

NODAL SETS OF RANDOM EIGENFUNCTIONS FOR THE ISOTROPIC HARMONIC OSCILLATOR

BORIS HANIN, STEVE ZELDITCH, PENG ZHOU

ABSTRACT. The expected hypersurface measure $\mathcal{H}^{d-1}(Z_{E,h} \cap B(r,x))$ of nodal sets of random eigenfunctions of eigenvalue E of the semi-classical isotropic harmonic oscillator in balls $B(r,x) \subset \mathbb{R}^d$ is determined as $h \rightarrow 0$. In the allowed region the volumes are of order h^{-1} , while in the forbidden region they are of order $h^{-\frac{1}{2}}$.

0. INTRODUCTION

This article is concerned with the semi-classical asymptotics of nodal (i.e. zero) sets of random eigenfunctions of the isotropic Harmonic Oscillator,

$$(0.1) \quad H_h = \sum_{j=1}^d \left(-\frac{h^2}{2} \frac{\partial^2}{\partial x_j^2} + \frac{x_j^2}{2} \right),$$

on $L^2(\mathbb{R}^d)$. Random isotropic Hermite functions of fixed degree have an $SO(d-1)$ symmetry and are in some ways analogous to random spherical harmonics of fixed degree on $L^2(S^d)$, whose nodal sets have been the subject of many recent studies (see e.g. [NS]).

However, there is a fundamentally new aspect to eigenfunctions of Schrödinger operators on \mathbb{R}^d , namely the existence of allowed and forbidden regions. In the allowed region, H_h behaves like an elliptic operator (with parameter h) and the nodal sets of eigenfunctions behave similarly to those of eigenfunctions of the Laplace operator on a Riemannian manifold. For instance the classical estimates of Donnelly-Fefferman [DF] for the hypersurface measure of nodal sets of Laplace eigenfunctions for real analytic metrics has an analogue for semiclassical eigenfunctions of Schrödinger operators with analytic metrics and potentials (see Long Jin [J]). In the C^∞ case one has lower bounds on hypersurface volumes of nodal sets in the allowed region which are similar to those for smooth metrics [ZZ]. However, there do not seem to exist prior results on nodal volumes in the forbidden region, although there do exist numerical and heuristic results on random eigenfunctions of (0.1) in Bies-Heller [BH]. In the forbidden region, eigenfunctions are exponentially decaying and it is not clear to what extent they oscillate and have zeros; in dimension one, eigenfunctions of the Harmonic oscillator

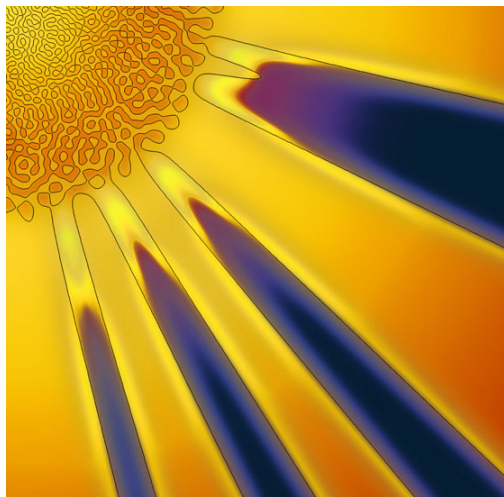


FIGURE 1

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have no zeros in the forbidden region. To gain insight into the behavior of nodal sets in the forbidden region, we randomize the problem and consider Gaussian random eigenfunctions of (0.1). Our main results show that the expected hypersurface measure of nodal sets in compact subsets of the allowed region are of order h^{-1} , parallel to that of Laplace eigenfunctions, while in the forbidden region they are of order $h^{-\frac{1}{2}}$.

To state our main result, Theorem 1, we introduce some notation and background. Acting on $L^2(\mathbb{R}^d, dx)$, H_h has an orthonormal basis of eigenfunctions

$$(0.2) \quad \phi_{\alpha,h}(x) = h^{-d/4} p_{\alpha}(x \cdot h^{-1/2}) e^{-x^2/2h},$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \geq (0, \dots, 0)$ is a d -dimensional multi-index and $p_{\alpha}(x)$ is the product $\prod_{j=1}^d (2^{\alpha_j} \alpha_j!)^{-1/2} \pi^{-1/4} p_{\alpha_j}(x_j)$ of the hermite polynomials p_k (of degree k) in one variable. The eigenvalue of $\phi_{\alpha,h}$ is given by

$$(0.3) \quad H_h \phi_{\alpha,h} = h(|\alpha| + d/2) \phi_{\alpha,h}.$$

The multiplicity of the eigenvalue $h(|\alpha| + d/2)$ is the partition function of $|\alpha|$, i.e. the number of $\alpha = (\alpha_1, \dots, \alpha_d) \geq (0, \dots, 0)$ with a fixed value of $|\alpha|$. The high multiplicity of the eigenvalues is similar in order of magnitude to that of the eigenvalues of Δ on the standard S^{d-1} .

The semi-classical asymptotics of eigenfunctions is the asymptotics as $h \rightarrow 0$ where the energy level E_h satisfies $E_h \rightarrow E$. This corresponds to fixing an energy level of the classical Hamiltonian

$$H(x, \xi) = \frac{1}{2}(|\xi|^2 + |x|^2) : T^*\mathbb{R}^m \rightarrow \mathbb{R}.$$

We refer to [Zw] for this and other background on semi-classical asymptotics of Schrödinger operators. For the remainder of this paper we fix $E > 0$ and set

$$h_N := \frac{E}{N + \frac{d}{2}}.$$

We will usually write $h = h_N$. We then consider the eigenspace

$$(0.4) \quad V_N = \text{Span}\{\phi_{\alpha,h_N}, |\alpha| = N\}.$$

Definition 1. A Gaussian random eigenfunction for H_h with eigenvalue E is the random series

$$\Phi_N(x) := \sum_{|\alpha|=N} a_{\alpha} \phi_{\alpha,h_N}(x),$$

for $a_{\alpha} \sim N(0, 1)_{\mathbb{R}}$ i.i.d. Equivalently, it is the Gaussian measure γ_N on V_N which is given by $e^{-\sum_{\alpha} |a_{\alpha}|^2/2} \prod da_{\alpha}$.

