we choose $\bar{r} = 1/r$ then in the $\{\bar{r}, \theta.\phi\}$ coordinates f^4h compactifies the infinity on adding the point $\{\bar{r} = 0\}$ to Σ (the orientation change due to inversion should be compensated by redefining it globally). At this end $f^4h = \left(1-4m^2a^{-4}\bar{r}^2+O(\bar{r}^3)\right)(d\bar{r}^2+\bar{r}^2d\sigma^2)+O(\bar{r}^2)$. Eq. (19) implies $\partial_r f = 1/(2\sqrt{ra})+o(r^{-1}), \partial_\theta f = O(r^{-3/2}), \partial_\phi f = O(r^{-3/2})$ and so

$$|\nabla f|_h^2 = (1/4)a^{-3}r + o(r^{1/2}) \text{ as } r \to \infty$$

It is a convenient point to explain why the definition of c-mass is not vacuous.

Remark 5.1. Since f^4h is asymptotically flat its mass will be well-defined by Theorem 4.3 of Bartnik [7] in case its Ricci curvature is in $L^p_{-2-\tau}$, $\tau \geq 1/2$ in Cartesian coordinates $\{x, y, z\}$ related to $\{r, \theta, \phi\}$. The Ricci curvature of h namely R_{ad} is given by

$$\mathrm{Ric}\left(f^4h\right)_{ad} = R_{ad} - 2f^{-1}f_{;da} + 6f^{-2}f_{;d}f_{;a} - 2f^{-2}|\nabla f|_h^2h_{ad} - 2f^{-1}h_{ad}\Delta_h f$$

where covariant derivatives are relative to h. One can easily check that $\mathrm{Ric}\left(f^4h\right)_{rr} = R_{rr} + o(r^{-5/2})$ but from the given asymptotic expansion of f without knowing the angular dependence in the $O(r^{-3/2})$ term $f_{;da}$ cannot be evaluated to required order because of the angular dependence of the Christoffel symbols of h. We therefore shall not attempt to express the necessary condition in terms of the Ricci curvature of h in the $\{x, y, z\}$ coordinates so that c-mass is defined. For completeness sake we remember that the cylindrical metric $dl^2 + l^2d\sigma^2$ has curvature components $R_{ll} = R_{l\theta} = R_{l\phi} = R_{\theta\phi} = 0, R_{\theta\theta} = 1, R_{\phi\phi} = \sin^2\theta, R = 2a^{-2}$.

Lemma 5.2. *Suppose Σ is diffeomorphic to $\mathbb{R}^3 \setminus \{Origin\}$, both f and $\bar{f}$ satisfies $\Delta_h u = (1/4)a^{-2}u$ and $f, \bar{f}$ satisfy the same asymptotic conditions Eq.* (17) *at the ends possibly with different constants m', m''. Then $f = \bar{f}$.*

Proof. We integrate the identity $\nabla_h((f-\bar{f})\nabla(f-\bar{f})) = |\nabla(f-\bar{f})|_h^2+(1/4)a^{-2}(f-\bar{f})^2$. We shall use r as asymptotic radial coordinates so that $r \to \infty$ at both ends. Let $\{\widetilde{r}, \theta, \phi\}$ be the spherical coordinates on Σ which is diffeomorphic to $\mathbb{R}^3 \setminus \{\text{Origin}\}$ so that $\widetilde{r} = 0$ corresponds to the origin and at the two ends $r = \widetilde{r}$ or $r = 1/\widetilde{r}$. We have

$$\iint_{\widetilde{r}\to 0}(f-\bar{f})(f-\bar{f})_{;A}n^A_{\rm out}+\iint_{\widetilde{r}\to\infty,}(f-\bar{f})(f-\bar{f})_{;A}n^A_{\rm out}$$
$$=\iiint\left(|\nabla(f-\bar{f})|^2_h+(1/4)a^{-2}(f-\bar{f})^2\right)$$

Now $f-\bar{f}=-(m'-m'')/(2\sqrt{ar})+o(r^{-3/2})$ as $r\to\infty$, surface measure relative to h is $(a^2+O(r^{-1}))d\theta d\phi$ and normal form $n_{\rm out}=\sqrt{h_{rr}}dr$. At both infinity and origin $(f-\bar{f})_{;A}n^A_{\rm out}=h^{rr}\partial_r(f-\bar{f})\sqrt{h_{rr}}=\sqrt{h^{rr}}\partial_r(f-\bar{f})=a^{-1}r^{1/2}\partial_r(f-\bar{f})+O(r^{-3/2})=(m'-m'')/(4a\sqrt{ar})+O(r^{-3/2})$. So surface integrals vanish as $r\to\infty$. RHS then gives $f=\bar{f}$ on Σ. □

Proof of Theorem 5.1. Let $\eta=f^4h$. Then $R_\eta f^5=R_hf-8\Delta_hf\geq\dfrac{2}{a^2}f-8\Delta_hf=0$. By virtue of Eq. (18) η is asymptotically flat with mass $m-m'$. Since the scalar curvature of η is nonnegative by the positive mass theorem we get $m\geq m'$. In case $m=m'$, $R_\eta=0$ since η is flat. Thus $0=R_hf-8\Delta_hf\geq 2a^{-2}f-8\Delta_hf=0$ giving $R_h=2a^{-2}$. In the metric η we have $\Delta_\eta f=(1/4)a^{-2}f^{-3}+2f^{-1}|\nabla f|^2_\eta$. Now $\sqrt{r/a}$ solves this equation for the Euclidean metric $dr^2+r^2d\sigma^2$ which is at present valid on the asymptotic region. Since f cannot attain interior maximum unless constant by maximum principle, $m'\geq 0$. Thus $m\geq 0$. We want to compactify all but one end using the coordinate inversion if $m=0$ or conformal transformation followed by the coordinate inversion if $m>0$ as explained above. In the later case the compactified metric $a^2r^{-2}h$ will have scalar curvature given by $R_{a^2r^{-2}h}=4(a^{-2}-r^{-2}h^{rr})=8a^{-2}m/r+O(r^{-2})$ which is positive as $r\to\infty$. Let $\tilde{h}$ denote the metric obtained from h after compactification of all but one end. Now the flat metric $\eta=f^4\tilde{h}$ will be the standard Euclidean metric globally. Let $\widetilde{r}$ be the global radial coordinate on the compactified Σ with origin at $\widetilde{r}=0$ at one compactified end. By Lemma 5.2, $f=\sqrt{\widetilde{r}/a}$. But this solution has $m'=0$. Hence $m=0$. In case (Σ,h) has two ends h is now isometric to $a^2\widetilde{r}^{-2}(d\widetilde{r}^2+\widetilde{r}^2d\sigma^2)=dl^2+a^2d\sigma^2$ where now $l=a\ln(\widetilde{r}/a)\in(-\infty,\infty)$. □

6 Conclusion

We showed the existence of harmonic spinors in some cylindrical 3-manifolds. Existence of such spinors are relevant in studying the uniqueness or rigidity theorems involving cylindrical geometry which is natural for a magnetic universe. Although the story is a simple one that the index of the elliptic operator is the same under some reasonable class of perturbations, in practice one needs to check huge amount of computation before one can apply it in a specific situation. Hopefully the method will be useful in extending the black hole uniqueness theorems for static and stationary solutions in a magnetic universe.

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