

Uniqueness of Kerr-Newman solution

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Abstract. We show that non-degenerate multiple black hole solution of Einstein-Maxwell equations in an asymptotically flat axisymmetric spacetime cannot be in stationary equilibrium. This extends the uniqueness of Kerr-Newman solution first proved by Bunting and Mazur in a much wider desirable class. Spin-spin interaction cannot hold the black hole apart even with electromagnetic forces.

Key words. Black hole uniqueness theorems, Kerr-Newman solution.

1 Introduction

We generalize the method used in [1],[2] for the uniqueness problem of Kerr-Newman solution for $M^2 > a^2 + e^2 + m^2$. Here m is the magnetic charge. For a single black hole case the results are due to Bunting [3] and Mazur [4] (also Carter [5]) using different techniques. Several extensions of these results have been obtained by Wells [6]. Regarding the possibility of multiple black holes including those in a vacuum spacetime several results are obtained by Weinstein [7, 8] (see also the review article by Beig and Chrusciel [9]), Neugebauer and Meinel [10], Chrusciel and Costa [11], Wong and Yu [12]. Wong and Yu does not need the axisymmetric assumption but assumed the solution to be close to Kerr-Newman solution in some sense. We consider only non-degenerate black hole boundary. Our technique involves tailoring suitably the spinorial proofs of the positive mass theorem of Schoen and Yau [13] due to Witten [14] and Bartnik [15].

Initially we suppose the spacetime metric is stationary and axisymmetric EM

black hole solution having the form

$${}^4g = {}^4g_{ab}dx^a dx^b = -Vdt^2 + 2Wdtd\phi + Xd\phi^2 + \bar{g} \quad (1)$$

Carter (Part II, [16]) showed that $\bar{g} = \Omega(d\rho^2 + dz^2)$ where ρ and z are conjugate harmonic functions. V, W, X and Ω are functions of ρ and z . Following Carter we take

$$W^2 + VX = \rho^2 \quad (2)$$

2 Ricci Curvature

We denote the 2-metric by $\bar{g} = {}^4g_{11}(dx^1)^2 + {}^4g_{22}(dx^2)^2 = {}^4g_{AB}dx^A dx^B$. We denote the induced 3-metric on a $t=\text{constant}$ hypersurface by

$$\widehat{g} = \bar{g} + Xd\phi^2 \quad (3)$$

Then $\det({}^4g_{ab}) = -(VX + W^2)\bar{g}_{11}\bar{g}_{22}$. Using Eq. (97) we get

$${}^4g^{ab} = \begin{bmatrix} -X\rho^{-2} & 0 & 0 & W\rho^{-2} \\ 0 & \bar{g}_{11}^{-1} & 0 & 0 \\ 0 & 0 & \bar{g}_{22}^{-1} & 0 \\ W\rho^{-2} & 0 & 0 & V\rho^{-2} \end{bmatrix}$$

Henceforth \bar{g}^{AB} is obtained from \bar{g}_{AB} by raising the indices with the 2-metric \bar{g} . The $t=\text{constant}$ surface has zero mean curvature. \widehat{g} is a Riemannian metric. We shall denote the Laplacian and the covariant derivative of the two-metric \bar{g} by $\bar{\Delta}$ and $\bar{\nabla}$. In general we shall use the metrics as subscripts in order to indicate w.r.t. which metric a norm or an operator is computed. Since we shall not use the usual formulations of a stationary axisymmetric vacuum spacetime it is better to give the expressions for the components of the Ricci curvature 4g for easy reference before equating them to zero using the vacuum Einstein equations. Ricci curvature of the four metric is

$${}^4R_{tt} = \frac{1}{2}\bar{\Delta}V + \frac{V}{4\rho^2}\langle\bar{\nabla}V, \bar{\nabla}X\rangle + \frac{V}{2\rho^2}|\bar{\nabla}W|^2 - \frac{W}{2\rho^2}\langle\bar{\nabla}V, \bar{\nabla}W\rangle - \frac{X}{4\rho^2}|\bar{\nabla}V|^2 \quad (4)$$

$${}^4R_{t\phi} = -\frac{1}{2}\bar{\Delta}W + \frac{V}{4\rho^2}\langle\bar{\nabla}W, \bar{\nabla}X\rangle - \frac{W}{2\rho^2}\langle\bar{\nabla}V, \bar{\nabla}X\rangle + \frac{X}{4\rho^2}\langle\bar{\nabla}V, \bar{\nabla}W\rangle \quad (5)$$

$${}^4R_{\phi\phi} = -\frac{1}{2}\bar{\Delta}X - \frac{X}{4\rho^2}\langle\bar{\nabla}V, \bar{\nabla}X\rangle + \frac{W}{2\rho^2}\langle\bar{\nabla}W, \bar{\nabla}X\rangle - \frac{X}{2\rho^2}|\bar{\nabla}W|^2 + \frac{V}{4\rho^2}|\bar{\nabla}X|^2 \quad (6)$$

$$R_{At}^{(4)} = 0 = R_{A\phi}^{(4)} \quad (7)$$

$${}^4R_{BD} = \frac{1}{2}\bar{R}\bar{g}_{BD} - \frac{1}{\rho}\bar{\nabla}_D\bar{\nabla}_B\rho + \frac{1}{2\rho^2}\bar{\nabla}_D W\bar{\nabla}_B W + \frac{1}{4\rho^2}\bar{\nabla}_D V\bar{\nabla}_B X + \frac{1}{4\rho^2}\bar{\nabla}_B V\bar{\nabla}_D X \quad (8)$$

3 Einstein equation

$${}^4R_{ab} = 8\pi \left(T_{ab} - (1/2)Tg_{ab} \right) \quad (9)$$

with energy-momentum tensor

$$T_{ab} = \frac{1}{4\pi} \left(F_{ac}F_{bd}g^{cd} - (1/4)g_{ab}F_{ij}F_{kl}g^{ik}g^{jl} \right) \quad (10)$$

F is the electromagnetic field tensor obtained from the electromagnetic potential one form \mathbf{A}

$$F_{ab} = \frac{\partial \mathbf{A}_b}{\partial x^a} - \frac{\partial \mathbf{A}_a}{\partial x^b} \quad (11)$$

Since $T = g^{ab} T_{ab} = 0$ Einstein equation becomes

$${}^4R_{ab} = 8\pi T_{ab} \quad (12)$$

Following Carter (Eq. 7.43 Part II [16]) we choose the electromagnetic potential to be of the form

$$\mathbf{A} = \xi dt + \psi d\phi \quad (13)$$

where ψ is a function of x^1 and x^2 . (Our sign in Eq. (11) is opposite to that of Eq.6.48 in Carter). Thus

One finds

$$T_{tt} = \frac{1}{8\pi} \left(|\bar{\nabla}\xi|^2 + |W\bar{\nabla}\xi + V\bar{\nabla}\psi|^2 \rho^{-2} \right) \quad (14)$$

$$T_{t\phi} = \frac{1}{8\pi} \left(2VX \langle \bar{\nabla}\xi, \bar{\nabla}\psi \rangle \rho^{-2} + WX|\bar{\nabla}\xi|^2 \rho^{-2} - WV|\bar{\nabla}\psi|^2 \rho^{-2} \right) \quad (15)$$

$$T_{\phi\phi} = \frac{1}{8\pi} \left(|\bar{\nabla}\psi|^2 + |X\bar{\nabla}\xi + W\bar{\nabla}\psi|^2 \rho^{-2} \right) \quad (16)$$

$$T_{AB} = \frac{1}{4\pi\rho^2} \left(-X \frac{\partial\xi}{\partial x^A} \frac{\partial\xi}{\partial x^B} + W \frac{\partial\xi}{\partial x^A} \frac{\partial\psi}{\partial x^B} + W \frac{\partial\psi}{\partial x^A} \frac{\partial\xi}{\partial x^B} + V \frac{\partial\psi}{\partial x^A} \frac{\partial\psi}{\partial x^B} \right) \quad (17)$$

$$+ \frac{X}{2} |\bar{\nabla}\xi|^2 \bar{g}_{AB} - W \langle \bar{\nabla}\xi, \bar{\nabla}\psi \rangle \bar{g}_{AB} - \frac{V}{2} |\bar{\nabla}\psi|^2 \bar{g}_{AB} \quad (18)$$

$$T_{At} = 0 = T_{A\phi} \quad (19)$$

Einstein equation Eq. (12) becomes

$${}^4 R_{tt} = |\bar{\nabla}\xi|^2 + \left| W\bar{\nabla}\xi + V\bar{\nabla}\psi \right|^2 \rho^{-2} \quad (20)$$

$${}^4 R_{t\phi} = 2VX \left\langle \bar{\nabla}\xi, \bar{\nabla}\psi \right\rangle \rho^{-2} + WX|\bar{\nabla}\xi|^2 \rho^{-2} - WV|\bar{\nabla}\psi|^2 \rho^{-2} \quad (21)$$

$${}^4 R_{\phi\phi} = |\bar{\nabla}\psi|^2 + \left| X\bar{\nabla}\xi + W\bar{\nabla}\psi \right|^2 \rho^{-2} \quad (22)$$

$${}^4 R_{BD} = \frac{2}{\rho^2} \left(-X \frac{\partial\xi}{\partial x^B} \frac{\partial\xi}{\partial x^D} + W \frac{\partial\xi}{\partial x^B} \frac{\partial\psi}{\partial x^D} + W \frac{\partial\psi}{\partial x^B} \frac{\partial\xi}{\partial x^D} + V \frac{\partial\psi}{\partial x^B} \frac{\partial\psi}{\partial x^D} \right) \quad (23)$$

$$+ \frac{X}{2} |\bar{\nabla}\xi|^2 \bar{g}_{BD} - W \left\langle \bar{\nabla}\xi, \bar{\nabla}\psi \right\rangle \bar{g}_{BD} - \frac{V}{2} |\bar{\nabla}\psi|^2 \bar{g}_{BD} \quad (24)$$

First three of the following five equations we get from Eqs. (4-6) and Eqs. (21-23) respectively. Last two equations are the non-trivial equations in the nontrivial set of Maxwell equations namely $F_{ab}{}^4{}_{;c} g^{bc} = 0$ for $a = 0$ and $a = 3$.

$$\bar{\Delta}V = -\frac{V}{2\rho^2} \left\langle \bar{\nabla}X, \bar{\nabla}V \right\rangle - \frac{V|\bar{\nabla}W|^2}{\rho^2} + \frac{W}{\rho^2} \left\langle \bar{\nabla}W, \bar{\nabla}V \right\rangle + \frac{X|\bar{\nabla}V|^2}{2\rho^2} + 2|\bar{\nabla}\xi|^2 + 2 \left| W\bar{\nabla}\xi + V\bar{\nabla}\psi \right|^2 \rho^{-2} \quad (25)$$

$$\bar{\Delta}W = \frac{V}{2\rho^2} \left\langle \bar{\nabla}X, \bar{\nabla}W \right\rangle - \frac{W}{\rho^2} \left\langle \bar{\nabla}V, \bar{\nabla}X \right\rangle + \frac{X}{2\rho^2} \left\langle \bar{\nabla}V, \bar{\nabla}W \right\rangle - \frac{4VX}{\rho^2} \left\langle \bar{\nabla}\xi, \bar{\nabla}\psi \right\rangle + \frac{2W}{\rho^2} \left(V|\bar{\nabla}\psi|^2 - X|\bar{\nabla}\xi|^2 \right) \quad (26)$$

$$\bar{\Delta}X = -\frac{X}{2\rho^2} \left\langle \bar{\nabla}V, \bar{\nabla}X \right\rangle + \frac{W}{\rho^2} \left\langle \bar{\nabla}W, \bar{\nabla}X \right\rangle - \frac{X|\bar{\nabla}W|^2}{\rho^2} + \frac{V|\bar{\nabla}X|^2}{2\rho^2} - 2|\bar{\nabla}\psi|^2 - 2 \left| X\bar{\nabla}\xi + W\bar{\nabla}\psi \right|^2 \rho^{-2} \quad (27)$$

$$\bar{\Delta}\xi = \frac{X}{2\rho^2} \left\langle \bar{\nabla}V, \bar{\nabla}\xi \right\rangle - \frac{V}{2\rho^2} \left\langle \bar{\nabla}X, \bar{\nabla}\xi \right\rangle + \frac{V}{\rho^2} \left\langle \bar{\nabla}W, \bar{\nabla}\psi \right\rangle - \frac{W}{\rho^2} \left\langle \bar{\nabla}V, \bar{\nabla}\psi \right\rangle \quad (28)$$

$$\bar{\Delta}\psi = -\frac{X}{2\rho^2} \left\langle \bar{\nabla}V, \bar{\nabla}\psi \right\rangle + \frac{V}{2\rho^2} \left\langle \bar{\nabla}X, \bar{\nabla}\psi \right\rangle - \frac{X}{\rho^2} \left\langle \bar{\nabla}W, \bar{\nabla}\xi \right\rangle + \frac{W}{\rho^2} \left\langle \bar{\nabla}X, \bar{\nabla}\xi \right\rangle \quad (29)$$

Last two equations are the same as in Bunting's thesis (replacing his functions E, F, A, B, C by $\xi, \psi, -V, X, W$) and are equivalent to the set given by Carter (p74, Part II [16]):

$$\begin{aligned} \bar{\nabla} \left(\left(X\bar{\nabla}\xi - W\bar{\nabla}\psi \right) \rho^{-1} \right) &= 0 \\ \bar{\nabla} \left(\rho X^{-1} \bar{\nabla}\psi + W(\rho X)^{-1} \left(X\bar{\nabla}\xi - W\bar{\nabla}\psi \right) \right) &= 0 \end{aligned}$$

From Eqs. (25-27) one can show that $\rho = \sqrt{VX + W^2}$ is a harmonic function i.e. $\bar{\Delta}\rho = 0$. This we are assuming from the start.

4 Remaining equations

Eqs. (8,24) give

$$\begin{aligned} & \frac{1}{2}\bar{R}\bar{g}_{BD} - \frac{1}{\rho}\bar{\nabla}_D\bar{\nabla}_B\rho + \frac{1}{2\rho^2}\frac{\partial W}{\partial x^D}\frac{\partial W}{\partial x^B} + \frac{1}{4\rho^2}\frac{\partial V}{\partial x^D}\frac{\partial X}{\partial x^B} + \frac{1}{4\rho^2}\frac{\partial V}{\partial x^B}\frac{\partial X}{\partial x^D} \\ &= \frac{2}{\rho^2}\left(-X\frac{\partial\xi}{\partial x^B}\frac{\partial\xi}{\partial x^D} + W\frac{\partial\xi}{\partial x^B}\frac{\partial\psi}{\partial x^D} + W\frac{\partial\psi}{\partial x^B}\frac{\partial\xi}{\partial x^D} + V\frac{\partial\psi}{\partial x^B}\frac{\partial\psi}{\partial x^D} + \frac{X}{2}|\bar{\nabla}\xi|^2\bar{g}_{BD} - W\langle\bar{\nabla}\xi, \bar{\nabla}\psi\rangle\bar{g}_{BD} - \frac{V}{2}|\bar{\nabla}\psi|^2\bar{g}_{BD}\right) \end{aligned} \quad (30)$$

Contracting we get

$$\bar{R} = -\frac{1}{2\rho^2}|\bar{\nabla}W|^2 - \frac{1}{2\rho^2}\langle\bar{\nabla}V, \bar{\nabla}X\rangle \quad (31)$$

Differentiating Eq. (97) we get $2\rho\bar{\nabla}\rho = V\bar{\nabla}X + X\bar{\nabla}V + 2W\bar{\nabla}W$ so that

$$2\rho\langle\bar{\nabla}\rho, \bar{\nabla}V\rangle = V\langle\bar{\nabla}X, \bar{\nabla}V\rangle + 2W\langle\bar{\nabla}W, \bar{\nabla}V\rangle + X|\bar{\nabla}V|^2 \quad (32)$$

$$2\rho\langle\bar{\nabla}\rho, \bar{\nabla}X\rangle = X\langle\bar{\nabla}V, \bar{\nabla}X\rangle + 2W\langle\bar{\nabla}W, \bar{\nabla}X\rangle + V|\bar{\nabla}X|^2 \quad (33)$$

$$2\rho\langle\bar{\nabla}\rho, \bar{\nabla}W\rangle = 2W|\bar{\nabla}W|^2 + V\langle\bar{\nabla}X, \bar{\nabla}W\rangle + X\langle\bar{\nabla}V, \bar{\nabla}W\rangle \quad (34)$$

Using the above relations and Eq. (31) we write Eqs. (25,27) as

$$\bar{\Delta}V = 2\bar{R}V + \langle\bar{\nabla}\ln\rho, \bar{\nabla}V\rangle + 2|\bar{\nabla}\xi|^2 + 2\left|W\bar{\nabla}\xi + V\bar{\nabla}\psi\right|^2\rho^{-2} \quad (35)$$

$$\bar{\Delta}W = 2\bar{R}W + \langle\bar{\nabla}\ln\rho, \bar{\nabla}W\rangle - \frac{4VX}{\rho^2}\langle\bar{\nabla}\xi, \bar{\nabla}\psi\rangle + \frac{2W}{\rho^2}\left(V|\bar{\nabla}\psi|^2 - X|\bar{\nabla}\xi|^2\right) \quad (36)$$

$$\bar{\Delta}X = 2\bar{R}X + \langle\bar{\nabla}\ln\rho, \bar{\nabla}X\rangle - 2|\bar{\nabla}\psi|^2 - 2\left|X\bar{\nabla}\xi + W\bar{\nabla}\psi\right|^2\rho^{-2} \quad (37)$$

Since it is well-known that the equations for Ω can be solved using its asymptotic value once we know the other functions we do not include the complicated equations for it.