We denote by

$$Z_{\Phi_N} = \{x : \Phi_N(x) = 0\}$$

the nodal set of Φ_N and by $|Z_{\Phi_N}|$ the random measure of integration over Z_{Φ_N} with respect to the Euclidean surface measure (the Hausdorff measure) of the nodal set. Thus for any ball $B \subset \mathbb{R}^d$,

$$|Z_{\Phi_N}|(B) = \mathcal{H}^{d-1}(B \cap Z_{\Phi_N}).$$

Thus $\mathbb{E}[|Z_{\Phi_N}|]$ is a measure on \mathbb{R}^n given by

$$\mathbb{E}[|Z_{\Phi_N}|](B) = \int_{V_N} \mathcal{H}^{d-1}(B \cap Z_{\Phi_N}) d\gamma_N.$$

The allowed region \mathcal{A}_E , resp. the forbidden region \mathcal{F}_E are defined respectively by

$$(0.5) \quad \mathcal{A}_E = \{x : |x|^2 < 2E\}, \quad \mathcal{F}_E = \{x : |x|^2 > 2E\}.$$

Thus, \mathcal{A}_E is the projection to \mathbb{R}^d of the energy surface $\{H = E\} \subset T^*\mathbb{R}^d$ and \mathcal{F}_E is its complement. The boundary of \mathcal{A}_E is known as the caustic set and is denoted $\partial\mathcal{A}_E$ or $\{|x| = 2E\}$. Our main result is:

Theorem 1. *Let $x \in \mathbb{R}^d$ such that $0 < |x| \neq \sqrt{2E}$. Then the measure $\mathbb{E}[|Z_{\Phi_N}|]$ has a density $F_N(x)$ with respect to Lebesgue measure given by*

$$\begin{cases} \text{If } x \in \mathcal{A}_E \setminus \{0\}, & F_N(x) \simeq h^{-1} \cdot c_d \sqrt{2E - |x|^2} (1 + O(h)) \\ \text{If } x \in \mathcal{F}_E, & F_N(x) \simeq h^{-1/2} \cdot C_d \frac{E^{1/2}}{|x|^{1/2}(|x|^2 - 2E)^{1/4}} (1 + O(h)) \end{cases},$$

where the implied constants in the ‘ O ’ symbols are uniform on compact subsets of the interiors of $\mathcal{A}_E \setminus \{0\}$ and \mathcal{F}_E , and where

$$c_d = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{d\pi}\Gamma\left(\frac{d}{2}\right)} \quad \text{and} \quad C_d = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}.$$

The novel aspect of Theorem 1 is the different growth rates in h for the density of zeros in the allowed and forbidden region. Let us explain briefly why this happens. As recalled in Lemma 1 of §1.3, $F_N(x)$ scales like the square root of the operator norm of the $d \times d$ matrix

$$(0.6) \quad (\Omega_{x,E})_{1 \leq j,k \leq d} = \frac{\Pi_{h,E}(x,x) \partial_{x_k} \partial_{y_j} |_{x=y} \Pi_{h,E}(x,y) - \partial_{x_k} |_{x=y} \Pi_{h,E}(x,y) \cdot \partial_{y_j} |_{x=y} \Pi_{h,E}(x,y)}{\Pi_{h,E}(x,x)^2},$$

where $\Pi_{h,E}$ is the spectral projector for H_h onto the eigenspace with eigenvalue E . Proposition 3 shows that $\Omega_{x,E}$ is a diagonal matrix times h^{-2} for $x \in \mathcal{A}_E$ $j = 2$ and h^{-1} when $x \in \mathcal{F}_E$.

These different powers of h in $\Omega_{x,E}$ come from Proposition 2, which gives different oscillatory integral representations for $\Pi_{h,E}(x,y)$ in the allowed and forbidden regions. Let us write them schematically as

$$\Pi_{h,E}(x,y) = \int A(\zeta) e^{\frac{i}{h} S(\zeta, x, y)} d\zeta.$$

The amplitude A is independent of x, y . Differentiating under the integral, we see that the first term in the numerator of (0.6), is

$$(0.7) \quad \frac{i}{h} \int \int A(\zeta_1) \cdot A(\zeta_2) [\partial_{x_k} \partial_{y_j} |_{x=y} S(\zeta_1, x, y)] e^{\frac{i}{h} (S(\zeta_1, x, y) + S(\zeta_2, x, y))} d\zeta_1 d\zeta_2$$

$$(0.8) \quad - \frac{1}{h^2} \int A(\zeta_2) e^{\frac{i}{h} S(\zeta_2, x, x)} d\zeta_2 \cdot \int A(\zeta_1) \partial_{x_k} |_{x=y} S(\zeta_1, x, y) \cdot \partial_{y_j} S(\zeta_1, x, y) e^{\frac{i}{h} S(\zeta_1, x, y)} d\zeta_1.$$

The other term, $-\partial_{x_k} |_{x=y} \Pi_{h,E}(x,y) \cdot \partial_{y_j} |_{x=y} \Pi_{h,E}(x,y)$, in the numerator of (0.6) is

$$(0.9) \quad \frac{1}{h^2} \int A(\zeta_1) \partial_{x_k} |_{x=y} S(\zeta_1, x, y) e^{\frac{i}{h} (S(\zeta_1, x, y))} d\zeta_1 \cdot \int A(\zeta_2) \partial_{y_j} S(\zeta_2, x, y) e^{\frac{i}{h} (S(\zeta_2, x, y))} d\zeta_2.$$

By the method of stationary phase, the above integrals localize to the critical point set of S . In the forbidden region \mathcal{F}_E , this critical point set has dimension 0 (see Lemma 7). The amplitudes in (0.8) and (0.9) therefore cancel to order h^{-2} , and their h^{-1} term together with the (0.7)’s h^{-1} term contribute to $\Omega_{x,E}$. In contrast, in the allowed region \mathcal{A}_E , the critical

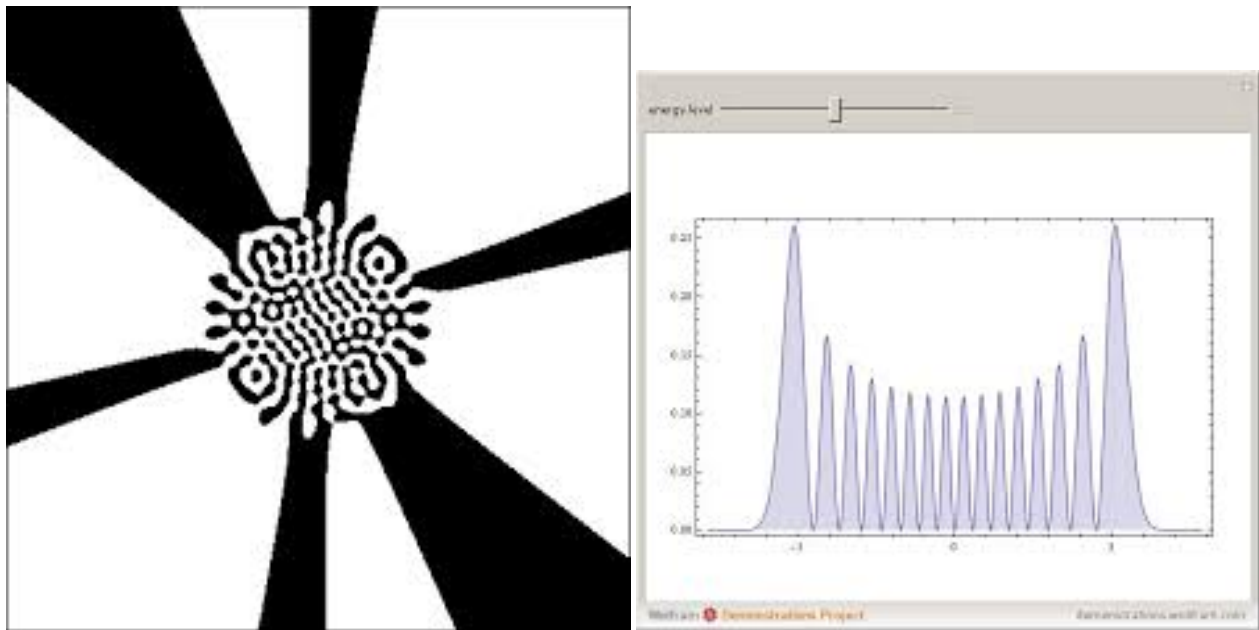


FIGURE 2. The boundary between the white and black region is the nodal set of a random Hermite function in dimension 2 is shown on the left. The figure on the right shows the graph of a Hermite function in dimension 1.

point set of S has dimension $d - 1$ (see Lemma 11). Each of the integrals in (0.9) vanishes when localized to the critical point set (see Equation (3.20)). The h^{-2} contribution from (0.8) gives the leading order of growth for $\Omega_{x,E}$ in \mathcal{A}_E .

Before giving the necessary background to prove Theorem 1, let us emphasize that our result does not cover the case of $|x| \in \{0, \sqrt{2E}\}$. Our model has the $SO(d-1)$ symmetry and the fixed point $x = 0$ is special. All odd degree Hermite functions vanish at $x = 0$ (for odd $|\alpha|$ the eigenfunctions are odd polynomials times the Gaussian factor). The Kac-Rice formula becomes singular there since $\Pi_{h_N,E}(x, x) = 0$ when $x = 0$. When N is even, $d_x \Pi_N(x, x) = 0$ at $x = 0$.

The caustic set $|x| = \sqrt{2E}$ is also special. It is the image of the projection $\pi : \{H = E\} \rightarrow \mathbb{R}^d$ along its singular set, where the projection has a fold singularity. As discussed in [KT] (see also [T]), this fold singularity causes a blow-up in L^p norms of eigenfunctions around the caustic set (as illustrated in the second figure in dimension one).

The caustic also causes anomalous behavior of the nodal set in a small ‘boundary layer’ around $\partial\mathcal{A}_E$. The nodal hypersurfaces in the forbidden region always cross the caustic set and connect with nodal hypersurfaces in the allowed region. In subsequent work we plan to rescale the nodal sets in $h^{2/3}$ -neighborhoods of $\partial\mathcal{A}_E$ and study their scaled distribution.

In semi-classical h -notation, the Donnelly-Fefferman result is that for Laplace eigenfunctions of real analytic compact Riemannian manifolds, with h^{-1} the eigenvalue of $\sqrt{\Delta}$,

$$c_g h^{-1} \leq \mathcal{H}^{d-1}(Z_{\phi_h}) \leq C_g h^{-1}.$$

Thus in the allowed region, the order of magnitude of the nodal set is the same as for Laplace eigenfunctions. As noted above, this has been proved for eigenfunctions of Schrödinger operators with real analytic metrics and potentials in [J]. The order of magnitude $h^{-\frac{1}{2}}$ in

the forbidden region is a new result. We hope to explain this result deterministically in subsequent work. It is evident from the graphics that the nodal domains in \mathcal{F}_E have some angular structure and that the ‘frequency’ of eigenfunctions in the forbidden region is lower than in the allowed region.

We expect that the results of this article generalize to all semi-classical Schrödinger operators with potentials of quadratic at infinity with evident modifications. In place of eigenspaces one would take linear combinations of eigenfunctions with eigenvalues from intervals of width $O(\hbar)$ corresponding to a fixed energy level. The case of radial potentials should be especially similar. But the difference “frequencies” of nodal sets in the allowed and forbidden regions should be a general phenomenon. We hope to take this up in subsequent work. It would also be interesting to generalize the methods and results of [NS] to random Hermite eigenfunctions.

Thanks to Long Jin for spotting a gap in the original version of this article, which led to a substantial revision of §3.1.

1. BACKGROUND

The calculation of the expected distribution of zeros is based on the Kac-Rice formula. In this formula the density of zeros of a Gaussian random function is expressed in terms of the covariance function

$$(1.1) \quad \Pi_{h_N, E}(x, y) := \mathbb{E}(\Phi_N(x)\Phi_N(y)) := \sum_{|\alpha|=N} \phi_{\alpha, h_N}(x)\phi_{\alpha, h_N}(y),$$

which (as is well known) is the orthogonal projection onto the eigenspace V_N . We will often write $\Pi_{h, E} = \Pi_{h_N, E}$. As in the case of spherical harmonics, a key input into the calculations is a relatively explicit formula for $\Pi_{h_N, E}$. In this section, we review the Mehler formulae and then the Kac-Rice formula. Further background may be found in [AT, BSZ].

1.1. Mehler Formula. The Mehler formula is an explicit formula for the Schwartz kernel $U_h(t, x, y)$ of the propagator, $e^{-\frac{i}{\hbar}tH_h}$. The Mehler formula [F] reads

$$(1.2) \quad U_h(t, x, y) = e^{-\frac{i}{\hbar}tH_h}(x, y) = \frac{1}{(2\pi i \hbar \sin t)^{d/2}} \exp\left(\frac{i}{\hbar} \left(\frac{|x|^2 + |y|^2 \cos t}{2 \sin t} - \frac{x \cdot y}{\sin t} \right)\right),$$

where $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$. The right hand side is singular at $t = 0$. It is well-defined as a distribution, however, with t understood as $t - i0$. Indeed, since H_h has a positive spectrum the propagator U_h is holomorphic in the lower half-plane and $U_h(t, x, y)$ is the boundary value of a holomorphic function in $\{\text{Im } t < 0\}$.