The $t = \text{constant}$ hypersurface has the topology $\Sigma^+ \cup \partial\Sigma^+$ where Σ^+ is an open 3-manifold and the boundary $\partial\Sigma^+$ is a finite number of disconnected 2-spheres. $X > 0$ in Σ^+ except on the axis. $(\partial\Sigma^+, \bar{g})$ is a smooth totally geodesic submanifold of the 3-dimensional Riemannian manifold with boundary $(\Sigma^+ \cup \partial\Sigma^+, \widehat{g})$. the 3-metric \widehat{g} has nonnegative scalar curvature, which is easy to see from Eq. (12), weak energy condition and doubly contracted Gauss equation for the maximal $t = \text{constant}$ hypersurface.

Let $\varrho^2 = r^2 + a^2 \cos^2 \theta$ and

$$M' = M - \frac{e^2 + m^2}{2r} \quad (38)$$

Kerr-Newman solution has the spacetime metric,

$$ds^2 = -\left(1 - \frac{2M'r}{\varrho^2}\right)dt^2 - \frac{4M'ra \sin^2 \theta}{\varrho^2}d\phi dt + \left((r^2 + a^2) \sin^2 \theta + \frac{2M'ra^2 \sin^4 \theta}{\varrho^2}\right)d\phi^2 + \varrho^2 \left(\frac{dr^2}{r^2 - 2M'r + a^2} + d\theta^2\right) \quad (39)$$

Soon after Kerr's discovery [17], Kerr-Newman solution was found by Newman and *et al.* [18]. The form given above is obtained from Carter (Eq. 5.54, Part I, [16]) by collecting the terms containing $dt^2, d\phi^2, dt d\phi$. This form includes the magnetic charge which can be removed by a duality transformation without changing the metric because the sum $e^2 + m^2$ remains constant under a duality transformation of the electromagnetic fields. The electromagnetic potential is

$$\mathbf{A}_K = -\frac{er + ma \cos \theta}{\varrho^2}dt + \frac{ear \sin^2 \theta + m(r^2 + a^2) \cos \theta}{\varrho^2}d\phi \quad (40)$$

We use subscript K for Kerr-Newman. Comparing with Eq. (1) we get

$$V_K = 1 - \frac{2M'r}{\varrho^2} \quad (41)$$

$$W_K = -\frac{2M'ra \sin^2 \theta}{\varrho^2} \quad (42)$$

$$X_K = (r^2 + a^2) \sin^2 \theta + \frac{2M'ra^2 \sin^4 \theta}{\varrho^2} \quad (43)$$

For Kerr-Newman solution we choose r, θ coordinates from Carter's ρ, z coordinates as follows. Let r, θ be solution of the following equations with $r \geq M + \sqrt{M^2 - e^2 - m^2 - a^2}$

$$\rho = \sqrt{r^2 - 2M'r + a^2} \sin \theta, \quad z = (r - M) \cos \theta \quad (44)$$

In the equation for z we use constant M because $(\partial\rho/\partial r) = (r^2 - 2M'r + a^2)^{-1/2}(r - M) \sin \theta$. This way $d\rho^2 + dz^2$ does not have a cross term containing $dr d\theta$. Expression for $d\rho^2 + dz^2$ is given in §6. $\rho = 0$ set which represents the horizon and the axis is now given by

$$r^2 - 2Mr + a^2 + e^2 + m^2 = 0, \text{ or } \sin \theta = 0 \quad (45)$$

For convenience we define

$$c^2 = M^2 - e^2 - m^2 - a^2, \quad c > 0 \quad (46)$$

The restriction on r now becomes $r \geq M + c$. In general (r, θ) coordinate system is defined away from the $\rho = 0$ set although the functions r and θ are defined on this set. Because of the restriction $r \geq M + c$ the equality is the only solution of the first equation of Eq. (45). The limiting set $r \downarrow M + c$ now contains the horizon and possibly some parts of the axis while $r > M + c, 0 \leq \theta \leq \pi$ represent the remaining part of Σ^+ .

5 Main Idea

We define some quantities which are crucial for the proof.

$$2\tilde{r} = r - M + \sqrt{r^2 - 2Mr + e^2 + m^2 + a^2} \quad (47)$$

$$\zeta = \tilde{r}^2 \varrho^{-2} \quad (48)$$

$$f = \tilde{r}^2 \sin^2 \theta \quad (49)$$

Significance of these quantities is that they transform the 2-metric

$$\bar{g}_K = \varrho^2 \left(\frac{dr^2}{r^2 - 2M'r + a^2} + d\theta^2 \right) \quad (50)$$

into the Euclidean 3-metric η_K in the spherical coordinates $\{\tilde{r}, \theta, \phi\}$ as follows

$$\eta_K = \zeta \bar{g}_K + f d\phi^2 = d\tilde{r}^2 + \tilde{r}^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (51)$$

Our aim is to show, by exploiting the field equations (25-30) and reasonable boundary conditions, that the general 3-metric η defined by

$$\eta = \zeta \bar{g} + f d\phi^2 \quad (52)$$

where ζ and f are the same functions of (r, θ) is the same Euclidean metric in the coordinates $\{\tilde{r}, \theta, \phi\}$ where \tilde{r} is the same function of r . In the actual process we get $X = X_K$ at first. Then one of Carter equations gives $W = W_K$. The rest are well-known. Let

$$r_{\text{out}} = \tilde{r}, \quad r_{\text{in}} = (1/2) \left(r - M - \sqrt{r^2 - 2Mr + e^2 + m^2 + a^2} \right) \quad (53)$$

If we take ζ and f as ζ^+ and f^+ and define

$$\zeta^- = r_{\text{in}}^2 \varrho^{-2} \quad (54)$$

$$f^- = r_{\text{in}}^2 \sin^2 \theta \quad (55)$$

then $\eta_{\text{K}}^- = \zeta^- \bar{g}_{\text{K}} + f^- d\phi^2 = dr_{\text{in}}^2 + r_{\text{in}}^2 (d\theta^2 + \sin^2 \theta d\phi^2)$ is also the Euclidean metric. Recalling Eq. (46) we note that $r_{\text{in}} = (1/4)c^2 r_{\text{out}}^{-1}$. So

$$f^- = (c^4/16)r_{\text{out}}^{-4}f = (16/c^4)r_{\text{in}}^4 f, \quad \zeta^- = (16/c^4)r_{\text{in}}^4 \zeta \quad (56)$$

We shall use spinor identities for the metric

$$\chi = \sigma^2 \zeta \bar{g} + U f d\phi^2 \quad (57)$$

where $U = U(r)$ is a solution of a first order ODE with appropriate boundary conditions to be specified later (see Eq. (113) and Lemma 10.3 below) and

$$\sigma = (X/X_{\text{K}})^{\frac{1}{4}} > 0 \quad (58)$$

We define $\chi^- = \sigma^2 \zeta^- \bar{g} + U f^- d\phi^2$. Then

$$(c^4/16)r_{\text{in}}^{-4} \chi^- = \chi^+ \equiv \chi \quad (59)$$

provided the same function U is used for both the metrics χ^\pm . The actual functions we shall use are not known to be the same initially. However Eq. (59) is useful in transforming formulas.

$r_{\text{out}} = r_{\text{in}}$ occurs at $r = M \pm c$. At these values $r_{\text{out/in}} = \pm c/2 = c/2$ neglecting the negative sign. For a Kerr-Newman solution η_{K}^- and $\eta_{\text{K}}^+ = \eta_{\text{K}}$ match on the boundary sphere of radius $r = M + c = M + \sqrt{M^2 - e^2 - m^2 - a^2}$ which corresponds to the outer Killing horizon. We have no business inside the outer Killing horizon.

For the general situation let $\eta^+ = \eta$ and let η^- be defined by replacing f and ζ in Eq. (52) with f^- and ζ^- . Asymptotic conditions ensure that η^+ is asymptotically flat with mass zero and η^- compactifies the infinity. So if we can show that these metric have nonnegative scalar curvature and they match smoothly at the inner boundaries, then positive mass theorem makes them Euclidean. Since we could not directly show that this scalar curvature is nonnegative we follow a detour. Keeping the spinorial proof of the positive mass theorem in mind we construct two spinor identities that solves difficult parts of the problem.

6 Computation in r, θ coordinates

If we define r, θ coordinates using Eqs (44) then we get

$$d\rho^2 + dz^2 = (r^2 - 2M'r + (M^2 - e^2 - m^2) \sin^2 \theta + a^2 \cos^2 \theta) \left[\frac{dr^2}{r^2 - 2M'r + a^2} + d\theta^2 \right] \quad (60)$$

Let $\Pi = (r^2 - 2M'r + a^2)^{-1} dr^2 + d\theta^2$. We have

$$\bar{g} = \Omega (r^2 - 2M'r + (M^2 - e^2 - m^2) \sin^2 \theta + a^2 \cos^2 \theta) \Pi \quad (61)$$

$$\bar{g}_{\theta\theta} = \Omega (r^2 - 2M'r + (M^2 - e^2 - m^2) \sin^2 \theta + a^2 \cos^2 \theta) = |\bar{\nabla}\theta|^{-2} \quad (62)$$

$$\bar{g}_{rr} = \bar{g}_{\theta\theta} (r^2 - 2M'r + a^2)^{-1} = |\bar{\nabla}r|^{-2} \quad (63)$$

Only nontrivial Christoffel symbol of Π is $\Gamma_{\Pi rr}^r = -(r^2 - 2M'r + a^2)^{-1} (r - M)$. All other Christoffel symbol of Π vanish. So using $\bar{\Delta}_{s\Pi} u = s^{-1} \bar{\Delta}_{\Pi} u$ and Eq. (61) we get

$$\bar{\Delta}r = (r - M)(r^2 - 2M'r + a^2)^{-1} |\bar{\nabla}r|^2 \quad (64)$$

$$\bar{\Delta}\theta = 0 \quad (65)$$

We note that

$$\begin{aligned} \bar{\Delta} \ln (f / \sin^2 \theta) &= \bar{\Delta} \ln r_{\text{out}} = 0 \\ \bar{\Delta} \ln f &= \bar{\Delta} \ln (\sin^2 \theta) = -2 \bar{g}^{\theta\theta} \csc^2 \theta \end{aligned} \quad (66)$$

The first equation follows because in \mathbb{R}^2 , $\ln r_{\text{out}}$ is a harmonic function. It can also be checked by explicit calculation using Eq. (64). Similarly the second equation follows because by virtue of Eq. (65), $\bar{\Delta} \ln f = -2 \csc^2 \theta |\bar{\nabla}\theta|^2$.

Eqs. (53) give $\frac{dr_{\text{out/in}}}{dr} = \pm \frac{r_{\text{out/in}}}{\sqrt{r^2 - 2M'r + a^2}}$. (In an earlier version of [2] there is an erroneous factor in equations corresponding to this equation and the following equation. Eq. 58 of that paper should have a factor 2 in the RHS. The factor of 1/4's before $(d \ln U / dr_{\text{out/in}})$ in Eqs.125-128 of that paper should be 1/2's and 1's inside the parentheses of those equations should be 2's.) For a differentiable

function $U = U(r)$ for $r > M + c$,

$$\lim_{r \rightarrow (M+c)^+} \frac{d \ln U}{dr_{\text{out/in}}} = \pm \lim_{r \rightarrow (M+c)^+} \frac{2 \sqrt{r^2 - 2M'r + a^2}}{c} \frac{d \ln U}{dr} \quad (67)$$

where + sign of \pm is for r_{out} . These equations need some clarification because finally we shall arrange such that at $r_{\text{out/in}} = c/2$, $\frac{d \ln U}{dr_{\text{out}}} = \frac{d \ln U}{dr_{\text{in}}}$. Thus in the RHS of Eq. (67), $U = U(r)$ are two different functions U^\pm of r unless $(d \ln U/dr)$ vanishes.

7 Scalar curvature of the 3-metric χ

We compute the scalar curvature R_χ of χ defined in Eq (57).

Lemma 7.1.

$$\sigma^2 \zeta R_\chi = \frac{1}{2} \left| \bar{\nabla} \ln U \right|^2 + \mathcal{P} - U^{-1} \left(\bar{\Delta} U + \left\langle \bar{\nabla} U, \bar{\nabla} \ln f \right\rangle + Q_{\bar{g}} U \right). \quad (68)$$

where $Q_{\bar{g}}$ and \mathcal{P} are as follows.

$$Q_{\bar{g}} = \left(4 \left\langle \bar{\nabla} \ln (X_K / \sqrt{\rho}), \bar{\nabla} \ln \sigma \right\rangle + f_{em} \right)^- \quad (69)$$

$$\mathcal{P} = \left(4 \left\langle \bar{\nabla} \ln (X_K / \sqrt{\rho}), \bar{\nabla} \ln \sigma \right\rangle + f_{em} \right)^+ + 8 \left| \bar{\nabla} \ln \sigma \right|^2 \quad (70)$$

Proof: For a given function \tilde{f} the scalar curvature R_γ of $\gamma = \bar{g} + \tilde{f} d\phi^2$ is given by

$$R_\gamma = \bar{R} - \tilde{f}^{-1} \bar{\Delta} \tilde{f} + \frac{1}{2} \left| \bar{\nabla} \ln \tilde{f} \right|^2 \quad (71)$$

Let $\tilde{f} = f \zeta^{-1}$. Then $\eta = \zeta \gamma$. So using the conformal transformation formula

$$\eta = \Psi^4 \gamma, \quad \Psi^4 R_\eta = R_\gamma - 8 \Psi^{-1} \Delta_\gamma \Psi = R_\gamma - 8 \Delta_\gamma \ln \Psi - 8 \left| \bar{\nabla} \ln \Psi \right|_\gamma^2 \quad (72)$$

and writing the Laplacian Δ_γ relative to the 3-metric γ in terms of the Laplacian of \bar{g} using

$$\Delta_\gamma u = \bar{\Delta} u + (1/2) \left\langle \bar{\nabla} \ln \tilde{f}, \bar{\nabla} u \right\rangle \quad (73)$$

we get $\zeta R_\eta = R_\gamma - 2\zeta^{-1}\bar{\Delta}\zeta + (3/2)\zeta^{-2}|\bar{\nabla}\zeta|^2 - \left\langle \bar{\nabla} \ln \tilde{f}, \bar{\nabla} \ln \zeta \right\rangle$. Using Eqs. (71,37) we then get

$$\begin{aligned} \zeta R_\eta &= (1/2)X^{-1}\bar{\Delta}X - (1/2)\left\langle \bar{\nabla} \ln \rho, \bar{\nabla} \ln X \right\rangle + X^{-1}|\bar{\nabla}\psi|^2 + X^{-1}\left|X\bar{\nabla}\xi + W\bar{\nabla}\psi\right|^2 \rho^{-2} \\ &\quad - \tilde{f}^{-1}\bar{\Delta}\tilde{f} + (1/2)|\bar{\nabla} \ln \tilde{f}|^2 - 2\zeta^{-1}\bar{\Delta}\zeta + (3/2)|\bar{\nabla} \ln \zeta|^2 - \left\langle \bar{\nabla} \ln \tilde{f}, \bar{\nabla} \ln \zeta \right\rangle \quad (74) \end{aligned}$$

Since for Kerr-Newman this gives

$$\begin{aligned} 0 &= (1/2)X_K^{-1}\bar{\Delta}X_K - (1/2)\left\langle \bar{\nabla} \ln \rho, \bar{\nabla} \ln X_K \right\rangle + X_K^{-1}|\bar{\nabla}\psi_K|^2 + X_K^{-1}\left|X_K\bar{\nabla}\xi_K + W_K\bar{\nabla}\psi_K\right|^2 \rho^{-2} \\ &\quad - \tilde{f}^{-1}\bar{\Delta}\tilde{f} + (1/2)|\bar{\nabla} \ln \tilde{f}|^2 - 2\zeta^{-1}\bar{\Delta}\zeta + (3/2)|\bar{\nabla} \ln \zeta|^2 - \left\langle \bar{\nabla} \ln \tilde{f}, \bar{\nabla} \ln \zeta \right\rangle \end{aligned}$$

writing

$$f_{\text{em}} = X^{-1}|\bar{\nabla}\psi|^2 + X^{-1}\left|X\bar{\nabla}\xi + W\bar{\nabla}\psi\right|^2 \rho^{-2} - X_K^{-1}|\bar{\nabla}\psi_K|^2 - X_K^{-1}\left|X_K\bar{\nabla}\xi_K + W_K\bar{\nabla}\psi_K\right|^2 \rho^{-2}$$

we get

$$\begin{aligned} \zeta R_\eta &= (1/2)X^{-1}\bar{\Delta}X - (1/2)\left\langle \bar{\nabla} \ln \rho, \bar{\nabla} \ln X \right\rangle + f_{\text{em}} - (1/2)X_K^{-1}\bar{\Delta}X_K + (1/2)\left\langle \bar{\nabla} \ln \rho, \bar{\nabla} \ln X_K \right\rangle \\ &= (1/2)\bar{\Delta} \ln X + (1/2)|\bar{\nabla} \ln X|^2 - (1/2)\bar{\Delta} \ln X_K - (1/2)|\bar{\nabla} \ln X_K|^2 - (1/2)\left\langle \bar{\nabla} \ln \rho, \bar{\nabla} \ln(X/X_K) \right\rangle + f_{\text{em}} \\ &= \frac{1}{2}\bar{\Delta} \ln(X/X_K) + (1/2)\left\langle \bar{\nabla} \ln(XX_K/\rho), \bar{\nabla} \ln(X/X_K) \right\rangle + f_{\text{em}} \\ &= (1/2)\bar{\Delta} \ln(X/X_K) + (1/2)|\bar{\nabla} \ln(X/X_K)|^2 + (1/2)\left\langle \bar{\nabla} \ln(X_K^2/\rho), \bar{\nabla} \ln(X/X_K) \right\rangle + f_{\text{em}} \\ &= 2\bar{\Delta} \ln \sigma + 8|\bar{\nabla} \ln \sigma|^2 + 2\left\langle \bar{\nabla} \ln(X_K^2/\rho), \bar{\nabla} \ln \sigma \right\rangle + f_{\text{em}} \end{aligned}$$

where in the last step we used Eq. (58). Remembering $Q_{\bar{g}}$ and \mathcal{P} we get

$$\zeta R_\eta = 2\bar{\Delta} \ln \sigma + \mathcal{P} - Q_{\bar{g}} \quad (75)$$

Next we compute the scalar curvature of

$$\vartheta = \sigma^2 \eta \quad (76)$$

$R_\vartheta = \sigma^{-2}R_\eta - 4\sigma^{-3}\Delta_\eta\sigma + 2\sigma^{-4}|\nabla\sigma|_\eta^2 = \sigma^{-2}R_\eta - 4\sigma^{-2}\Delta_\eta \ln \sigma - 2\sigma^{-2}|\nabla \ln \sigma|_\eta^2$. Using Eq. (52) and the formula Eq. (73) we have

$$\Delta_\eta \ln \sigma = \bar{\Delta}_{\xi\bar{g}} \ln \sigma + \frac{1}{2f} \left\langle \bar{\nabla} f, \bar{\nabla} \ln \sigma \right\rangle_{\xi\bar{g}}$$

Thus $\zeta\sigma^2R_\theta = \zeta R_\eta - 4\bar{\Delta}\ln\sigma - 2\langle\bar{\nabla}\ln f, \bar{\nabla}\ln\sigma\rangle - 2|\bar{\nabla}\ln\sigma|^2$. Using Eq. (75) we get

$$\zeta\sigma^2R_\theta = -2\bar{\Delta}\ln\sigma + \mathcal{P} - Q_{\bar{g}} - 2\langle\bar{\nabla}\ln\sigma, \bar{\nabla}\ln f\rangle - 2|\bar{\nabla}\ln\sigma|^2 \quad (77)$$

Finally we write χ as $\vartheta + \varpi d\phi^2$. $\chi_{\phi\phi} = Uf = \sigma^2f + \varpi \Rightarrow U = f^{-1}(\sigma^2f + \varpi)$. We recall that if $h = \bar{G} + \varphi d\phi^2$, and $\widehat{h} = \bar{G} + \widehat{\varphi}d\phi^2$, where \bar{G} is a 2-dimensional metric on the $\phi = \text{constant}$ surfaces and $\varphi, \widehat{\varphi}$ are independent of ϕ , then

$$R_h = R_{\widehat{h}} + \bar{\Delta}_{\bar{G}}\ln\frac{\widehat{\varphi}}{\varphi} - \frac{1}{2}|\bar{\nabla}\ln\varphi|_{\bar{G}}^2 + \frac{1}{2}|\bar{\nabla}\ln\widehat{\varphi}|_{\bar{G}}^2 \quad (78)$$

Taking $\bar{G} = \sigma^2\zeta\bar{g}$, $\varphi = \sigma^2f + \varpi = \chi_{\phi\phi}$ and $\widehat{\varphi} = \sigma^2f = \vartheta_{\phi\phi}$ we get

$$\begin{aligned} \sigma^2\zeta R_\chi &= \sigma^2\zeta\left(R_\theta + \bar{\Delta}_{\sigma^2\zeta\bar{g}}\ln\frac{\sigma^2f}{\sigma^2f + \varpi} - (1/2)|\bar{\nabla}\ln(\sigma^2f + \varpi)|_{\bar{G}}^2 + (1/2)|\bar{\nabla}\ln(\sigma^2f)|_{\bar{G}}^2\right) \\ &= \sigma^2\zeta\left(R_\theta + \sigma^{-2}\zeta^{-1}\bar{\Delta}\ln\frac{\sigma^2f}{\sigma^2f + \varpi} - (1/2)\sigma^{-2}\zeta^{-1}|\bar{\nabla}\ln(\sigma^2f + \varpi)|^2 + (1/2)\sigma^{-2}\zeta^{-1}|\bar{\nabla}\ln(\sigma^2f)|^2\right) \\ &= \sigma^2\zeta R_\theta + \bar{\Delta}\ln(\sigma^2U^{-1}) - (1/2)|\bar{\nabla}\ln(Uf)|^2 + (1/2)|\bar{\nabla}\ln(\sigma^2f)|^2 \\ &= \sigma^2\zeta R_\theta + \bar{\Delta}\ln(\sigma^2U^{-1}) - (1/2)|\bar{\nabla}\ln U|^2 + 2|\bar{\nabla}\ln\sigma|^2 - \langle\bar{\nabla}\ln U, \bar{\nabla}\ln f\rangle + 2\langle\bar{\nabla}\ln\sigma, \bar{\nabla}\ln f\rangle \\ &= -2\bar{\Delta}\ln\sigma + \mathcal{P} - Q_{\bar{g}} + \bar{\Delta}\ln(\sigma^2U^{-1}) - (1/2)|\bar{\nabla}\ln U|^2 - \langle\bar{\nabla}\ln U, \bar{\nabla}\ln f\rangle \end{aligned}$$

where in the last step we used Eq. (77). Thus we get Eq. (68). |

Similarly we find the scalar curvature of $\chi^- = \sigma^2\zeta^-\bar{g} + Uf^-d\phi^2$ to be

$$R_{\chi^-} = (\zeta^-)^{-1}\sigma^{-2}\left(\mathcal{P} - Q_{\bar{g}} - U^{-1}\bar{\Delta}U + \frac{1}{2}|\bar{\nabla}\ln U|^2 - \langle\bar{\nabla}\ln U, \bar{\nabla}\ln f^-\rangle\right) \quad (79)$$

We can also derive this formula by conformal transformation $\chi^- = (16/c^4)r_{\text{in}}^4\chi$ (see Eq. (59)) and the fact that $\ln r_{\text{in}}$ is a harmonic function in the 2-metric \bar{g} .

8 Finding a Spinor

Let r_0 be a constant. Let $\oint\!\!\!\int_{r=r_0, \eta}$ represents the surface integral on the $r = r_0$ surface relative to the 2-metric induced from η and $\iint_{r=r_0}$ represents ordinary double integral. All integrations are done on subsets of $\Sigma^+ \cup \partial\Sigma^+$ unless indicated otherwise.

We need a $SU(2)$ -spinor Θ_θ on Σ with the following properties.

$$D_\theta \Theta_\theta = 0 \quad (80)$$

$$\|\Theta_\theta\| = 1 + O(r^{-1}) \text{ as } r \rightarrow \infty \quad (81)$$

$$\frac{\partial \|\Theta_\theta\|^2}{\partial r} = o(r^{-1}) \text{ as } r \rightarrow \infty \quad (82)$$

$$\Theta_\theta \text{ is independent of } \phi \quad (83)$$

$$\text{As } r_0 \downarrow M + c \text{ on the horizon } \int_{r=r_0} \sqrt{r^2 - 2M'r + a^2} \frac{\partial \|\Theta_\theta\|^2}{\partial r} d\theta = 0 \quad (84)$$

Here D_θ is the Dirac operator of the metric ϑ . Such a spinor exists. On the double the 3-metric \widehat{g} has nonnegative scalar curvature, and it is asymptotically flat. So we can use Bartnik's proof for the existence of a spinor $\Theta_{\widehat{g}}$ harmonic relative to \widehat{g} . Because of the axisymmetry we can choose $\Theta_{\widehat{g}}$ to be independent of ϕ . Θ_θ can be obtained $\Theta_{\widehat{g}}$ by what we called a 2+1 conformal transformation. It is explained below. We have outlined the proof of the following lemma in the appendix.

Lemma 8.1. *Let $\overline{G} = \overline{G}_{11}(d(x^1)^2 + d(x^2)^2)$, $g_1 = \overline{G} + f_1 d\phi^2$ and $g_2 = \overline{G} + q f_1 d\phi^2$. All functions and metrics are independent of ϕ . If Θ is a spinor satisfying the Dirac equation $D_{g_1} \Theta = 0$ and Θ is independent of ϕ , then*

$$D_{g_2} \left(q^{-\frac{3}{8}} \Theta \right) = 0$$

We also have the conformal transformation formula. Let $\xi_{\psi^{-2}\widehat{\chi}}$ be a fixed spinor satisfying Dirac equation relative to the metric $\psi^{-2}\widehat{\chi}$. Then the spinor $\xi_{\widehat{\chi}} = \psi^{-1} \xi_{\psi^{-2}\widehat{\chi}}$ satisfies Dirac equation relative to the conformal metric $\widehat{\chi}$ (Lichnerowicz [19], Branson, T., Kosmann-Schwarzbach [20]).

To find the spinor Θ_θ from $\Theta_{\widehat{g}}$ we take $g_1 = \widehat{g} = \overline{g} + X d\phi^2$ and $g_2 = \sigma^{-2} \zeta^{-1} \vartheta = \overline{g} + \zeta^{-1} f d\phi^2$ in Lemma 8.1. That is we put $f_1 = X$ and $q = f \zeta^{-1} X^{-1}$. Then $D_{\sigma^{-2} \zeta^{-1} \vartheta} \left((f \zeta^{-1} X^{-1})^{-3/8} \Theta_{\widehat{g}} \right) = 0$. So the spinor

$$\Theta_\theta = \sigma^{-1} \zeta^{-1/2} (f \zeta^{-1} X^{-1})^{-3/8} \Theta_{\widehat{g}} \quad (85)$$

satisfies $D_\theta \Theta_\theta = 0$. We note that

$$\|\Theta_\theta\|^2 = \sigma^{-2} \zeta^{-1/4} X^{3/4} f^{-3/4} \|\Theta_{\widehat{g}}\|^2 \quad (86)$$

Now $8\pi M = - \iint_{\lambda_0 \rightarrow \infty, \widehat{g}} \left\langle \overline{\nabla} \|\Theta_{\widehat{g}}\|^2, n_{\widehat{g}} \right\rangle_{\widehat{g}}$ so that (because of the asymptotic regularity in the existence proof of the spinor)

$$\|\Theta_{\widehat{g}}\|^2 = 1 - 2M/r + O(r^{-2}) \quad (87)$$

which by virtue of the asymptotic conditions on σ, f, ζ, X gives

$$\|\Theta_{\theta}\|^2 = 1 + O(r^{-2}) \quad (88)$$

Thus Θ_{θ} satisfies properties 80-83. To see Eq. (84) we note that

$\int_{r=r_0} \sqrt{r^2 - 2M'r + a^2} \frac{\partial \|\Theta_{\theta}\|^2}{\partial r} d\theta = - \oint_{r_0, \overline{\theta}} \left\langle \overline{\nabla} \|\Theta_{\theta}\|^2, n_{\overline{\theta}} \right\rangle_{\overline{\theta}}$ where $n_{\overline{\theta}}$ is the unit normal form on the $r = \text{constant}$ loop on a $\phi = \text{constant}$ surface with the normal vector pointing towards decreasing r . $\Theta_{\widehat{g}}$ is a harmonic spinor on the double $\Sigma^+ \cup \Sigma^- \cup \partial\Sigma^+$ relative to the metric \widehat{g} producing the same contribution at each end for $\lim_{r_0 \uparrow \infty} \iint_{r_0, \widehat{g}} \left\langle \overline{\nabla} \|\Theta_{\widehat{g}}\|^2, n_{\widehat{g}} \right\rangle_{\widehat{g}}$. Thus on the horizon $\iint_{\widehat{g}} \left\langle \overline{\nabla} \|\Theta_{\widehat{g}}\|^2, n_{\widehat{g}} \right\rangle_{\widehat{g}} = 0$ because of the symmetry across the totally geodesic boundary where $n_{\widehat{g}}$ is the unit normal form on the relevant surface. Since the derivative of $\sigma^{-2} \zeta^{-1/4} X^{3/4} f^{-3/4}$ is regular on the horizon we get Eq. (84).

Now we find a spinor satisfying $D_{\chi} \Theta_{\chi} = 0$. Recalling Eqs. (76,52,57) for χ and θ we apply Lemma 8.1 again with $g_1 = \theta, f_1 = f\sigma^2, g_2 = \chi$ and $q = U\sigma^{-2}$. Thus we take

$$\Theta_{\chi} = \sigma \frac{3}{4} U \overline{\frac{3}{8}} \Theta_{\theta} \quad (89)$$

For the harmonic spinor Θ_{χ} one has the identity,

$$2\Delta_{\chi} \|\Theta_{\chi}\|^2 = R_{\chi} \|\Theta_{\chi}\|^2 + 4\|\nabla_{\chi} \Theta_{\chi}\|^2 \quad (90)$$

Using the expression for R_{χ} from Eq. (68) we get

$$2\Delta_{\chi} \|\Theta_{\chi}\|^2 = \left((1/2) \left\langle \overline{\nabla} \ln U, \overline{\nabla} \ln U \right\rangle + \mathcal{P} - U^{-1} \left(\overline{\Delta} U + \left\langle \overline{\nabla} U, \overline{\nabla} \ln f \right\rangle + Q_{\widehat{g}} U \right) \right) \sigma^{-2} \zeta^{-1} \|\Theta_{\chi}\|^2 + 4\|\nabla_{\chi} \Theta_{\chi}\|^2 \quad (91)$$

For complex U , $|\overline{\nabla} \ln U|^2$ is not nonnegative definite. So we have replaced it in Eq. (68) by $\left\langle \overline{\nabla} \ln U, \overline{\nabla} \ln U \right\rangle$ to remove confusion. Our next aim is to write the above identity using 2-dimensional Laplacian because that way we can easily tackle integration if U becomes complex.

9 Two spinor identities on Σ^\pm

Let $L = (r, \theta)$

$$|U|^{-\frac{3}{4}} / \|\Theta_\chi\|^2 = L \quad (92)$$

L is not defined on the possible zero set of $\|\Theta_\chi\|$ in case the known spinor $\Theta_{\bar{g}}$ can vanish. By Eq. (89), $\sigma^{3/2}L\|\Theta_\theta\|^2 = 1$. We need to introduce L on Σ^\pm only for $L_{ave} \leq 4/3$. This will be clear later from Eqs. (115,120). $L_{ave} = L_{ave}(r)$ is defined as follows.

$$L_{ave}(r_0) \int_{r=r_0} \sigma^{3/2} \|\Theta_\theta\|^2 \sin^2 \theta d\theta = \int_{r=r_0} \sigma^{3/2} L \|\Theta_\theta\|^2 \sin^2 \theta d\theta \quad (93)$$

We shall use L either in the form $L\|\Theta_\theta\|^2$ or in L_{ave} . We removed the θ dependence from $L = L(r, \theta)$ by averaging L on the $r = \text{constant}$ loops on a $\phi = \text{constant}$ surface. For future reference we note that L and $L_{ave} \rightarrow 1$ as $r \rightarrow \infty$. This follows from Eq. (81).