In the future, we write

$$(1.3) \quad S(t, x, y) = \frac{|x|^2 + |y|^2 \cos t}{2 \sin t} - \frac{x \cdot y}{\sin t}$$

for the phase in the Mehler formula (1.2).

1.2. Spectral projections. The second fact we use is that the spectrum of H_h is easily related to the integers $|\alpha|$. The operator with the same eigenfunctions as H_h and eigenvalues $h|\alpha|$ is often called the number operator, $h\mathcal{N}$. If we replace $U_h(t)$ by $e^{-\frac{it}{\hbar}\mathcal{N}}$ then the spectral

projections $\Pi_{h,E}$ are simply the Fourier coefficients of $e^{-\frac{it}{h}\mathcal{N}}$. In Lemma 4 we will derive the related formula,

$$(1.4) \quad \Pi_{h_N,E}(x, y) = \int_{-\pi}^{\pi} U_h(t - i\epsilon, x, y) e^{\frac{i}{h}(t-i\epsilon)E} \frac{dt}{2\pi}.$$

The integral is independent of ϵ . Using the Mehler formula (1.2) we obtain a rather explicit integral representation of (1.1).

1.3. Kac-Rice Formula. Next we recall the Kac-Rice formula. We refer to [BSZ, AT] for further background and proofs of the Kac-Rice formula in a general context that applies to the setting of this article. In fact we state the result on a general manifold for future applications to more general Schrödinger operators.

Let (M, g) be a smooth Riemannian manifold of dimension m and dV_g be the induced volume form on M . Consider $f : M \rightarrow \mathbb{R}$, a smooth random function so that at each $x \in M$ the density $\text{Den}_{f(x)}$ with respect to Lebesgue measure exists. Let us write $|Z_f|$ for the (random) hypersurface measure on the nodal set $f^{-1}(0)$.

Proposition 1 (Kac-Rice). $\mathbb{E}[|Z_f|]$ has a density F with respect to dV_g given by

$$(1.5) \quad F(x) = \text{Den}_{f(x)}(0) \cdot \mathbb{E} \left[|df(x)|_g \mid f(x) = 0 \right].$$

In order to rewrite this expression for F for our purposes, suppose that f is a centered 1-dimensional Gaussian field on M . This means that for every $x \in M$, the random variable $f(x)$ is a real-valued Gaussian with mean 0. Recall that the covariance kernel of f is defined by

$$\Pi_f(x, y) := \mathbb{E}[f(x)f(y)].$$

The law of any centered Gaussian field on M is determined uniquely by its covariance kernel. In particular, we may rewrite the general Kac-Rice formula of Lemma 1 only in terms of $\Pi_f(x, y)$ as follows.

Lemma 1 (Kac-Rice for Gaussian Fields). *Let f be a smooth centered Gaussian field on M . Fix $x \in M$. In a geodesic normal coordinate chart centered at x ,*

$$(1.6) \quad F(x) = (2\pi)^{-\frac{d+1}{2}} \int_{\mathbb{R}^d} |\Omega_x^{1/2} \xi| e^{-|\xi|^2/2} d\xi,$$

where Ω_x is the $d \times d$ matrix

$$(1.7) \quad \begin{aligned} (\Omega_x)_{1 \leq j, k \leq d} &= \partial_{x_j} \partial_{y_k} |_{x=y} \log \Pi_f(x, y) \\ &= \frac{\Pi_f(x, x) \partial_{x_k} \partial_{y_j} |_{x=y} \Pi_f(x, y) - \partial_{x_k} |_{x=y} \Pi_f(x, y) \cdot \partial_{y_j} |_{x=y} \Pi_f(x, y)}{\Pi_f(x, x)^2}. \end{aligned}$$

Proof. Fix $x \in M$. The pair $(f(x), df(x))$ is a centered Gaussian vector. The so-called regression formula states that if (v, w) is any centered Gaussian vector with covariance

$$\text{Cov}(v, w) = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

then w conditioned on $v = 0$ is again a centered Gaussian with covariance $C - B^*A^{-1}B$. For the vector $(f(x), df(x))$, we have

$$A = \Pi_f(x, y), \quad (B)_{1, 1 \leq j \leq d} = \partial_{x_j} |_{x=y} \Pi_f(x, y), \quad (C)_{1 \leq k, j \leq d} = \partial_{x_j} \partial_{y_k} |_{x=y} \Pi_f(x, y).$$

Hence, the vector $df(x)$ conditioned on $f(x) = 0$ is a centered Gaussian vector with covariance matrix

$$\frac{\Pi_f(x, y) \partial_{x_j} \partial_{y_k} |_{x=y} \Pi_f(x, y) - \partial_{x_j} |_{x=y} \Pi_f(x, y) \partial_{y_k} |_{x=y} \Pi_f(x, y)}{\Pi_f(x, x)},$$

which equals $\Pi_f(x, x) \cdot \partial_{x_j} \partial_{y_k} |_{x=y} \log \Pi_f(x, y)$. Note that

$$\text{Den}_{f(x)}(0) = (2\pi \Pi_f(x, x))^{-1/2}.$$

Observe that $\Pi_f(x, x)^{-1/2} \cdot df(x)$ is a centered Gaussian vector with covariance matrix

$$(\Omega_x)_{jk} = \partial_{x_j} \partial_{y_k} |_{x=y} \log \Pi_f(x, y).$$

Although the matrix Ω_x is non-negative definite, it need not be positive definite. Up to an orthogonal change of coordinates, we may write it as

$$\Omega_x = \begin{pmatrix} \tilde{\Omega}_x & 0 \\ 0 & 0 \end{pmatrix}$$

for some positive definite matrix $\tilde{\Omega}_x$ matrix of size $k \times k$ for some $1 \leq k \leq n$. The density of a centered Gaussian η on \mathbb{R}^k with positive definite covariance matrix $\tilde{\Omega}_x$ is then given by

$$\frac{1}{(2\pi)^{k/2} \det \tilde{\Omega}_x^{1/2}} e^{-\frac{1}{2} \langle \tilde{\Omega}_x^{-1} \eta, \eta \rangle} d\eta.$$