We denote a $\phi = \text{constant}$ surface in Σ^+ by Σ_2^+ . Let $U = |U|e^{i\omega}$, ω being real. We assume that U, ω are functions of r only.

Lemma 9.1. *On $(\Sigma_2^+, \bar{\chi})$ for $U = U(r)$ wherever $U = U(r)$ is twice differentiable,*

$$\begin{aligned} \bar{\nabla}_{\bar{\chi}} \left(\sin^2 \theta \left(2\bar{\nabla}\|\Theta_\chi\|^2 + 2\|\Theta_\chi\|^2 \bar{\nabla} \ln r_{out} - 2|U|^{-\frac{3}{4}} \bar{\nabla} \ln r_{out} + \|\Theta_\chi\|^2 \bar{\nabla} \ln U \right) - (\|\Theta_\chi\|^2 - 1) \bar{\nabla} \sin^2 \theta \right) = \\ - \|\Theta_\chi\|^2 \bar{\Delta}_{\bar{\chi}} \sin^2 \theta + \bar{\Delta}_{\bar{\chi}} \sin^2 \theta + 4 \sin^2 \theta \|\bar{\nabla}_\chi \Theta_\chi\|^2 - \frac{3}{4} iL \left(\bar{\nabla} \omega, \bar{\nabla} \ln f \right)_{\bar{\chi}} \|\Theta_\chi\|^2 \sin^2 \theta + \\ \sin^2 \theta \left(\mathcal{P} - \frac{1}{2} \left(\bar{\nabla} \ln U, \bar{\nabla} \ln U \right) - \frac{1}{4} (4 - 3L) \left(\bar{\nabla} \ln U, \bar{\nabla} \ln f \right) - \mathcal{Q}_{\bar{g}} \right) \zeta^{-1} \|\Theta_\chi\|^2 \end{aligned} \quad (94)$$

Proof: Writing the 3-Laplacian Δ_χ relative to $\chi = \sigma^2 \zeta \bar{g} + U f d\phi^2$ in terms of the Laplacian of the 2-metric $\bar{\chi} = \sigma^2 \zeta \bar{g}$ we get $2\Delta_\chi \|\Theta_\chi\|^2 = 2\bar{\Delta}_{\bar{\chi}} \|\Theta_\chi\|^2 + \left(\bar{\nabla} \ln(Uf), \bar{\nabla} \|\Theta_\chi\|^2 \right)_{\bar{\chi}}$. We also have $U^{-1} \|\Theta_\chi\|^2 \bar{\Delta}_{\bar{\chi}} U = \bar{\nabla}_{\bar{\chi}} \left(\|\Theta_\chi\|^2 U^{-1} \bar{\nabla} U \right) - \left(\bar{\nabla} \|\Theta_\chi\|^2, \bar{\nabla} \ln U \right)_{\bar{\chi}} + \left(\bar{\nabla} \ln U, \bar{\nabla} \ln U \right)_{\bar{\chi}} \|\Theta_\chi\|^2$. So Eq. (91) gives $2\bar{\Delta}_{\bar{\chi}} \|\Theta_\chi\|^2 + \left(\bar{\nabla} \ln f, \bar{\nabla} \|\Theta_\chi\|^2 \right)_{\bar{\chi}} + \bar{\nabla}_{\bar{\chi}} \left(\|\Theta_\chi\|^2 U^{-1} \bar{\nabla} U \right) = \left(\mathcal{P} - (1/2) \left(\bar{\nabla} \ln U, \bar{\nabla} \ln U \right) - \left(\bar{\nabla} \ln U, \bar{\nabla} \ln f \right) - \mathcal{Q}_{\bar{g}} \right) \sigma^{-2} \zeta^{-1} \|\Theta_\chi\|^2 + 4 \|\bar{\nabla}_\chi \Theta_\chi\|^2 \Rightarrow 2\bar{\Delta}_{\bar{\chi}} \|\Theta_\chi\|^2 + \bar{\nabla}_{\bar{\chi}} \left(\|\Theta_\chi\|^2 \bar{\nabla} \ln f - 2|U|^{-3/4} \bar{\nabla} \ln r_{out} + \|\Theta_\chi\|^2 \bar{\nabla} \ln U \right) = 4 \|\bar{\nabla}_\chi \Theta_\chi\|^2 + \|\Theta_\chi\|^2 \sigma^{-2} \zeta^{-1} \bar{\Delta} \ln f + (3/2)|U|^{-3/4} \left(\bar{\nabla} \ln |U|, \bar{\nabla} \ln r_{out} \right) + \left(\mathcal{P} - (1/2) \left(\bar{\nabla} \ln U, \bar{\nabla} \ln U \right) - \left(\bar{\nabla} \ln U, \bar{\nabla} \ln f \right) - \mathcal{Q}_{\bar{g}} \right) \sigma^{-2} \zeta^{-1} \|\Theta_\chi\|^2$. Now we use

Eq. (66) to get

$$\begin{aligned} \bar{\nabla}_{\bar{\chi}} \left(2\bar{\nabla}\|\Theta_{\chi}\|^2 + \|\Theta_{\chi}\|^2\bar{\nabla}\ln f - 2|U|^{-3/4}\bar{\nabla}\ln r_{\text{out}} + \|\Theta_{\chi}\|^2\bar{\nabla}\ln U \right) &= 4\|\nabla_{\chi}\Theta_{\chi}\|^2 - 2\|\Theta_{\chi}\|^2\sigma^{-2}\zeta^{-1}\bar{g}^{\theta\theta}\csc^2\theta + \\ (3/2)|U|^{-3/4}\left\langle\bar{\nabla}\ln|U|, \bar{\nabla}\ln r_{\text{out}}\right\rangle_{\bar{\chi}} + \left(\mathcal{P} - (1/2)\left\langle\bar{\nabla}\ln U, \bar{\nabla}\ln U\right\rangle - \left\langle\bar{\nabla}\ln U, \bar{\nabla}\ln f\right\rangle - \mathcal{Q}_{\bar{g}}\right)\sigma^{-2}\zeta^{-1}\|\Theta_{\chi}\|^2 \end{aligned} \quad (95)$$

For $U = U(r)$, $2\left\langle\bar{\nabla}\ln|U|, \bar{\nabla}\ln r_{\text{out}}\right\rangle = \left\langle\bar{\nabla}\ln|U|, \bar{\nabla}\ln f\right\rangle = \left\langle\bar{\nabla}\ln U, \bar{\nabla}\ln f\right\rangle - i\left\langle\bar{\nabla}\omega, \bar{\nabla}\ln f\right\rangle$. So RHS of Eq. (95) simplifies to

$$\begin{aligned} 4\|\nabla_{\chi}\Theta_{\chi}\|^2 - 2\|\Theta_{\chi}\|^2\sigma^{-2}\zeta^{-1}\bar{g}^{\theta\theta}\csc^2\theta - (3/4)iL\left\langle\bar{\nabla}\omega, \bar{\nabla}\ln f\right\rangle_{\bar{\chi}}\|\Theta_{\chi}\|^2 + \\ \left(\mathcal{P} - (1/2)\left\langle\bar{\nabla}\ln U, \bar{\nabla}\ln U\right\rangle - \frac{1}{4}(4 - 3L)\left\langle\bar{\nabla}\ln U, \bar{\nabla}\ln f\right\rangle - \mathcal{Q}_{\bar{g}}\right)\sigma^{-2}\zeta^{-1}\|\Theta_{\chi}\|^2 \end{aligned}$$

Now we multiply both sides of Eq. (95) by $\sin^2\theta$. LHS becomes (since U, r_{out} are functions of r only),

$$\begin{aligned} \bar{\nabla}_{\bar{\chi}} \left(\sin^2\theta \left(2\bar{\nabla}\|\Theta_{\chi}\|^2 + \|\Theta_{\chi}\|^2\bar{\nabla}\ln f - 2|U|^{-3/4}\bar{\nabla}\ln r_{\text{out}} + \|\Theta_{\chi}\|^2\bar{\nabla}\ln U \right) \right) - \\ 2\left\langle\bar{\nabla}\sin^2\theta, \bar{\nabla}\|\Theta_{\chi}\|^2\right\rangle_{\bar{\chi}} - \left\langle\bar{\nabla}\sin^2\theta, \bar{\nabla}\ln(\sin^2\theta)\right\rangle_{\bar{\chi}}\|\Theta_{\chi}\|^2 = \\ \bar{\nabla}_{\bar{\chi}} \left(\sin^2\theta \left(2\bar{\nabla}\|\Theta_{\chi}\|^2 + \|\Theta_{\chi}\|^2\bar{\nabla}\ln f - 2|U|^{-3/4}\bar{\nabla}\ln r_{\text{out}} + \|\Theta_{\chi}\|^2\bar{\nabla}\ln U \right) \right) - 2\bar{\nabla}_{\bar{\chi}} \left(\|\Theta_{\chi}\|^2\bar{\nabla}_{\bar{\chi}}\sin^2\theta \right) + 2\|\Theta_{\chi}\|^2\bar{\Delta}_{\bar{\chi}}\sin^2\theta \\ - 4\sigma^{-2}\zeta^{-1}\bar{g}^{\theta\theta}\|\Theta_{\chi}\|^2\cos^2\theta = \\ \bar{\nabla}_{\bar{\chi}} \left(\sin^2\theta \left(2\bar{\nabla}\|\Theta_{\chi}\|^2 + 2\|\Theta_{\chi}\|^2\bar{\nabla}\ln r_{\text{out}} + 2\|\Theta_{\chi}\|^2\bar{\nabla}\ln\sin\theta - 2|U|^{-3/4}\bar{\nabla}\ln r_{\text{out}} + \|\Theta_{\chi}\|^2\bar{\nabla}\ln U \right) \right) - 2\bar{\nabla}_{\bar{\chi}} \left(\|\Theta_{\chi}\|^2\bar{\nabla}_{\bar{\chi}}\sin^2\theta \right) \\ + 2\|\Theta_{\chi}\|^2\bar{\Delta}_{\bar{\chi}}\sin^2\theta - 4\sigma^{-2}\zeta^{-1}\bar{g}^{\theta\theta}\|\Theta_{\chi}\|^2\cos^2\theta = \\ \bar{\nabla}_{\bar{\chi}} \left(\sin^2\theta \left(2\bar{\nabla}\|\Theta_{\chi}\|^2 + 2\|\Theta_{\chi}\|^2\bar{\nabla}\ln r_{\text{out}} - 2|U|^{-3/4}\bar{\nabla}\ln r_{\text{out}} + \|\Theta_{\chi}\|^2\bar{\nabla}\ln U \right) - \|\Theta_{\chi}\|^2\bar{\nabla}\sin^2\theta \right) + 2\|\Theta_{\chi}\|^2\bar{\Delta}_{\bar{\chi}}\sin^2\theta \\ - 4\sigma^{-2}\zeta^{-1}\bar{g}^{\theta\theta}\|\Theta_{\chi}\|^2\cos^2\theta = \\ \bar{\nabla}_{\bar{\chi}} \left(\sin^2\theta \left(2\bar{\nabla}\|\Theta_{\chi}\|^2 + 2\|\Theta_{\chi}\|^2\bar{\nabla}\ln r_{\text{out}} - 2|U|^{-3/4}\bar{\nabla}\ln r_{\text{out}} + \|\Theta_{\chi}\|^2\bar{\nabla}\ln U \right) - \|\Theta_{\chi}\|^2\bar{\nabla}\sin^2\theta \right) - 4\sigma^{-2}\zeta^{-1}\bar{g}^{\theta\theta}\|\Theta_{\chi}\|^2\sin^2\theta \end{aligned}$$

where we used

$$\bar{\Delta}\sin^2\theta = 2\bar{g}^{\theta\theta}\cos(2\theta) \quad (96)$$

Thus Eq. (95) becomes

$$\begin{aligned} \bar{\nabla}_{\bar{\chi}} \left(\sin^2\theta \left(2\bar{\nabla}\|\Theta_{\chi}\|^2 + 2\|\Theta_{\chi}\|^2\bar{\nabla}\ln r_{\text{out}} - 2|U|^{-3/4}\bar{\nabla}\ln r_{\text{out}} + \|\Theta_{\chi}\|^2\bar{\nabla}\ln U \right) - \|\Theta_{\chi}\|^2\bar{\nabla}\sin^2\theta \right) = \\ - 2\sigma^{-2}\zeta^{-1}\bar{g}^{\theta\theta}\|\Theta_{\chi}\|^2\cos(2\theta) + 4\sin^2\theta\|\nabla_{\chi}\Theta_{\chi}\|^2 - (3/4)iL\left\langle\bar{\nabla}\omega, \bar{\nabla}\ln f\right\rangle_{\bar{\chi}}\|\Theta_{\chi}\|^2\sin^2\theta \\ + \sin^2\theta \left(\mathcal{P} - \frac{1}{2}\left\langle\bar{\nabla}\ln U, \bar{\nabla}\ln U\right\rangle - (1/4)(4 - 3L)\left\langle\bar{\nabla}\ln U, \bar{\nabla}\ln f\right\rangle - \mathcal{Q}_{\bar{g}} \right)\sigma^{-2}\zeta^{-1}\|\Theta_{\chi}\|^2 \end{aligned} \quad (97)$$

Hence using Eq. (96) again and adding $\bar{\Delta}_{\bar{\chi}}\sin^2\theta$ to both sides we get Eq. (94). \blacksquare

On $(\Sigma^-, \bar{\chi}^-)$ we get a similar identity as Eq. (94) using the spinor $\Theta_{\chi^-} = \sigma^{3/4}U^{-3/8}\Theta_{\vartheta^-}$ (compare Eq. (89)) where $\vartheta^- = \sigma^2\eta^- = \sigma^2\zeta^-\bar{g} + \sigma^2f^-d\phi^2$ and Θ_{ϑ^-} we define replacing ζ, f by ζ^-, f^- in Eq. (85). Then by Eq. (56), $\Theta_{\chi^-} = (c^2/4)r_{\text{in}}^{-2}\sigma^{3/4}U^{-3/8}\Theta_{\vartheta^-} =$

$(c^2/4)r_{\text{in}}^{-2}\Theta_\chi$. This is Θ_χ on Σ^- and it may differ from Θ_χ on Σ^+ because in general $U(r)$ are different functions on Σ^\pm . However to keep notation simple we use U for U^\pm . We also recall $\bar{\chi}^- = (16/c^4)r_{\text{in}}^4\bar{\chi}$. The singular factor r_{in}^{-2} in Θ_{χ^-} makes it difficult to manipulate. So we shall write the identity for Σ^- using Θ_χ on Σ^- . This is done below using several transformation formulas which are straightforward to check.

$$\|\nabla_{\chi^-}(r_{\text{in}}^{-2}\Theta_\chi)\|^2 = 4r_{\text{in}}^{-4}|\nabla \ln r_{\text{in}}|_{\chi^-}^2\|\Theta_\chi\|^2 + r_{\text{in}}^{-4}\|\nabla_{\chi^-}\Theta_\chi\|^2 - 2r_{\text{in}}^{-4}\langle \nabla \ln r_{\text{in}}, \nabla\|\Theta_\chi\|^2 \rangle_{\chi^-} \quad (98)$$

$$(16/c^4)\|\nabla_{\chi^-}\Theta_\chi\|^2 = r_{\text{in}}^{-4}\|\nabla_\chi\Theta_\chi\|^2 - r_{\text{in}}^{-4}\langle \nabla \ln r_{\text{in}}, \nabla\|\Theta_\chi\|^2 \rangle_\chi + 2r_{\text{in}}^{-4}|\nabla \ln r_{\text{in}}|_\chi^2\|\Theta_\chi\|^2 \quad (99)$$

$$\Delta_{\chi^-} \ln r_{\text{in}} = (1/2)\langle \bar{\nabla} \ln(Uf^-), \bar{\nabla} \ln r_{\text{in}} \rangle_{\chi^-} \quad (100)$$

As usual when not specified explicitly norms and inner products for gradients of functions are w.r.t. \bar{g} . In order to tackle the contribution from the term $\|\Theta_\chi\|^2\langle \bar{\nabla} \ln U, \bar{\nabla} \ln r_{\text{in}} \rangle_{\chi^-}$ coming from Eq. (100) we shall need the following lemma. The lemma is proved in the appendix.

Lemma 9.2. *For the metric $\gamma = \sigma^2\zeta\bar{g} + |U|^{-4}Ufd\phi^2$ and spinor $\Theta_\gamma = |U|^{1/2}\Theta_\chi$,*

$$\|\nabla_\chi\Theta_\chi\|^2 - (1/2)\sigma^{-2}\zeta^{-1}\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f \rangle\|\Theta_\chi\|^2 = I + |U|^{-1}\|\nabla_\gamma\Theta_\gamma\|^2 \quad (101)$$

where I is a real function. In case U is a positive real function $I = 0$.

The spinor Θ_{χ^-} satisfies $D_{\chi^-}\Theta_{\chi^-} = 0$ by the conformal transformation formula. For it Eq. (90) becomes after dividing out by $c^4/16$,

$$2\Delta_{\chi^-}\|r_{\text{in}}^{-2}\Theta_\chi\|^2 = R_{\chi^-}\|r_{\text{in}}^{-2}\Theta_\chi\|^2 + 4\|\nabla_{\chi^-}(r_{\text{in}}^{-2}\Theta_\chi)\|^2 \quad (102)$$

Using Eqs. (98-100) we obtain

$$2\Delta_{\chi^-}\|r_{\text{in}}^{-2}\Theta_\chi\|^2 - 4\|\nabla_{\chi^-}(r_{\text{in}}^{-2}\Theta_\chi)\|^2 = -4r_{\text{in}}^{-4}\|\Theta_\chi\|^2\langle \bar{\nabla} \ln U^-, \bar{\nabla} \ln r_{\text{in}} \rangle_{\chi^-} + 2r_{\text{in}}^{-4}\Delta_{\chi^-}\|\Theta_\chi\|^2 - 4(c^4/16)r_{\text{in}}^{-8}\|\nabla_\chi\Theta_\chi\|^2 - 4r_{\text{in}}^{-4}\langle \nabla \ln r_{\text{in}}, \nabla\|\Theta_\chi\|^2 \rangle_{\chi^-} \quad (103)$$

So Eq. (102) gives

$$2\Delta_{\chi^-}\|\Theta_\chi\|^2 = R_{\chi^-}\|\Theta_\chi\|^2 + 4(M^4/16)r_{\text{in}}^{-4}\|\nabla_\chi\Theta_\chi\|^2 + 4\|\Theta_\chi\|^2\langle \bar{\nabla} \ln U, \bar{\nabla} \ln r_{\text{in}} \rangle_{\chi^-} + 4\langle \nabla \ln r_{\text{in}}, \nabla\|\Theta_\chi\|^2 \rangle_{\chi^-} \quad (104)$$

Finally Eq. (79) gives

$$2\Delta_{\chi^-}\|\Theta_\chi\|^2 = 4(c^4/16)r_{\text{in}}^{-4}\|\nabla_\chi\Theta_\chi\|^2 + 4\|\Theta_\chi\|^2\langle \bar{\nabla} \ln U, \bar{\nabla} \ln r_{\text{in}} \rangle_{\chi^-} + 4\langle \nabla \ln r_{\text{in}}, \nabla\|\Theta_\chi\|^2 \rangle_{\chi^-} + \sigma^{-2}(\zeta^-)^{-1}\left(\mathcal{P} - \mathcal{Q}_{\bar{g}} - U^{-1}\bar{\Delta}U + \frac{1}{2}\bar{\nabla} \ln U|^2 - \langle \bar{\nabla} \ln U, \bar{\nabla} \ln f^- \rangle\right)\|\Theta_\chi\|^2 \quad (105)$$

Identity for Σ^- is given in the following lemma.

Lemma 9.3.

$$\begin{aligned}
& \bar{\nabla}_{\mathcal{X}^-} \left(\sin^2 \theta \left(2\bar{\nabla} \|\Theta_\chi\|^2 - 2\|\Theta_\chi\|^2 \bar{\nabla} \ln r_{in} + 2|U|^{-3/4} \bar{\nabla} \ln r_{in} + \|\Theta_\chi\|^2 \bar{\nabla} \ln U \right) - (\|\Theta_\chi\|^2 - 1) \bar{\nabla} \sin^2 \theta \right) \\
&= \left(4(c^4/16)r_{in}^{-4} \sigma^2 \zeta^- (\mathcal{R} + iI) - (3/2)L^- \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f^- \right\rangle + i(3/4)|U|^{-3/4} \left\langle \bar{\nabla} \omega, \bar{\nabla} \ln f^- \right\rangle \right) \sigma^{-2} (\zeta^-)^{-1} \sin^2 \theta \\
&\quad + \sigma^{-2} (\zeta^-)^{-1} \left(\mathcal{P} - \mathcal{Q}_{\bar{g}} - (1/2) \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln U \right\rangle - (1/4)(4 - 3L^-) \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f^- \right\rangle \right) \|\Theta_\chi\|^2 \sin^2 \theta \\
&\quad - 2\|\Theta_\chi\|^2 \sigma^{-2} (\zeta^-)^{-1} \bar{g}^{\theta\theta} \cos(2\theta) + \bar{\Delta}_{\mathcal{X}^-} \sin^2 \theta \quad (106)
\end{aligned}$$

where I is as in Lemma 9.2 and $\mathcal{R} = |U|^{-1} \|\nabla_\gamma \Theta_\gamma\|^2 \geq 0$.

Proof: Writing the 3-Laplacian $\Delta_{\mathcal{X}^-}$ relative to $\mathcal{X}^- = \sigma^2 \zeta^- \bar{g} + U f^- d\phi^2$ in terms of the Laplacian of the 2-metric $\bar{\mathcal{X}}^- = \sigma^2 \zeta^- \bar{g}$ we get

$$\begin{aligned}
2\Delta_{\mathcal{X}^-} \|\Theta_\chi\|^2 &= 2\bar{\Delta}_{\bar{\mathcal{X}}^-} \|\Theta_\chi\|^2 + \left\langle \bar{\nabla} \ln(U f^-), \bar{\nabla} \|\Theta_\chi\|^2 \right\rangle_{\bar{\mathcal{X}}^-}. \text{ We also have } U^{-1} \|\Theta_\chi\|^2 \bar{\Delta}_{\bar{\mathcal{X}}^-} U = \bar{\nabla}_{\mathcal{X}^-} \left(\|\Theta_\chi\|^2 U^{-1} \bar{\nabla} U \right) - \\
&\left\langle \bar{\nabla} \|\Theta_\chi\|^2, \bar{\nabla} \ln U \right\rangle_{\bar{\mathcal{X}}^-} + \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln U \right\rangle_{\bar{\mathcal{X}}^-} \|\Theta_\chi\|^2. \text{ So Eq. (105) gives}
\end{aligned}$$

$$\begin{aligned}
& 2\bar{\Delta}_{\bar{\mathcal{X}}^-} \|\Theta_\chi\|^2 + \left\langle \bar{\nabla} \ln f^-, \bar{\nabla} \|\Theta_\chi\|^2 \right\rangle_{\bar{\mathcal{X}}^-} + \bar{\nabla}_{\mathcal{X}^-} \left(\|\Theta_\chi\|^2 U^{-1} \bar{\nabla} U \right) = 4(c^4/16)r_{in}^{-4} \|\nabla_\chi \Theta_\chi\|^2 + 4\|\Theta_\chi\|^2 \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln r_{in} \right\rangle_{\bar{\mathcal{X}}^-} \\
&\quad + 4 \left\langle \bar{\nabla} \ln r_{in}, \bar{\nabla} \|\Theta_\chi\|^2 \right\rangle_{\bar{\mathcal{X}}^-} + \sigma^{-2} (\zeta^-)^{-1} \left(\mathcal{P} - \mathcal{Q}_{\bar{g}} - (1/2) \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln U \right\rangle - \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f^- \right\rangle \right) \|\Theta_\chi\|^2 \Rightarrow \\
& 2\bar{\Delta}_{\bar{\mathcal{X}}^-} \|\Theta_\chi\|^2 - 2 \left\langle \bar{\nabla} \ln r_{in}, \bar{\nabla} \|\Theta_\chi\|^2 \right\rangle_{\bar{\mathcal{X}}^-} + \bar{\nabla}_{\mathcal{X}^-} \left(\|\Theta_\chi\|^2 U^{-1} \bar{\nabla} U \right) = 4(c^4/16)r_{in}^{-4} \|\nabla_\chi \Theta_\chi\|^2 + 4\|\Theta_\chi\|^2 \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln r_{in} \right\rangle_{\bar{\mathcal{X}}^-} \\
&\quad - \left\langle \bar{\nabla} \ln \sin^2 \theta, \bar{\nabla} \|\Theta_\chi\|^2 \right\rangle_{\bar{\mathcal{X}}^-} + \sigma^{-2} (\zeta^-)^{-1} \left(\mathcal{P} - \mathcal{Q}_{\bar{g}} - (1/2) \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln U \right\rangle - \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f^- \right\rangle \right) \|\Theta_\chi\|^2 \Rightarrow \\
& 2\bar{\Delta}_{\bar{\mathcal{X}}^-} \|\Theta_\chi\|^2 + \bar{\nabla}_{\mathcal{X}^-} \left(-2\|\Theta_\chi\|^2 \bar{\nabla} \ln r_{in} + 2|U|^{-3/4} \bar{\nabla} \ln r_{in} + \|\Theta_\chi\|^2 \bar{\nabla} \ln U \right) = \\
&\quad 4(c^4/16)r_{in}^{-4} \|\nabla_\chi \Theta_\chi\|^2 - (3/2)|U|^{-3/4} \left\langle \bar{\nabla} \ln |U|, \bar{\nabla} \ln r_{in} \right\rangle_{\bar{\mathcal{X}}^-} + 4\|\Theta_\chi\|^2 \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln r_{in} \right\rangle_{\bar{\mathcal{X}}^-} \\
&\quad - \left\langle \bar{\nabla} \ln \sin^2 \theta, \bar{\nabla} \|\Theta_\chi\|^2 \right\rangle_{\bar{\mathcal{X}}^-} + \sigma^{-2} (\zeta^-)^{-1} \left(\mathcal{P} - \mathcal{Q}_{\bar{g}} - (1/2) \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln U \right\rangle - \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f^- \right\rangle \right) \|\Theta_\chi\|^2
\end{aligned}$$

Multiplying both sides by $\sin^2 \theta$ and using that U, r_{in} are functions of r only we get

$$\begin{aligned}
& \bar{\nabla}_{\mathcal{X}^-} \left(\sin^2 \theta \left(2\bar{\nabla} \|\Theta_\chi\|^2 - 2\|\Theta_\chi\|^2 \bar{\nabla} \ln r_{in} + 2|U|^{-3/4} \bar{\nabla} \ln r_{in} + \|\Theta_\chi\|^2 \bar{\nabla} \ln U \right) \right) - 2 \left\langle \bar{\nabla} \sin^2 \theta, \bar{\nabla} \|\Theta_\chi\|^2 \right\rangle_{\bar{\mathcal{X}}^-} \\
&= 4 \sin^2 \theta (c^4/16)r_{in}^{-4} \|\nabla_\chi \Theta_\chi\|^2 - (3/2) \sin^2 \theta |U|^{-3/4} \left\langle \bar{\nabla} \ln |U|, \bar{\nabla} \ln r_{in} \right\rangle_{\bar{\mathcal{X}}^-} + 4 \sin^2 \theta \|\Theta_\chi\|^2 \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln r_{in} \right\rangle_{\bar{\mathcal{X}}^-} \\
&\quad - \left\langle \bar{\nabla} \sin^2 \theta, \bar{\nabla} \|\Theta_\chi\|^2 \right\rangle_{\bar{\mathcal{X}}^-} + \sigma^{-2} (\zeta^-)^{-1} \left(\mathcal{P} - \mathcal{Q}_{\bar{g}} - (1/2) \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln U \right\rangle - \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f^- \right\rangle \right) \|\Theta_\chi\|^2 \sin^2 \theta \Rightarrow \\
& \bar{\nabla}_{\mathcal{X}^-} \left(\sin^2 \theta \left(2\bar{\nabla} \|\Theta_\chi\|^2 - 2\|\Theta_\chi\|^2 \bar{\nabla} \ln r_{in} + 2|U|^{-3/4} \bar{\nabla} \ln r_{in} + \|\Theta_\chi\|^2 \bar{\nabla} \ln U \right) \right) - \bar{\nabla}_{\mathcal{X}^-} \left(\|\Theta_\chi\|^2 \bar{\nabla} \sin^2 \theta \right) \\
&= 4 \sin^2 \theta (c^4/16)r_{in}^{-4} \|\nabla_\chi \Theta_\chi\|^2 - (3/2) \sin^2 \theta |U|^{-3/4} \left\langle \bar{\nabla} \ln |U|, \bar{\nabla} \ln r_{in} \right\rangle_{\bar{\mathcal{X}}^-} + 4 \sin^2 \theta \|\Theta_\chi\|^2 \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln r_{in} \right\rangle_{\bar{\mathcal{X}}^-} + \\
&\quad \sigma^{-2} (\zeta^-)^{-1} \left(\mathcal{P} - \mathcal{Q}_{\bar{g}} - (1/2) \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln U \right\rangle - \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f^- \right\rangle \right) \|\Theta_\chi\|^2 \sin^2 \theta - \|\Theta_\chi\|^2 \bar{\Delta}_{\bar{\mathcal{X}}^-} \sin^2 \theta \Rightarrow
\end{aligned}$$

$$\begin{aligned}
& \bar{\nabla}_{\chi^-} \left(\sin^2 \theta \left(2\bar{\nabla} \|\Theta_\chi\|^2 - 2\|\Theta_\chi\|^2 \bar{\nabla} \ln r_{\text{in}} + 2|U|^{-3/4} \bar{\nabla} \ln r_{\text{in}} + \|\Theta_\chi\|^2 \bar{\nabla} \ln U \right) - (\|\Theta_\chi\|^2 - 1) \bar{\nabla} \sin^2 \theta \right) \\
&= \sigma^{-2} (\zeta^-)^{-1} \left(4(c^4/16)r_{\text{in}}^{-4} \sigma^2 \zeta^- \|\nabla_\chi \Theta_\chi\|^2 - (3/4)|U|^{-3/4} \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f^- \right\rangle + i(3/4)|U|^{-3/4} \left\langle \bar{\nabla} \omega, \bar{\nabla} \ln f^- \right\rangle \right) + \\
& 4\|\Theta_\chi\|^2 \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln r_{\text{in}} \right\rangle + \left(\mathcal{P} - \mathcal{Q}_{\bar{g}} - (1/2) \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln U \right\rangle - \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f^- \right\rangle \right) \|\Theta_\chi\|^2 \sin^2 \theta - (\|\Theta_\chi\|^2 - 1) \bar{\Delta}_{\chi^-} \sin^2 \theta
\end{aligned}$$

Using $|U|^{-3/4} = L^- \|\Theta_\chi\|^2$ and Eq. (96) we get

$$\begin{aligned}
& \bar{\nabla}_{\chi^-} \left(\sin^2 \theta \left(2\bar{\nabla} \|\Theta_\chi\|^2 - 2\|\Theta_\chi\|^2 \bar{\nabla} \ln r_{\text{in}} + 2|U|^{-3/4} \bar{\nabla} \ln r_{\text{in}} + \|\Theta_\chi\|^2 \bar{\nabla} \ln U \right) - (\|\Theta_\chi\|^2 - 1) \bar{\nabla} \sin^2 \theta \right) \\
&= \sigma^{-2} (\zeta^-)^{-1} \left(4(c^4/16)r_{\text{in}}^{-4} \sigma^2 \zeta^- \|\nabla_\chi \Theta_\chi\|^2 - (3/2)L^- \|\Theta_\chi\|^2 \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f^- \right\rangle + i(3/4)|U|^{-3/4} \left\langle \bar{\nabla} \omega, \bar{\nabla} \ln f^- \right\rangle \right) + \\
& 4\|\Theta_\chi\|^2 \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln r_{\text{in}} \right\rangle + \left(\mathcal{P} - \mathcal{Q}_{\bar{g}} - (1/2) \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln U \right\rangle - (1/4)(4 - 3L^-) \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f^- \right\rangle \right) \|\Theta_\chi\|^2 \sin^2 \theta \\
& \quad - 2\|\Theta_\chi\|^2 \sigma^{-2} (\zeta^-)^{-1} \bar{g}^{\theta\theta} \cos(2\theta) + \bar{\Delta}_{\chi^-} \sin^2 \theta
\end{aligned}$$