Thus, using (1.5), we find

$$\begin{aligned} F(x) &= (2\pi)^{-1/2} \mathbb{E} \left[\Pi_f(x, x)^{-1/2} \cdot |df(x)|_g \mid f(x) = 0 \right] \\ &= (2\pi)^{-1/2} \int_{\mathbb{R}^k} \frac{|\eta|}{(2\pi)^{k/2} \det \tilde{\Omega}_x^{1/2}} e^{-\frac{1}{2} \langle \tilde{\Omega}_x^{-1} \eta, \eta \rangle} d\eta \\ &= (2\pi)^{-\frac{k+1}{2}} \int_{\mathbb{R}^k} |\tilde{\Omega}_x^{1/2} \tilde{\xi}| e^{-|\tilde{\xi}|^2/2} d\tilde{\xi} \\ &= (2\pi)^{-\frac{d+1}{2}} \int_{\mathbb{R}^d} |\Omega_x^{1/2} \xi| e^{-|\xi|^2/2} d\xi, \end{aligned}$$

as claimed. \square

Let us denote $\omega_{d-1} = \text{Vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$. In the course of proving Theorem 1, we will need the following identity for the expected value of the absolute value of a standard Gaussian:

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|v|}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}} dv &= \frac{\omega_{d-1}}{(2\pi)^{d/2}} \cdot \int_0^\infty r^d e^{-\frac{r^2}{2}} \\ (1.8) \qquad \qquad \qquad &= \sqrt{2} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})}. \end{aligned}$$

1.4. Stationary Phase with Non-Degenerate Critical Manifolds. We will also need the method of stationary phase for non-degenerate critical manifolds, and recall the statement here. For further background we refer to [DSj, GrSj, Hor, Zw]. Let $S, a \in C^\infty(\mathbb{R}^N)$, and consider

$$I(h) = (2\pi i h)^{-N/2} \int_{\mathbb{R}^N} e^{iS(x)/h} a(x) dx.$$

One says that S is Bott-Morse if the critical points of S form a non-degenerate critical manifold, i.e. the transverse Hessian is non-degenerate.

Lemma 2. *If S is a Bott-Morse function with connected critical manifold W of dimension n , then there are constants c_j such that*

$$(1.9) \quad I(h) = (2\pi i h)^{-n/2} e^{-\frac{1}{2}\pi i \nu} e^{iS(W)/h} \left(\sum_{k=0}^{\infty} c_k h^k \right) + O(h^\infty)$$

with

$$(1.10) \quad c_0 = \int_W a(y) d\mu_W,$$

where ν is the Morse index of S along W and $d\mu_W$ is the Leray measure on W induced by its defining function dS .

Equivalently, $d\mu_W$ is the quotient of $|dx|$ by the Riemannian measure on the normal bundle associated to S :

$$d\mu_W = \frac{|dx|}{|\det Hess^\perp S|^{1/2} |dz|},$$

where $Hess^\perp S$ is the normal Hessian of S and z is a coordinate on the normal bundle to S . In addition to Lemma 2, we will need an explicit expression for the sub-leading terms of the stationary phase expansion in the case when S is quadratic and has a single non-degenerate critical point.

Lemma 3 ([Hor] Theorem 7.7.5). *Suppose $a, S \in \mathcal{S}(\mathbb{R})$ and S is a complex-valued phase function such that $\text{Im } S|_{\text{supp}(a)} \geq 0$ with a unique non-degenerate critical point at $t_0 \in \text{supp}(a)$ satisfying $\text{Im } S(t_0) = 0$. Then*

$$(1.11) \quad I(h) = C(S) \left[a(t_0) + \frac{h}{i} \left(-\frac{a''}{2S''} + \frac{S'''' \cdot a}{8(S'')^2} + \frac{S''' \cdot a'}{2(S'')^2} - \frac{5(S''')^2 \cdot a}{24(S'')^3} \right) \Big|_{t=t_0} + O(h^2) \right],$$

where

$$(1.12) \quad C(S) = e^{i\frac{\pi}{4} \text{sgn } S''(t_0)} \left(\frac{2\pi h}{|S''(t_0)|} \right)^{1/2}.$$

2. SEMI-CLASSICAL PROPAGATOR AND SPECTRAL PROJECTIONS

As mentioned in the beginning of §1, the covariance kernel of the Gaussian field Φ_N is $\Pi_{h_N, E}(x, y)$, the kernel of the spectral projector for H_h onto the E -eigenspace. The Kac-Rice formula (1.6) and equation (1.7) show the density of zeros of Φ_N is controlled by $\Pi_{h_N, E}(x, y)$ and its derivatives evaluated on the diagonal $x = y$. The main result of this section (Proposition 2) gives a representation $\Pi_{h_N, E}$ as a semi-classical oscillatory integral.

First we use the periodicity of the propagator $U_h(t)$ to express the spectral projections as Fourier coefficients of the propagator as in (1.4).

Lemma 4. *For each N , we abbreviate $h_N = h$. For every $\epsilon > 0$, we may write*

$$(2.1) \quad \Pi_{h_N, E}(x, y) = \int_{-\pi}^{\pi} U_h(t - i\epsilon, x, y) e^{\frac{i}{h}(t - i\epsilon)E} \frac{dt}{2\pi},$$

where U_h is defined in (1.2). The integral is independent of ϵ .

Proof. From the definition of the kernel of $e^{-\frac{i}{h}tH}$, we have

$$U_h(t - i\epsilon, x, y) = \sum_{\alpha} e^{-\frac{i}{h}(t-i\epsilon)h(|\alpha|+d/2)} \phi_{\alpha,h}(x) \phi_{\alpha,h}(y),$$

where $\alpha \in \mathbb{Z}_{\geq 0}^d$ is a multi-index, and $\epsilon > 0$ ensures the absolute convergence of the series for fixed x, y . Using that $E = h_N(N + d/2)$, we get

$$\begin{aligned} \int_{-\pi}^{\pi} U_h(t - i\epsilon, x, y) e^{\frac{i}{h}(t-i\epsilon)E} \frac{dt}{2\pi} &= \int_{-\pi}^{\pi} \sum_{\alpha} e^{-i(t-i\epsilon)(|\alpha|-N)} \phi_{\alpha,h}(x) \phi_{\alpha,h}(y) \frac{dt}{2\pi} \\ &= \sum_{\alpha} \phi_{\alpha,h}(x) \phi_{\alpha,h}(y) \int_{-\pi}^{\pi} e^{-i(t-i\epsilon)(|\alpha|-N)} \frac{dt}{2\pi} \\ &= \sum_{|\alpha|=N} \phi_{\alpha,h}(x) \phi_{\alpha,h}(y) = \Pi_{h,E}(x, y). \end{aligned}$$

□

We then use Mehler's formula (1.2) to obtain an oscillatory integral formula. It will prove to be convenient to break up the integral using two cutoff functions. First, for any $\delta \in (0, \frac{\pi}{8})$, define a smooth function $\chi_{\delta} : S^1 \rightarrow [0, 1]$ satisfying

$$\chi_{\delta}(t) = \begin{cases} 1, & \text{if } t \in (-\delta, \delta) \cup (\pi - \delta, \pi + \delta) \\ 0, & \text{if } t \notin (-2\delta, 2\delta) \cup (\pi - 2\delta, \pi + 2\delta) \end{cases}.$$

where $t \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Second, define the smooth function $\tilde{\chi}_E : \mathbb{R}^d \rightarrow [0, 1]$ satisfying

$$\tilde{\chi}_E(p) = \begin{cases} 1, & \text{if } |p| \leq 3E \\ 0, & \text{if } |p| > 4E \end{cases}.$$

We fix $\epsilon \in (0, 1)$ and combine Lemma 4 with (1.2) to obtain,

$$(2.2) \quad \begin{aligned} \Pi_{h,E}(x, y) &= \lim_{\epsilon \rightarrow 0^+} \int_0^{2\pi} \frac{e^{-i(t-i\epsilon)d/2} \chi_{\delta}(t)}{(2\pi i h \sin(t - i\epsilon))^{d/2}} e^{\frac{i}{h}(S(t-i\epsilon, x, y) + (t-i\epsilon)E)} \frac{dt}{2\pi} \\ &\quad + \lim_{\epsilon \rightarrow 0^+} \int_0^{2\pi} \frac{(1 - \chi_{\delta}(t)) e^{-i(t-i\epsilon)d/2}}{(2\pi i h \sin(t - i\epsilon))^{d/2}} e^{\frac{i}{h}(S(t-i\epsilon, x, y) + (t-i\epsilon)E)} \frac{dt}{2\pi} \end{aligned}$$

The second equal sign is valid, since the integral in the first line is independent of ϵ . The first term is problematic due to the singularity of the integrand at $t = 0$. We therefore rewrite it by taking the Fourier transform of the Mehler formula in the y variable. We also define the integration-by-parts operator

$$(2.3) \quad L_{h,t} := \frac{h}{i} \frac{1}{E + \partial_t \widehat{S}(t, x, p)} \partial_t$$

and will write $L_{h,t}^*$ for its adjoint. We now give the representation of $\Pi_{h_N, E}(x, y)$ as a semi-classical oscillatory integral operator that will be used in the Kac-Rice calculations.

Proposition 2. *Further, for every $\delta \in (0, \frac{\pi}{8})$ and each integer $k > d + 1$*

$$(2.4) \quad \begin{aligned} \Pi_{h_N, E}(x, y) &= \int_{S^1 \times \mathbb{R}^d} \frac{\chi_\delta(t)}{(\cos t)^{d/2}} e^{\frac{i}{h}(\widehat{S}(t, x, p) - y \cdot p + tE)} \widetilde{\chi}_R(p) \frac{dp}{(2\pi h)^d} \frac{dt}{2\pi} \\ &+ \int_{S^1 \times \mathbb{R}^d} \left[(L_{h, t}^*)^k \left(\frac{\chi_\delta(t)}{(\cos t)^{d/2}} \right) \right] e^{\frac{i}{h}(\widehat{S}(t, x, p) - y \cdot p + tE)} (1 - \widetilde{\chi}_R(p)) \frac{dp}{(2\pi h)^d} \frac{dt}{2\pi} \\ &+ \int_{S^1} \frac{1 - \chi_\delta(t)}{(2\pi h i \sin t)^{d/2}} \cdot e^{\frac{i}{h}(S(t, x, y) + tE)} \frac{dt}{2\pi}. \end{aligned}$$

Remark 1. *We will prove in Lemma 6 below that*

$$(2.5) \quad \left| E + \partial_t \widehat{S}(t, x, p) \right| \geq c(1 + |p|^2)$$

with $c > 0$ when $|x|^2 + |p|^2 \geq 3E$. In particular, $(1 - \widetilde{\chi}_E(p)) L_{h, t}$ is a well-defined operator. Lemma 6 shows that the second term in (2.4) is well-defined.

Proof. Let \mathcal{F}_h denote the semiclassical Fourier transform. $U_h(t - i\epsilon, x, y)$ is a Schwartz function in the y variable and we may take its Fourier transform

$$\widehat{U}_h(t - i\epsilon, x, p) := \mathcal{F}_{h, y \rightarrow p} U_h(t - i\epsilon, x, y).$$

Lemma 5. *Fix any $\epsilon \in (0, 1)$. The semiclassical Fourier transform of U_h in y is*

$$\widehat{U}_h(t - i\epsilon, x, p) = \frac{1}{(\cos(t - i\epsilon))^{d/2}} e^{\frac{i}{h} \widehat{S}(t - i\epsilon, x, p)},$$

where

$$(2.6) \quad \widehat{S}(t, x, p) = -\frac{|x|^2 + |p|^2}{2} \frac{\sin t}{\cos t} + \frac{x \cdot p}{\cos t}.$$

Proof. We will use the Mehler Formula (1.2) for U_h . First note that the prefactor $(2\pi i h \sin(t - i\epsilon))$ never vanishes and that $U_h(t - i\epsilon, x, y)$ is a smooth function. To show $U_h(t - i\epsilon, x, y)$ has fast decay in y under the condition in the Lemma, it suffices to check that $\text{Im } S(t - i\epsilon, x, y) > cy^2$ for some $c(\epsilon) > 0$ and all large enough y . From the definition of S , we have

$$\text{Im } S(t - i\epsilon, x, y) = \frac{|x|^2 + |y|^2}{2} \text{Im}(\cot(t - i\epsilon)) - x \cdot y \text{Im}(\csc(t - i\epsilon)).$$

Note that

$$\text{Im}(\cot(t - i\epsilon)) = \frac{e^{2\epsilon} - e^{-2\epsilon}}{|e^\epsilon e^{it} - e^{-\epsilon} e^{-it}|^2} > 0.$$

We thus have that $U(t - i\epsilon, x, y)$ is a Schwartz function for all t with the Schwartz seminorms depending only on ϵ . Thus, the Fourier transform in y is well-defined.