Since $4\sigma^{-2} (\zeta^-)^{-1} \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln r_{\text{in}} \right\rangle = -2\sigma^{-2} (\zeta^-)^{-1} \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln f \right\rangle = -2(c^4/16)r_{\text{in}}^{-4} \left\langle \bar{\nabla} \ln U, \bar{\nabla} \ln r_{\text{in}} \right\rangle_{\bar{\chi}}$ we get Eq. (106) using Lemma 9.2. \blacksquare

10 Main theorem

We separate the positive part of the contribution of the integral of $\|\Theta_\chi\|^2 \sigma^{-2} (\zeta^\pm)^{-1} \bar{g}^{\theta\theta} \cos(2\theta)$ on Σ^\pm . Although this integral will be zero in case $\|\Theta_\chi\|^2$ is constant the integrand is not identically zero even in the case of Kerr-Newman solution. This follows from Eq. (96). Let

$$\mathcal{A}(r_0) = \frac{\int_{r=r_0} \sigma^{3/2} \|\Theta_\theta\|^2 \cos(2\theta) d\theta}{\int_{r=r_0} \sigma^{3/2} \|\Theta_\theta\|^2 \sin^2 \theta d\theta} \quad (107)$$

First we show

$$\int_{(r_1, r_2), \bar{\chi}^\pm} \|\Theta_\chi\|^2 \sigma^{-2} (\zeta^\pm)^{-1} \bar{g}^{\theta\theta} \cos(2\theta) = \int_{(r_1, r_2), \bar{\chi}^\pm} \sin^2 \theta \|\Theta_\chi\|^2 \sigma^{-2} (\zeta^\pm)^{-1} \bar{g}^{\theta\theta} \mathcal{A} \quad (108)$$

$$\begin{aligned}
\text{LHS} &= \iint_{(r_1, r_2)} |U|^{-3/4} \|\Theta_\theta\|^2 \sigma^{3/2-2} (\zeta^\pm)^{-1} \bar{g}^{\theta\theta} \cos(2\theta) \sqrt{\chi_{r\bar{r}}^\pm \chi_{\theta\bar{\theta}}^\pm} dr d\theta \\
&= \iint_{(r_1, r_2)} |U|^{-3/4} \sigma^{3/2} \|\Theta_\theta\|^2 \cos(2\theta) (r^2 - 2M'r + a^2)^{-1/2} dr d\theta \\
&= \int_{r_1}^{r_2} |U|^{-3/4} (r^2 - 2M'r + a^2)^{-1/2} \left(\int \sigma^{3/2} \|\Theta_\theta\|^2 \cos(2\theta) d\theta \right) dr \\
&= \int_{r_1}^{r_2} |U|^{-3/4} (r^2 - 2M'r + a^2)^{-1/2} \mathcal{A} \left(\int \sigma^{3/2} \|\Theta_\theta\|^2 \sin^2 \theta d\theta \right) dr \\
&= \iint_{(r_1, r_2)} \sin^2 \theta |U|^{-3/4} \sigma^{3/2} \|\Theta_\theta\|^2 (r^2 - 2M'r + a^2)^{-1/2} \mathcal{A} dr d\theta
\end{aligned}$$

$$\begin{aligned}
&= \iint_{(r_1, r_2)} \sin^2 \theta |U|^{-3/4} \sigma^{3/2} \|\Theta_\theta\|^2 (r^2 - 2M'r + a^2)^{-1/2} \mathcal{A} \sqrt{\chi^{rr} \chi^{\theta\theta}} \sqrt{\chi_{rr} \chi_{\theta\theta}} dr d\theta \\
&= \int_{(r_1, r_2), \bar{\chi}^\pm} \sin^2 \theta \|\Theta_\chi\|^2 \sigma^{-2} (\zeta^\pm)^{-1} \bar{g}^{\theta\theta} \mathcal{A}. \text{ Having checked Eq. (108) we now decom-} \\
&\text{pose } \mathcal{A} \text{ into positive and negative parts. Positive part we add with } Q_{\bar{g}} \text{ since there} \\
&\text{is a negative sign in front of } 2\|\Theta_\chi\|^2 (\zeta^-)^{-1} \bar{g}^{\theta\theta} \cos(2\theta) \text{ in Eqs. (94,106). We define}
\end{aligned}$$

$$\widehat{Q}_{\bar{g}} = Q_{\bar{g}} + 2\bar{g}^{\theta\theta} \mathcal{A}^+ \quad (109)$$

$\widehat{Q}_{\bar{g}}$ is nonnegative. This will be important in the proof of the main theorem. We define $Q_{\Pi} = \left(4 \left\langle \bar{\nabla} \ln (X_K / \sqrt{\rho}), \bar{\nabla} \ln \sigma \right\rangle + f_{\text{em}} \right)_{\Pi}$. Then from Eqs. (61,62,69) we get $Q_{\bar{g}} = \bar{g}^{\theta\theta} Q_{\Pi}$. Similarly we write Eq. (109) as

$$\widehat{Q}_{\bar{g}} = \bar{g}^{\theta\theta} \widehat{Q}_{\Pi} \quad (110)$$

where $\widehat{Q}_{\Pi} = Q_{\Pi} + 2\mathcal{A}^+$. We now define an average of \widehat{Q}_{Π} on a $r = \text{constant}$ loop on a $\phi = \text{constant}$ surface.

$$Q_{\text{ave}}(r_0) = \frac{\int_{r=r_0} \sigma^{3/2} \widehat{Q}_{\Pi} \|\Theta_\theta\|^2 \sin^2 \theta d\theta}{\int_{r=r_0} \sigma^{3/2} \|\Theta_\theta\|^2 \sin^2 \theta d\theta} + 2\mathcal{A}^+ \quad (111)$$

Note that $\int_{r=r_0} \sigma^{3/2} \|\Theta_\theta\|^2 \cos(2\theta) d\theta = \int_{r=r_0} (1 + O(r^{-1})) \cos(2\theta) d\theta = O(r^{-1})$. Thus as $r \rightarrow \infty$

$$Q_{\text{ave}}(r_0) = O(r^{-1}) \quad (112)$$

Lemma 10.1. *Suppose $U = U(r)$ and*

$$(r^2 - 2M'r + a^2) \left(\frac{d \ln U}{dr} \right)^2 + (4 - 3L_{\text{ave}}) \sqrt{r^2 - 2M'r + a^2} \frac{d \ln U}{dr} + 2Q_{\text{ave}} = 0 \quad (113)$$

$$\text{Then } \iint_{\Sigma_{\bar{g}}^\pm} \left(2 \langle \bar{\nabla} \ln U, \bar{\nabla} \ln U \rangle + (4 - 3L) \left(\bar{\nabla} \ln U, \bar{\nabla} \ln f \right) + 4\widehat{Q}_{\bar{g}} \right) \sigma^{-2} \zeta^{-1} \sin^2 \theta \|\Theta_\chi\|^2 = 0 \quad (114)$$

Proof: Using Eqs. (89,57,52) we see that the LHS of Eq. (114) is

$$\begin{aligned}
&\iint_{(r_1, r_2)} \sin^2 \theta \left(2(r^2 - 2M'r + a^2) \left(\frac{d \ln U}{dr} \right)^2 + (4 - 3L)(r^2 - 2M'r + a^2) \frac{d \ln U}{dr} \frac{\partial \ln f}{\partial r} + 4\widehat{Q}_{\Pi} \right) \bar{g}^{\theta\theta} \sigma^{-2} \zeta^{-1} \|\Theta_\chi\|^2 \sqrt{\chi_{rr} \chi_{\theta\theta}} dr d\theta \\
&= \iint_{(r_1, r_2)} \sin^2 \theta \left(2(r^2 - 2M'r + a^2) \left(\frac{d \ln U}{dr} \right)^2 + (4 - 3L)(r^2 - 2M'r + a^2) \frac{d \ln U}{dr} \frac{\partial \ln f}{\partial r} + 4\widehat{Q}_{\Pi} \right) |U|^{-3/4} \sigma^{3/2} \|\Theta_\theta\|^2 \\
&(r^2 - 2M'r + a^2)^{-1/2} dr d\theta. \text{ We use the definition of } Q_{\text{ave}} \text{ given in Eq. (111). Also from} \\
&\text{(93) and noting that } (\partial \ln f / \partial r) = 2(r^2 - 2M'r + a^2)^{-1/2} \text{ we get}
\end{aligned}$$

$$\int L \sigma^{3/2} (r^2 - 2M'r + a^2) \frac{d \ln U}{dr} \frac{\partial \ln f}{\partial r} \|\Theta_\theta\|^2 \sin^2 \theta d\theta = 2L_{\text{ave}} \sqrt{r^2 - 2M'r + a^2} \frac{d \ln U}{dr} \int \sigma^{3/2} \|\Theta_\theta\|^2 \sin^2 \theta d\theta$$

So replacing \widehat{Q}_Π and L by their respective average values Q_{ave} and L_{ave} we write the LHS of Eq. (114) as

$$\iint_{(r_1, r_2)} \sin^2 \theta \left(2(r^2 - 2M'r + a^2) \left(\frac{d \ln U}{dr} \right)^2 + 2(4 - 3L_{\text{ave}}) \sqrt{r^2 - 2M'r + a^2} \frac{d \ln U}{dr} + 4Q_{\text{ave}} \right) |U|^{-3/4} \sigma^{3/2} \|\Theta\|^2 (r^2 - 2M'r + a^2)^{-1/2} dr d\theta \text{ which is 0 by Eq. (113).} \quad \blacksquare$$

Thus if U is a solution of (113) then integrating (94) we get

$$\iint_{\Sigma_2^+ \bar{x}} \left(\sin^2 \theta \left(4\|\nabla_{\bar{x}} \Theta_\chi\|^2 + \|\Theta_\chi\|^2 \sigma^{-2} \zeta^{-1} \left(\mathcal{P} + 2\mathcal{A} \bar{g}^{\theta\theta} - (3/4)iL \left(\bar{\nabla} \omega, \bar{\nabla} \ln f \right) \right) \right) \right) = \lim_{r_1 \rightarrow 2M} \int_{r=r_1} \mathcal{B} d\theta + \lim_{r_2 \rightarrow \infty} \int_{r=r_2} \mathcal{B} d\theta$$

where we wrote the boundary integrands with respect to outward normals by \mathcal{B} . The axis part given by $\theta = 0$ or $\theta = \pi$ alone does not appear as a limiting boundary. If we consider a $\theta = \text{constant} = \theta_0$ line for θ_0 close to 0 or π as a boundary then the LHS of Eq. (94) gives on this boundary $\int_{\theta_0} \left(2 \sin^2 \theta \bar{\nabla} \|\Theta_\chi\|^2 - (\|\Theta_\chi\|^2 - 1) \bar{\nabla} \sin^2 \theta, \bar{n} \right)_{\bar{x}}$ where \bar{n} is the unit normal form on this line. All other terms are orthogonal to \bar{n} . The integral vanishes as $\theta_0 \rightarrow 0$ or π because it is bounded by $C \sin \theta \int (r^2 - 2M'r + a^2)^{-1/2} r^{-1} dr$ for some constant C . r^{-1} factor comes from $\|\Theta_\chi\|^2 - 1$ for large values of r . This factor is necessary only to make the constant C independent of r . But we can also take the $\theta_0 \rightarrow 0$ or π limit for a fixed large r first and then let $r \rightarrow \infty$. The boundary term from $\bar{\Delta}_{\bar{x}} \sin^2 \theta$ in the RHS of Eq. (94) vanishes.

Now we want to compute the boundary integrals. But in order to evaluate the boundary integral at $r = M + c$ and not on the axis we are forced to consider the identity in Eq. (106) on Σ^- and to match U suitably across the smooth parts of the limiting surface at $r = M + c$. We proceed as follows.

A solution of Eq. (113) for $4 - 3L_{\text{ave}} \geq 0$ is given by the first case of the following equation.

$$\frac{d \ln U}{dr} = \begin{cases} \frac{-2 + (3/2)L_{\text{ave}} + \sqrt{(2 - (3/2)L_{\text{ave}})^2 - 2Q_{\text{ave}}}}{\sqrt{r^2 - 2M'r + a^2}} & \text{if } 4 - 3L_{\text{ave}} \geq 0, \\ i \sqrt{2Q_{\text{ave}} / (r^2 - 2M'r + a^2)} & \text{otherwise} \end{cases} \quad (115)$$

For this solution for $r > M + c$, $\text{Re} \frac{d \ln U}{dr} \leq 0$. Writing for this solution

$$-\iint_{\Sigma_2^+ \bar{x}} \sigma^{-2} \zeta^{-1} \sin^2 \theta \left((1/2) \langle \nabla \ln U, \nabla \ln U \rangle + (1/4)(4 - 3L) \left(\bar{\nabla} \ln U, \bar{\nabla} \ln f \right) + \widehat{Q}_{\bar{x}} \right) \|\Theta_\chi\|^2 = i \iint_{\Sigma_2^+ \bar{x}} I_1 \sigma^{-2} \zeta^{-1} \bar{g}^{\theta\theta} \sin^2 \theta \|\Theta_\chi\|^2 \quad (116)$$

we see that I_1 is a real function on Σ^+ . Absorbing the other pure imaginary (or zero) term of Eq. (94) we define

$$i\mathcal{I}_{\text{out}}\zeta^{-1} = iI_1\zeta^{-1}g^{\theta\theta} - (3/4)\zeta^{-1}iL|U|^{-3/4}\left\langle\bar{\nabla}\omega, \bar{\nabla}\ln f\right\rangle \quad (117)$$

Similarly on Σ^- we seek U such that

$$2\langle\nabla\ln U, \nabla\ln U\rangle + (4 - 3L^-)\left\langle\bar{\nabla}\ln U, \bar{\nabla}\ln f^-\right\rangle + 4\widehat{Q}_{\bar{g}} = 0 \quad (118)$$

for $4 - 3L_{\text{ave}} \geq 0$.

Since $(\partial\ln f^-/\partial r) = -2(r^2 - 2M'r + a^2)^{-1/2}$ this time we need

$$(r^2 - 2M'r + a^2)\left(\frac{d\ln U}{dr}\right)^2 - (4 - 3L_{\text{ave}}^-)\frac{d\ln U}{dr} + 2Q_{\text{ave}} = 0 \quad (119)$$

A solution of Eq. (119) for $4 - 3L_{\text{ave}}^- \geq 0$ is given by the first case of

$$\frac{d\ln U}{dr} = \begin{cases} \frac{2 - (3/2)L_{\text{ave}}^- - \sqrt{(2 - (3/2)L_{\text{ave}}^-)^2 - 2Q_{\text{ave}}}}{\sqrt{r^2 - 2Mr}} & \text{if } 4 - 3L_{\text{ave}}^- \geq 0, \\ -i\sqrt{2Q_{\text{ave}}}/(r^2 - 2Mr) & \text{otherwise} \end{cases} \quad (120)$$

For this solution for $r > M+c$, $\text{Re} \frac{d\ln U}{dr} \geq 0$. In particular $\text{Re} L^- \left\langle\bar{\nabla}\ln U, \bar{\nabla}\ln f^-\right\rangle \leq 0$. Thus in Eq. (106) for Σ^- we write

$$\begin{aligned} & \iint_{\Sigma_2^-\bar{x}^-} \sigma^{-2}(\zeta^-)^{-1} \sin^2\theta \left(4(c^4/16)r_{\text{in}}^{-4}\sigma^2\zeta^-(\mathcal{R} + iI) - (1/2)\langle\nabla\ln U, \nabla\ln U\rangle - (1/4)(4 - 3L^-)\left\langle\bar{\nabla}\ln U, \bar{\nabla}\ln f^-\right\rangle - \widehat{Q}_{\bar{g}} \right. \\ & \left. - (3/2)L^- \left\langle\bar{\nabla}\ln U, \bar{\nabla}\ln f^-\right\rangle + i(3/4)L^- \left\langle\bar{\nabla}\omega, \bar{\nabla}\ln f^-\right\rangle\right) \|\Theta_\chi\|^2 = \iint_{\Sigma_2^-\bar{x}^-} \sigma^{-2}(\zeta^-)^{-1} \sin^2\theta (\mathcal{R}_{\text{in}} + iI_{\text{in}}) \|\Theta_\chi\|^2 \quad (121) \end{aligned}$$

where both \mathcal{R}_{in} and I_{in} are real and $\mathcal{R}_{\text{in}} \geq 0$.

Since on Σ^- , $|U|^{-3/4} = L^- \|\Theta_\chi\|^2$ and $\|\Theta_\chi\|^2 = \sigma^{3/2}|U|^{-3/4} \|\Theta_\theta\|^2$ we have $\sigma^{3/2}L^- \|\Theta_\theta\|^2 = 1$. Since $\|\Theta_\theta\|^2$ is continuous across $r = M + c$, at $r = M + c$, $L^- = L$ and hence $L_{\text{ave}}^- = L_{\text{ave}}$. It now follows from Eqs. (115,120) that

$$\left.\frac{d\ln U}{dr_{\text{out}}}\right|_{r=M+c} = \left.\frac{d\ln U}{dr_{\text{in}}}\right|_{r=M+c} \quad (122)$$

The above equation holds even before we match U on both Σ_2^\pm at $r = M + c$ using the constant of integration.