To get explicit formula for $\widehat{U}_h(t - i\epsilon, x, p)$, recall that if Q is a $d \times d$ non-degenerate symmetric real matrix, then

$$\mathcal{F}_{y \rightarrow p} \left(e^{\frac{i}{2h} \langle Qy, y \rangle} \right) (p) = \frac{(2\pi h)^{d/2} e^{i\pi \text{sgn}(Q)/4}}{|\det Q|^{1/2}} e^{-\frac{i}{2h} \langle Q^{-1}p, p \rangle}.$$

Thus, using (1.2),

$$\begin{aligned} \mathcal{F}_{y \rightarrow p} U_h(t - i\epsilon, x, p) &= \frac{(2\pi i h)^{d/2}}{(\cot(t - i\epsilon))^{d/2}} \cdot (2\pi i h \sin(t - i\epsilon))^{-d/2} \cdot e^{-i \frac{\tan(t - i\epsilon)}{2h} |x|^2} \cdot e^{\frac{i}{h} \frac{x \cdot p}{\cos(t - i\epsilon)}} \cdot e^{-\frac{i}{2h} \tan(t - i\epsilon) |p|^2} \\ &= (\cos(t - i\epsilon))^{-d/2} e^{-\frac{i}{h} \left(\frac{|x|^2 + |p|^2}{2} \tan(t - i\epsilon) - \frac{x \cdot p}{\cos(t - i\epsilon)} \right)}, \end{aligned}$$

proving the Lemma. \square

Using the Fourier inversion formula, we may write the first term in (2.2) as

$$\lim_{\epsilon \rightarrow 0^+} \int_0^{2\pi} \int_{\mathbb{R}^d} \frac{\chi_\delta(t)}{(\cos(t - i\epsilon))^{d/2}} e^{\frac{i}{h} (\widehat{S}(t - i\epsilon, x, y) - y \cdot p + (t - i\epsilon)E)} \frac{dp}{(2\pi h)^d} \frac{dt}{2\pi}$$

Writing $1 = \widetilde{\chi}_E(p) + (1 - \widetilde{\chi}_E(p))$, the integral in the previous line becomes

$$(2.7) \quad \lim_{\epsilon \rightarrow 0^+} \int_0^{2\pi} \int_{\mathbb{R}^d} \frac{\chi_\delta(t)}{(\cos(t - i\epsilon))^{d/2}} e^{\frac{i}{h} (\widehat{S}(t - i\epsilon, x, p) - y \cdot p + (t - i\epsilon)E)} \widetilde{\chi}_E(p) \frac{dp}{(2\pi h)^d} \frac{dt}{2\pi}$$

$$(2.8) \quad + \lim_{\epsilon \rightarrow 0^+} \int_0^{2\pi} \int_{\mathbb{R}^d} \frac{\chi_\delta(t)}{(\cos(t - i\epsilon))^{d/2}} e^{\frac{i}{h} (\widehat{S}(t - i\epsilon, x, p) - y \cdot p + (t - i\epsilon)E)} (1 - \widetilde{\chi}_E(p)) \frac{dp}{(2\pi h)^d} \frac{dt}{2\pi}.$$

The dominated convergence theorem allows us to take the limit as $\epsilon \rightarrow 0$ in (2.7), which gives the first term in (2.4). Finally, to study (2.8), we need the following result.

Lemma 6. *For each integer $k \geq 0$,*

$$(L_{h,t}^*)^k = h^k \sum_{i=0}^k a_{k,i}(t, x, p) \partial_t^i$$

with $|a_{k,i}(t, x, p)| \leq c_{k,i} (1 + |p|^2)^{-k}$ for all $|p| > 3E$ and some $c > 0$. The operator $L_{h,t}^*$ is the adjoint of $L_{h,t}$, which is defined in (2.3).

Proof. Let us write

$$f(t, x, p, E) := E + \partial_t \widehat{S} = E - \frac{|x|^2 + |p|^2}{2 \cos^2 t} + \frac{x \cdot p \sin(t)}{\cos^2 t}.$$

Note that when $t \in \text{supp } \chi_\delta$, we have that

$$(2.9) \quad |f(t, x, p, E)| > c (1 + |p|^2)$$

for some $c > 0$ as long as $|p| > 3E$. To control L^* , let us write $M_{1/f}$ for the multiplication operator by $1/f$, which is well-defined when $t \in \text{supp } \chi_\delta$ and $|p| > 3E$. We have

$$(L_{h,t}^*)^k = \left(\frac{h}{i} \right)^k (\partial_t \circ M_{1/f})^k,$$

which we may write as a sum of finitely many terms of the form

$$(2.10) \quad \left(\frac{h}{i} \right)^k C_{k_1, \dots, k_r} \frac{\partial_t^{k_1}(f) \cdots \partial_t^{k_r}(f)}{f^{k+r}} \partial_t^{k_0}$$

where $k_0 + \cdots + k_r = k$ and C_{k_0, \dots, k_r} are some constants. Note that for any $j \geq 1$

$$(2.11) \quad |\partial_t^j f(t, x, p, E)| \leq c_j \cdot (1 + |p|^2)$$

for some constants $c_j \in \mathbb{R}$ that are uniform in $t \in \text{supp } \chi_\delta$. Combining (2.9)-(2.11) completes the proof. \square

Lemma 6 allows us to integrate by parts using the operator $L_{h,t}^k$ in the integral (2.8). The resulting integrand is $L^1(\mathbb{R})$ uniformly in ϵ . We therefore send $\epsilon \rightarrow 0$ and again use the dominated convergence theorem to obtain the second term in the stated formula (2.4). This completes the proof of Proposition 2. \square

Remark 2. We note that the estimate for $L_{h,t}^*$ is first used by Chazarain [Ch], and a similar result can be obtained for a more general class of potential with quadratic growth at infinity.

3. PROOF OF THEOREM 1

Lebesgue measure on \mathbb{R}^d is the volume form for the flat metric, so by the Kac-Rice formula (1.6), $\mathbb{E}[|Z_{\Phi_N}|]$ has a density $F_N(x)$ given by (1.6). In order to use this formula, we need to understand the $d \times d$ matrix

$$\Omega_{x,E} := \frac{\Pi_{h,E}(x, x) \partial_{x_k} \partial_{y_j}|_{x=y} \Pi_{h,E}(x, y) - \partial_{x_k}|_{x=y} \Pi_{h,E}(x, y) \cdot \partial_{y_j}|_{x=y} \Pi_{h,E}(x, y)}{\Pi_{h,E}(x, x)^2}.$$

The main step of the proof of Theorem 1 is the following Proposition, which gives an explicit formula for $\Omega_{x,E}$.