Lemma 10.2. *Let $n_{\bar{\chi}}$ be the unit normal form relative to the metric $\bar{\chi}$ on the loop $r = r_0$ and the corresponding vector points in the direction of decreasing r . Let $U = U(r)$. For $M + c < r_0 < \infty$,*

$$\begin{aligned} & \oint_{r_0, \bar{\chi}} \left(2 \sin^2 \theta \bar{\nabla} \|\Theta_\chi\|^2 + 2 \sin^2 \theta \|\Theta_\chi\|^2 \bar{\nabla} \ln r_{out} - 2 \sin^2 \theta |U|^{-3/4} \bar{\nabla} \ln r_{out} + \sin^2 \theta \|\Theta_\chi\|^2 \bar{\nabla} \ln U, n_{\bar{\chi}} \right)_{\bar{\chi}} = \frac{1}{|U(r_0)|^{3/4}} \\ & \int_{r=r_0} \left(\sigma^{3/2} \left(\|\Theta\|^2 \left((1/2) r_{out} \frac{d \ln U}{dr_{out}} - i(3/2) \sqrt{r^2 - 2M'r + a^2} \frac{d\omega}{dr} - 2 \right) - 2 \sqrt{r^2 - 2M'r + a^2} \frac{\partial \|\Theta\|^2}{\partial r} \right) + 2 \right) \sin^2 \theta d\theta \end{aligned} \quad (123)$$

Similarly on Σ^- with $n_{\bar{\chi}^-}$ being the unit normal form relative to the metric $\bar{\chi}^-$ on the loop $r = r_0$ and the corresponding vector pointing in the direction of decreasing r ,

$$\begin{aligned} & \oint_{r_0, \bar{\chi}^-} \left(2 \sin^2 \theta \bar{\nabla} \|\Theta_\chi\|^2 + 2 \sin^2 \theta \|\Theta_\chi\|^2 \bar{\nabla} \ln r_{in} - 2 \sin^2 \theta |U|^{-3/4} \bar{\nabla} \ln r_{in} + \sin^2 \theta \|\Theta_\chi\|^2 \bar{\nabla} \ln U, n_{\bar{\chi}^-} \right)_{\bar{\chi}^-} = -\frac{1}{|U(r_0)|^{3/4}} \\ & \int_{r=r_0} \left(\sigma^{3/2} \left(\|\Theta\|^2 \left((1/2) r_{in} \frac{d \ln U}{dr_{in}} + i(3/2) \sqrt{r^2 - 2M'r + a^2} \frac{d\omega}{dr} - 2 \right) + 2 \sqrt{r^2 - 2M'r + a^2} \frac{\partial \|\Theta\|^2}{\partial r} \right) + 2 \right) \sin^2 \theta d\theta \end{aligned} \quad (124)$$

In case U is real near infinity, U^{-1} is bounded and $\frac{d \ln U}{dr} = o(r^{-1})$ as $r \rightarrow \infty$, right hand sides of Eqs. (123, 124) vanish as $r_0 \rightarrow \infty$.

Proof: First we note that $\bar{\nabla} \|\Theta_\chi\|^2 = \bar{\nabla} (\sigma^{3/2} |U|^{-3/4} \|\Theta_\theta\|^2) = \bar{\nabla} (\sigma^{3/2} U^{-3/4} e^{i(3/4)\omega} \|\Theta_\theta\|^2)$
 $= \sigma^{3/2} e^{i(3/4)\omega} \left((3/2) U^{-3/4} \|\Theta_\theta\|^2 \bar{\nabla} \ln \sigma - (3/4) U^{-3/4} \|\Theta_\theta\|^2 \bar{\nabla} \ln U + i(3/4) U^{-3/4} \|\Theta_\theta\|^2 \bar{\nabla} \omega + U^{-3/4} \bar{\nabla} \|\Theta_\theta\|^2 \right)$
 $= (3/2) \sigma^{3/2} |U|^{-3/4} \|\Theta_\theta\|^2 \bar{\nabla} \ln \sigma - (3/4) \sigma^{3/2} |U|^{-3/4} \|\Theta_\theta\|^2 \bar{\nabla} \ln U + i(3/4) \sigma^{3/2} |U|^{-3/4} \|\Theta_\theta\|^2 \bar{\nabla} \omega + \sigma^{3/2} |U|^{-3/4} \bar{\nabla} \|\Theta_\theta\|^2.$
Now $n_{\bar{\chi}, r} = -\sqrt{\chi_{rr}} = -\sigma \sqrt{\zeta \bar{g}_{rr}}$. So for any differentiable function F on the $r = r_0$ loop, $\left(\bar{\nabla} F, n_{\bar{\chi}} \right)_{\bar{\chi}} \sqrt{\chi_{\theta\theta}} = \bar{\chi}^{rr} \frac{\partial F}{\partial r} n_{\bar{\chi}, r} \sqrt{\chi_{\theta\theta}} = -\sqrt{\bar{g}^{rr}} \frac{\partial F}{\partial r} \sqrt{g_{\theta\theta}} = -\sqrt{r^2 - 2M'r + a^2} \frac{\partial F}{\partial r}$. Thus using $\|\Theta_\chi\|^2 = \sigma^{3/2} |U|^{-3/4} \|\Theta_\theta\|^2$ we get

$$\begin{aligned} & \oint_{r_0, \bar{\chi}} \left(2 \sin^2 \theta \bar{\nabla} \|\Theta_\chi\|^2 + 2 \sin^2 \theta \|\Theta_\chi\|^2 \bar{\nabla} \ln r_{out} - 2 \sin^2 \theta |U|^{-3/4} \bar{\nabla} \ln r_{out} + \sin^2 \theta \|\Theta_\chi\|^2 \bar{\nabla} \ln U, n_{\bar{\chi}} \right)_{\bar{\chi}} = \\ & -\frac{\sqrt{r_0^2 - 2M'r_0 + a^2}}{|U(r_0)|^{3/4}} \int_{r=r_0} \left(\sigma^{3/2} \left(\|\Theta_\theta\|^2 \left(-(1/2) \frac{d \ln U}{dr} + i(3/2) \frac{d\omega}{dr} + 2 \frac{d \ln r_{out}}{dr} \right) + 2 \frac{\partial \|\Theta_\theta\|^2}{\partial r} \right) - 2 \frac{d \ln r_{out}}{dr} \right) \sin^2 \theta d\theta = \\ & \frac{1}{|U(r_0)|^{3/4}} \int_{r=r_0} \left(\sigma^{3/2} \left(\|\Theta_\theta\|^2 \left((1/2) r_{out} \frac{d \ln U}{dr_{out}} - i(3/2) \sqrt{r^2 - 2M'r + a^2} \frac{d\omega}{dr} - 2 \right) - 2 \sqrt{r^2 - 2M'r + a^2} \frac{\partial \|\Theta_\theta\|^2}{\partial r} \right) + 2 \right) \sin^2 \theta d\theta \end{aligned}$$

Similarly on Σ^- with $n_{\bar{\chi}^-}$ being the unit normal form relative to the metric $\bar{\chi}^-$ on

the loop $r = r_0$ and the corresponding vector pointing in the direction of decreasing r ,

$$\begin{aligned} & \oint_{r_0, \bar{\chi}} \left(2 \sin^2 \theta \bar{\nabla} \|\Theta_\chi\|^2 + 2 \sin^2 \theta \|\Theta_\chi\|^2 \bar{\nabla} \ln r_{\text{in}} - 2 \sin^2 \theta |U|^{-3/4} \bar{\nabla} \ln r_{\text{in}} + \sin^2 \theta \|\Theta_\chi\|^2 \bar{\nabla} \ln U, \bar{\chi}^- \right)_{\bar{\chi}^-} = \\ & - \frac{\sqrt{r_0^2 - 2M'r_0 + a^2}}{|U(r_0)|^{3/4}} \int_{r=r_0} \left(\sigma^{3/2} \left(\|\Theta_\theta\|^2 \left(-(1/2) \frac{d \ln U}{dr} + i(3/2) \frac{d\omega}{dr} + 2 \frac{d \ln r_{\text{in}}}{dr} \right) + 2 \frac{\partial \|\Theta_\theta\|^2}{\partial r} \right) - 2 \frac{d \ln r_{\text{in}}}{dr} \right) \sin^2 \theta d\theta = \\ & - \frac{1}{|U(r_0)|^{3/4}} \int_{r=r_0} \left(\sigma^{3/2} \left(\|\Theta_\theta\|^2 \left((1/2) r_{\text{in}} \frac{d \ln U}{dr_{\text{in}}} + i(3/2) \sqrt{r^2 - 2M'r + a^2} \frac{d\omega}{dr} - 2 \right) + 2 \sqrt{r^2 - 2M'r + a^2} \frac{\partial \|\Theta_\theta\|^2}{\partial r} \right) + 2 \right) \sin^2 \theta d\theta \end{aligned}$$

As $r_0 \rightarrow \infty$, $\|\Theta_\theta\|^2 - 1 = O(r^{-1})$. The contribution of the remaining terms vanish as $r_0 \rightarrow \infty$ by the hypotheses on U and Eq. (82). \blacksquare

We now integrate Eqs. (94,106) on Σ^\pm . For both cases the boundary integrals at $r = M + c$ (equivalently $2r_{\text{in}} = 2r_{\text{out}} = c$) are evaluated w.r.t. to the normal vector pointing in the direction of decreasing r .

Lemma 10.3. *Suppose $U = U(r)$ is globally C^1 and U^{-1} is globally bounded and $(d \ln U/dr) = o(r^{-1})$ as $r \rightarrow \infty$ and U is a solution of Eq. (115) in Σ^+ and Eq. (120) in Σ^- such that $(d \ln U/dr)$ is locally bounded in $(M + c, \infty)$, and fails to be differentiable at most at finite number of points. Then*

$$\begin{aligned} & \iint_{\Sigma_2^+, \bar{\chi}} \left(\mathcal{P} \sigma^{-2} \zeta^{-1} \|\Theta_\chi\|^2 + 4 \|\nabla_\chi \Theta_\chi\|^2 + \|\Theta_\chi\|^2 \sigma^{-2} \zeta^{-1} (2\mathcal{A}^- \bar{g}^{\theta\theta} + i\mathcal{I}_{\text{out}}) \right) \sin^2 \theta = \\ & \lim_{r_0 \downarrow M+c} \frac{1}{|U(r_0)|^{3/4}} \int_{r=r_0} \left(\sigma^{3/2} \left(\|\Theta_\theta\|^2 \left((1/2) r_{\text{out}} \frac{d \ln U}{dr_{\text{out}}} - i(3/2) \sqrt{r^2 - 2M'r + a^2} \frac{d\omega}{dr} - 2 \right) + 2 \right) \sin^2 \theta d\theta \quad (125) \end{aligned}$$

The corresponding integral on $(\Sigma_2^-, \bar{\chi}^-)$ is

$$\begin{aligned} & \iint_{\Sigma_2^-, \bar{\chi}^-} \left(\mathcal{P} \sigma^{-2} \zeta^{-1} \|\Theta_\chi\|^2 + 4 \|\nabla_\chi \Theta_\chi\|^2 + \|\Theta_\chi\|^2 \sigma^{-2} \zeta^{-1} (2\mathcal{A}^- \bar{g}^{\theta\theta} + \mathcal{R}_{\text{in}} + i\mathcal{I}_{\text{in}}) \right) \sin^2 \theta = \\ & - \lim_{r_0 \downarrow M+c} \frac{1}{|U(r_0)|^{3/4}} \int_{r=r_0} \left(\sigma^{3/2} \left(\|\Theta_\theta\|^2 \left((1/2) r_{\text{in}} \frac{d \ln U}{dr_{\text{in}}} + i(3/2) \sqrt{r^2 - 2M'r + a^2} \frac{d\omega}{dr} - 2 \right) + 2 \right) \sin^2 \theta d\theta \quad (126) \end{aligned}$$

The limiting sets for the integrals in the RHS of both equations are restricted to the horizon only.

Proof: Left hand sides result from the right hand sides of Eqs. (94,106) for the solutions U of Eqs. (115,120) after we modify $\underline{Q}_{\bar{g}}$ to $\widehat{Q}_{\bar{g}}$ in Eq. (109) and replace the latter and L by their averages and use Eqs. (116,121). Right hand sides come from the right hand sides of Eqs. (123,124). We note that in the right hand sides

of Eqs. (125,126) on the horizon $\sqrt{r^2 - 2M'r + a^2} \frac{\partial \|\Theta_\theta\|^2}{\partial r}$ drops out because of Eq. (84). Axis parts do not contribute in the limit $r_0 \downarrow M + c$ because in the limit these are two copies of line segments with normal pointing in the opposite directions. **|**

Theorem 10.4. *There is no analytic multiple black hole solution of asymptotically flat stationary axisymmetric Einstein-Magnetic Maxwell equation with non-degenerate event horizon and the single black hole solution belongs to the Kerr-Newman family.*

Proof: $L_{\text{ave}} \rightarrow 1$ as $r \rightarrow \infty$. As $r \rightarrow \infty$, $Q_{\text{ave}} = O(r^{-1})$ by Eq. (112). Thus for large r , $(d \ln U/dr)$ is real and $\frac{d \ln U}{dr} = \frac{-1 + \sqrt{1 - O(r^{-1})}}{2\sqrt{r^2 - 2M'r + a^2}} = O(r^{-2})$. So near infinity U is real and U and its derivative have the appropriate decay needed for Lemma 10.3. Near $r = M + c$, U could be imaginary but its real and imaginary parts have the necessary decay. Near $r = M + c$, Q_{ave} is of the form $C_1 + C_2 r$ where C_1 and C_2 are real constants. Thus near $r = M + c$, U is an approximate solution of $\frac{d \ln U}{dr} = \frac{-C_3 - C_4 r}{\sqrt{r^2 - 2M'r + a^2}}$ for some complex constants C_3 and C_4 . This gives $U \approx \exp\left(C_0 + \int \frac{-C_3 - C_4 r}{\sqrt{r^2 - 2M'r + a^2}} dr\right) = \exp\left(C_5 - C_4 \sqrt{r^2 - 2M'r + a^2} - C_3 \ln(r - M + \sqrt{r^2 - 2M'r + a^2})\right)$. U is never zero. For an analytic spacetime metric L_{ave} and Q_{ave} are analytic so the radicands in Eqs. (115,120) can vanish either at finite number of values of r or for all r . So $(d \ln U/dr)$ can fail to be differentiable only for finite number of values of r . We can now add Eqs. (126,125) to find that the real parts of the LHS of both equations are zero by virtue of Eq. (122) provided we take the same value of $|U|$ at $M + c$ for U on both sides. The real part has nonnegative definite integrand. This shows $\mathcal{P} = 0$. $\mathcal{P} = 0$ gives $\sigma = 1$ identically. Thus by asymptotic conditions $X = X_K$. It is now well-known that $W = W_K$, $V = V_K$ and $\Omega = \Omega_K$ and the spacetime has Kerr-Newman metric. **|**

Remark 10.1. The analyticity assumption is used only to show that $(d \ln U/dr)$ can fail to be differentiable only for finite number of values of r to prove Lemma 10.3 so that there is no contribution in the right hand sides of Eqs.(125,126) from the boundaries at singular values. This assumption can possibly be avoided by considering the identities in weak form.

11 Conclusion

We showed that spin-spin interaction cannot hold black holes apart in stationary equilibrium in an analytic asymptotically flat axisymmetric spacetime even in the presence of electromagnetic fields. The way we modified the method step by stem from the application of positive mass theorem in Bunting and Masood-ul-Alam [21] gives us hope that the method can be further modified to drop the axisymmetry assumption. This would however be a huge program because our method only shows $X = X_K$. The equations to show the rest (including the equation for Ω) are changed without the assumption of axisymmetry. A more manageable problem the solution of which will also be a significant progress is as follows. In stead of Eq. (1) one starts with a non-axisymmetric spacetime metric of the form $-(V + \epsilon_t)dt^2 + 2(W + \epsilon_t\phi)dtd\phi + 2\epsilon_{t,x^A}dtdx^A + (X + \epsilon_{\phi\phi})d\phi^2 + \bar{g} + \epsilon_{\alpha\beta}dx^\alpha dx^\beta$ where ϵ is small and has appropriate boundary properties. In this case one expects that the error from the deviation from axisymmetry can be absorbed in a modified Q_{ave} having the required boundary properties for our method to work.

12 Appendix

We outline the proofs of Lemma 8.1 and Lemma 9.2. If necessary further details can be found in Appendix I of [2]. We denote ∇_{g_1} by $\overset{1}{\nabla}$ and ∇_{g_2} by $\overset{2}{\nabla}$. Let $\{e(1)^i\}_{i=1,2,3}$ be orthonormal frame field of one forms for $g_1 = \bar{G} + f_1 d\phi^2$. Let $\{e(2)^i\}$ be the corresponding orthonormal one forms for $g_2 = \bar{G} + qf_1 d\phi^2$. We take $e(2)^\phi = \sqrt{qf_1}d\phi$ because $qf_1 \langle d\phi, d\phi \rangle_{g_2} = qf_1 g_2^{\phi\phi} = 1$. Similarly $e(1)^\phi = \sqrt{f_1}d\phi$. Thus $\sqrt{q}e(1)^\phi = e(2)^\phi$. Note $e(1)^A = e(2)^A$, $A = 1, 2$. Corresponding orthonormal frame field of vectors are $e(1)_A = e(2)_A$, $A = 1, 2$. $\frac{1}{\sqrt{q}}e(1)_\phi = e(2)_\phi$. $e(2)_A = \sqrt{\bar{G}^{11}} \frac{\partial}{\partial x^A}$, since $\langle e(2)_A, e(2)_B \rangle_{g_2} = \delta_{AB}$ but $\left\langle \frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^A} \right\rangle_{g_2} = \bar{G}_{AB}$. However at a single point we can arrange \bar{G}_{AB} to be δ_{AB} . We also note that $\langle e(2)_\phi, e(2)_\phi \rangle_{g_2} = qf_1 (e(2)_\phi)^\phi (e(2)_\phi)^\phi = f_1 (e(1)_\phi)^\phi (e(1)_\phi)^\phi = \langle e(1)_\phi, e(1)_\phi \rangle_{g_1}$. $\left\langle \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi} \right\rangle_{g_2} = qf_1 \Rightarrow \dot{e}_\phi = \frac{1}{\sqrt{qf_1}} \frac{\partial}{\partial \phi} \cdot \left\langle \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi} \right\rangle_{g_1} = f_1 \Rightarrow e(1)_\phi = \frac{1}{\sqrt{f_1}} \frac{\partial}{\partial \phi}$. First

we compute the Christoffel symbols. Let Λ_{AD}^C be the Christoffel symbols of \bar{G} . Let $\overset{1}{\Gamma}_{\beta\gamma}^\mu$ and $\overset{2}{\Gamma}_{\beta\gamma}^\mu$ be the Christoffel symbols of g_1 and g_2 . These symbols are w.r.t. $\{x^A, \phi\}$ coordinates. They are not the connection coefficients related to the one frame fields.

$$\overset{2}{\Gamma}_{\phi\phi}^\phi = 0, \overset{2}{\Gamma}_{AB}^\phi = 0, \overset{2}{\Gamma}_{B\phi}^A = 0, \overset{2}{\Gamma}_{\phi A}^\phi = \frac{1}{2} \frac{\partial \ln(qf_1)}{\partial x^A}, \overset{2}{\Gamma}_{\phi\phi}^A = -\frac{1}{2} \frac{\partial(qf_1)}{\partial x^A}, \overset{2}{\Gamma}_{BC}^A = \Lambda_{BC}^A$$

$$\overset{1}{\Gamma}_{\phi\phi}^\phi = 0, \overset{1}{\Gamma}_{AB}^\phi = 0, \overset{1}{\Gamma}_{B\phi}^A = 0, \overset{1}{\Gamma}_{\phi A}^\phi = \frac{1}{2} \frac{\partial \ln f_1}{\partial x^A}, \overset{1}{\Gamma}_{\phi\phi}^A = -\frac{1}{2} \frac{\partial f_1}{\partial x^A}, \overset{1}{\Gamma}_{BC}^A = \Lambda_{BC}^A$$

Now we calculate the connection coefficients $\overset{2}{C}$ and $\overset{1}{C}$ in the frame fields $\{e(2)_i\}$ and $\{e(1)_i\}$ for the two metrics respectively. On $\overset{2}{C}$ and $\overset{1}{C}$ the indices refer to the frame fields not coordinates.

$$\begin{aligned} \overset{1}{C}_{\phi\phi\phi} &= \langle e(1)_\phi, \overset{1}{\nabla}_{e(1)_\phi} e(1)_\phi \rangle_{g_1} = 0, \quad \overset{1}{C}_{\phi\phi A} = \left\langle e(1)_\phi, \overset{1}{\nabla}_{e(1)_\phi} e(1)_A \right\rangle_{g_1} = (1/2) \frac{\partial \ln f_1}{\partial x^A}, \\ \overset{1}{C}_{\phi AB} &= \left\langle e(1)_\phi, \overset{1}{\nabla}_{e(1)_A} e(1)_B \right\rangle_{g_1} = 0, \quad \overset{1}{C}_{A\phi B} = \left\langle e(1)_A, \overset{1}{\nabla}_{e(1)_\phi} e(1)_B \right\rangle_{g_1} = 0, \\ \overset{1}{C}_{A\phi\phi} &= \left\langle e(1)_A, \overset{1}{\nabla}_{e(1)_\phi} e(1)_\phi \right\rangle_{g_1} = -(1/2) \frac{\partial \ln f_1}{\partial x^A}, \quad \overset{1}{C}_{AB\phi} = \left\langle e(1)_A, \overset{1}{\nabla}_{e(1)_B} e(1)_\phi \right\rangle_{g_1} = 0, \\ \overset{1}{C}_{ABC} &= \left\langle e(1)_A, \overset{1}{\nabla}_{e(1)_B} e(1)_C \right\rangle_{g_1} = \Lambda_{ABC} + \bar{G}_{AC} \frac{\partial}{\partial x^B}. \end{aligned}$$

We get $\overset{2}{C}$ replacing f_1 in expressions for $\overset{1}{C}$ by qf_1 .

$$\overset{2}{C}_{\phi\phi\phi} = 0 = \overset{2}{C}_{\phi AB} = \overset{2}{C}_{AB\phi}, \quad \overset{2}{C}_{\phi\phi A} = (1/2) \frac{\partial \ln(qf_1)}{\partial x^A} = -\overset{2}{C}_{A\phi\phi}, \quad \overset{2}{C}_{ABC} = \overset{1}{C}_{ABC}$$

In the following Clifford multiplication by $e(1)^i$ and $e(2)^i$ are denoted by \cdot . Distinction is not necessary because it is multiplication by the same matrix. Clifford relation is $e(1)^i \cdot e(1)^k + e(1)^k \cdot e(1)^i = -2\delta^{ij}$. For the $SU(2)$ spinor $\xi \in \mathbb{C}^2$, using $\overset{1}{\nabla}_{e(1)_k} \xi = e(1)_k(\xi) - \frac{1}{4} \left\langle e(1)_i, \overset{1}{\nabla}_{e(1)_k} e(1)_j \right\rangle_{g_1} e(1)^i \cdot e(1)^j \cdot \xi$, we get

$$\begin{aligned} \overset{1}{\nabla}_{e(1)_B} \xi &= e(1)_B(\xi) - \frac{1}{4} \overset{1}{C}_{ijB} e(1)^i \cdot e(1)^j \cdot \xi \\ &= e(2)_B(\xi) + (1/4) \overset{1}{C}_{\phi\phi B} \xi - (1/4) \overset{1}{C}_{ACB} e(1)^A \cdot e(1)^C \cdot \xi \\ &= e(2)_B(\xi) + \frac{1}{4} \overset{2}{C}_{\phi\phi B} \xi - (1/4) \overset{2}{C}_{ACB} e(2)^A \cdot e(2)^C \cdot \xi - (1/8) \left(\frac{\partial \ln q}{\partial x^B} \right) \xi \\ &= \overset{2}{\nabla}_{e(2)_B} \xi - (1/8) \left(\frac{\partial \ln q}{\partial x^B} \right) \xi \end{aligned}$$

Similarly for $e(2)_\phi(\xi) = 0 = e(1)_\phi(\xi)$,

$$\begin{aligned}\overset{1}{\nabla}_{e(1)_\phi}\xi &= e(1)_\phi(\xi) - (1/4)\overset{1}{C}_{ij\phi}e(1)^i \cdot e(1)^j \cdot \xi = (1/4)\left(\frac{\partial \ln f_1}{\partial x^A}\right)e(1)^A \cdot e(1)^\phi \cdot \xi \\ &= (1/4)\left(\frac{\partial \ln(qf_1)}{\partial x^A}\right)e(1)^A \cdot e(1)^\phi \cdot \xi - (1/4)\left(\frac{\partial \ln q}{\partial x^A}\right)e(1)^A \cdot e(1)^\phi \cdot \xi \\ &= \overset{2}{\nabla}_{e(2)_\phi}\xi - (1/4)\left(\frac{\partial \ln q}{\partial x^A}\right)e(2)^A \cdot e(2)^\phi \cdot \xi\end{aligned}$$

So $D_{g_2}\xi = Dg_1\xi + \frac{3}{8}\left(\frac{\partial \ln q}{\partial x^B}\right)e(2)_B \cdot \xi$ giving $D_{g_2}\left(q^{-\frac{3}{8}}\xi\right) = q^{-\frac{3}{8}}D_{g_1}\xi$. This proves Lemma 8.1.

To prove Lemma 9.2 we also need

$$\overset{2}{\nabla}_{e(2)_\phi}\xi = \frac{1}{4}\left(\frac{\partial \ln(qf_1)}{\partial x^A}\right)e(2)^A \cdot e(2)^\phi \cdot \xi$$

Then

$$\begin{aligned}\|\overset{1}{\nabla}\xi\|^2 &= \|\overset{2}{\nabla}\xi\|^2 + \sum_B \left(- (1/8)\left(\frac{\partial \ln q}{\partial x^B}\right)\left(\left\langle \overset{2}{\nabla}_{e(2)_B}\xi, \xi \right\rangle + \left\langle \xi, \overset{2}{\nabla}_{e(2)_B}\xi \right\rangle\right) + \frac{1}{64}\left(\frac{\partial \ln q}{\partial x^B}\right)\left(\frac{\partial \ln q}{\partial x^B}\right)\|\xi\|^2 \\ &\quad - (1/16)\left(\frac{\partial \ln q}{\partial x^A}\right)\left(\frac{\partial \ln(qf_1)}{\partial x^B}\right)\left(\left\langle e(2)^B \cdot e(2)^\phi \cdot \xi, e(2)^A \cdot e(2)^\phi \cdot \xi \right\rangle + \left\langle e(2)^A \cdot e(2)^\phi \cdot \xi, e(2)^B \cdot e(2)^\phi \cdot \xi \right\rangle\right) \\ &\quad + (1/16)\left(\frac{\partial \ln q}{\partial x^A}\right)\left(\frac{\partial \ln q}{\partial x^B}\right)\left\langle e(2)^A \cdot e(2)^\phi \cdot \xi, e(2)^B \cdot e(2)^\phi \cdot \xi \right\rangle \quad (127)\end{aligned}$$

Now $\left\langle e(2)^B \cdot \xi, e(2)^A \cdot \xi \right\rangle + \left\langle e(2)^A \cdot \xi, e(2)^B \cdot \xi \right\rangle = 2\|\xi\|^2$. Also $\left\langle e(2)^A \cdot \xi, e(2)^B \cdot \xi \right\rangle = -\left\langle e(2)^B \cdot e(2)^A \cdot \xi, \xi \right\rangle = 2\delta^{AB}\|\xi\|^2 - \left\langle e(2)^B \cdot \xi, e(2)^A \cdot \xi \right\rangle$. So Eq. (127) gives

$$\|\overset{1}{\nabla}\xi\|^2 = \|\overset{2}{\nabla}\xi\|^2 - \frac{1}{8}\left\langle \bar{\nabla} \ln q, \bar{\nabla} \|\xi\|^2 \right\rangle_G - \frac{3}{64}|\bar{\nabla} \ln q|_G^2\|\xi\|^2 - \frac{1}{8}\left\langle \bar{\nabla} \ln q, \bar{\nabla} \ln f_1 \right\rangle_G \|\xi\|^2 \quad (128)$$

We take

$$\begin{aligned}g_1 &= \chi = \sigma^2 \zeta \bar{g} + U f d \phi^2 \\ g_2 &= \gamma = \sigma^2 \zeta \bar{g} + |U|^{-4} U f d \phi^2 \\ f_1 &= U f, \quad q = |U|^{-4}\end{aligned}$$

Then Eq. (128) gives

$$\|\nabla_\chi \Theta_\gamma\|^2 = \|\nabla_\gamma \Theta_\gamma\|^2 + (1/2)\sigma^{-2}\zeta^{-1}\left\langle \bar{\nabla} \ln |U|, \bar{\nabla} \|\Theta_\gamma\|^2 \right\rangle - (3/4)\sigma^{-2}\zeta^{-1}\left\langle \bar{\nabla} \ln |U|, \bar{\nabla} \ln |U| \right\rangle \|\Theta_\gamma\|^2 +$$

$(1/2)\sigma^{-2}\zeta^{-1}\left\langle\bar{\nabla}\ln|U|,\bar{\nabla}\ln(Uf)\right\rangle\|\Theta_\gamma\|^2$ where norms of vectors or forms and inner product of vectors and forms are with respect to 2-metric \bar{g} .

We have $\Theta_\gamma = |U|^{1/2}\Theta_\chi$. Recalling $\bar{\nabla} = \nabla_\chi$ we get,

$$\begin{aligned}\|\nabla_\chi\Theta_\gamma\|^2 &= \|\nabla_\chi(|U|^{1/2}\Theta_\chi)\|^2 = \|(1/2)|U|^{1/2}(\nabla\ln|U|)\Theta_\chi + |U|^{1/2}\nabla_\chi\Theta_\chi\|^2 \\ &= (1/4)\sigma^{-2}\zeta^{-1}\left\langle\bar{\nabla}\ln|U|,\bar{\nabla}\ln|U|\right\rangle\|\Theta_\gamma\|^2 + |U|\|\nabla_\chi\Theta_\chi\|^2 + (1/2)|U|\left\langle\bar{\nabla}\ln|U|,\nabla\|\Theta_\chi\|^2\right\rangle_\chi.\end{aligned}$$

Thus for some pure imaginary function (or zero) Im we get

$$\begin{aligned}(1/4)\sigma^{-2}\zeta^{-1}\left\langle\bar{\nabla}\ln|U|,\bar{\nabla}\ln|U|\right\rangle\|\Theta_\gamma\|^2 &+ |U|\|\nabla_\chi\Theta_\chi\|^2 + (1/2)|U|\left\langle\bar{\nabla}\ln|U|,\nabla\|\Theta_\chi\|^2\right\rangle_\chi + Im \\ &= \|\nabla_\gamma\Theta_\gamma\|^2 + (1/2)\sigma^{-2}\zeta^{-1}\left\langle\bar{\nabla}\ln|U|,\bar{\nabla}\|\Theta_\gamma\|^2\right\rangle - (1/4)\sigma^{-2}\zeta^{-1}\left\langle\bar{\nabla}\ln|U|,\bar{\nabla}\ln|U|\right\rangle\|\Theta_\gamma\|^2 \\ &\quad + (1/2)\sigma^{-2}\zeta^{-1}\left\langle\bar{\nabla}\ln|U|,\bar{\nabla}\ln f\right\rangle\|\Theta_\gamma\|^2\end{aligned}$$

Now since $\|\Theta_\chi\|^2 = |U|^{-1}\|\Theta_\gamma\|^2$, $\bar{\nabla}\|\Theta_\chi\|^2 = -|U|^{-1}\|\Theta_\gamma\|^2\bar{\nabla}\ln|U| + |U|^{-1}\bar{\nabla}\|\Theta_\gamma\|^2$ which gives

$$\left\langle\bar{\nabla}\ln|U|,\bar{\nabla}\|\Theta_\chi\|^2\right\rangle = -\|\Theta_\chi\|^2\left\langle\bar{\nabla}\ln|U|,\bar{\nabla}\ln|U|\right\rangle + |U|^{-1}\left\langle\bar{\nabla}\ln|U|,\bar{\nabla}\|\Theta_\gamma\|^2\right\rangle$$

Thus we get

$$|U|\|\nabla_\chi\Theta_\chi\|^2 + Im = \|\nabla_\gamma\Theta_\gamma\|^2 + (1/2)\sigma^{-2}\zeta^{-1}\left\langle\bar{\nabla}\ln|U|,\bar{\nabla}\ln f\right\rangle\|\Theta_\gamma\|^2$$

which gives Lemma 9.2.

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