Proposition 3. *Suppose $|x|^2 \in \mathcal{F}_E$. Then*

$$(3.1) \quad (\Omega_{x,E})_{kj} = h^{-1} \frac{(\delta_{kj} - \hat{x}_k \hat{x}_j) E}{|x| \sqrt{|x|^2 - 2E}} + O(1),$$

where $\hat{x}_k := \frac{x_k}{|x|}$. Suppose $x \in \mathcal{A}_E \setminus \{0\}$. Then

$$(3.2) \quad (\Omega_{x,E})_{kj} = h^{-2} \delta_{kj} \cdot \frac{\omega_{d-2}}{d \cdot \omega_{d-1}} (2E - |x|^2) (1 + O(h^2))$$

The implied constants in the ‘O’ error terms in (3.2) and (3.1) are uniform on compact subsets of the interiors in $\mathcal{A}_E \setminus \{0\}$ and \mathcal{F}_E .

Theorem 1 follows easily by substituting (3.2) and (3.1) into (1.6) and using the identities (1.8). We will prove (3.1) in §3.1 and (3.2) in §3.2.

3.1. Proof of Proposition 3 in the Forbidden Region. We fix $x \in \mathbb{R}^d$ with $|x|^2 > 2E$. Our goal is to prove Equation (3.1). Recall from (1.4) and (1.2) that

$$\Pi_{h_N, E}(x, y) = \int_{-\pi - i\epsilon}^{\pi - i\epsilon} \frac{1}{(2\pi i h \sin t)^{d/2}} e^{\frac{i}{h}(S(t, x, y) + tE)} \frac{dt}{2\pi}$$

is an absolutely convergent integral for all $\epsilon > 0$ that is independent of the value of ϵ . Equation (3.1) will follow from Lemma 7, Equations (3.10)-(3.11), and Lemma 8.

Lemma 7. *The phase $S(t, x, x) + Et$ has no critical points in the real domain. In the complex domain, it has two critical points $\pm i\beta$, which are the two distinct solutions to*

$$\cosh(\beta/2) = \frac{|x|}{\sqrt{2E}}.$$

These critical points are non-degenerate.

Proof. Note that $S(t, x, x) = -\tan\left(\frac{t}{2}\right) |x|^2$. Hence, $\partial_t(S(t, x, x) + Et) = 0$ is equivalent to

$$\cos\left(\frac{t}{2}\right) = \pm \frac{|x|}{\sqrt{2E}}.$$

Since we have assumed $\frac{|x|^2}{2E} > 1$, the phase $S(t, x, x) + Et$ has no real critical points. Setting $t = \alpha + i\beta$, the critical point equation is equivalent to

$$\cos\left(\frac{\alpha}{2}\right) \cos\left(\frac{i\beta}{2}\right) - \sin\left(\frac{\alpha}{2}\right) \sin\left(\frac{i\beta}{2}\right) = \pm \frac{|x|}{\sqrt{2E}}.$$

Since the right hand side is real and $\sin(it) = i \sinh(t)$, we conclude that $\alpha = 0$. Using that \cosh is positive, the equation therefore reduces to

$$(3.3) \quad \cosh\left(\frac{\beta}{2}\right) = \frac{|x|}{\sqrt{2E}},$$

which has two distinct solutions $\pm\beta$ for some $\beta > 0$. It remains to check that these critical points are non-degenerate:

$$\partial_{tt}S(\pm i\beta, x, x) = \mp iE \tanh \beta/2,$$

which is non-zero since $\beta \neq 0$. □

Lemma 7 shows that to evaluate the integral (1.4), we should set $\epsilon = \beta$. Let us abbreviate

$$S_{j,k}(t) = \partial_{x_j} \partial_{y_k} S(t, x, y) \Big|_{x=y}$$

with the convention that $S_{j,0}(t) = \partial_{x_j} S(t, x, y) \Big|_{x=y}$ and $S_{0,0}(t) = S(t, x, x)$. We will continue to write ' for derivatives with respect to t .

Lemma 8. *For each $x \in \mathbb{R}^d$ and $1 \leq j, k \leq d$, we have*

$$(3.4) \quad (\Omega_{x,E})_{j,k} = \frac{i}{h} \left(S_{j,k}(-i\beta) - \frac{S'_{j,0}(-i\beta) S'_{0,k}(-i\beta)}{S''_{0,0}(-i\beta)} \right) + O(1)$$

Proof. Let us write

$$\Pi_{h,E}^{j,k} = \partial_{x_j} \partial_{y_k} \Big|_{x=y} \Pi_{h,E}(x, y),$$

again with the understanding that $\Pi_{h,E}^{j,0}$ means no derivative in y and so on. We then have

$$(3.5) \quad (\Omega_{x,E})_{j,k} = \frac{\Pi_{h,E}(x, x) \Pi_{h,E}^{j,k} - \Pi_{h,E}^{j,0} \Pi_{h,E}^{0,k}}{\Pi_{h,E}(x, x)^2}.$$

Each term in the numerator and denominator is an oscillatory integral with the same phase $S(t, x, x) + tE$. Indeed, if we abbreviate

$$A(t) = \frac{1}{(2\pi i h \sin t)^{d/2}} \cdot \frac{dt}{2\pi},$$

then

$$(3.6) \quad \Pi_{h,E}(x, x) = \int_{-\pi-i\epsilon}^{\pi-i\epsilon} e^{\frac{i}{h}(S(t,x,x)+tE)} A(t)$$

$$(3.7) \quad \Pi_{h,E}^{j,k}(x, x) = \int_{-\pi-i\epsilon}^{\pi-i\epsilon} \left(\frac{i}{h} S_{j,k} - \frac{1}{h^2} S_{j,0} \cdot S_{0,k} \right) e^{\frac{i}{h}(S(t,x,x)+tE)} A(t)$$

$$(3.8) \quad \Pi_{h,E}^{j,0}(x, x) = \int_{-\pi-i\epsilon}^{\pi-i\epsilon} \left(\frac{i}{h} S_{j,0} \right) e^{\frac{i}{h}(S(t,x,x)+tE)} A(t)$$

$$(3.9) \quad \Pi_{h,E}^{0,k}(x, x) = \int_{-\pi-i\epsilon}^{\pi-i\epsilon} \left(\frac{i}{h} S_{0,k} \right) e^{\frac{i}{h}(S(t,x,x)+tE)} A(t)$$

We now apply Lemma 3 to each term. Let us rewrite (1.11) schematically as

$$\int a e^{\frac{i}{h}S} = C (a + h \cdot f_1(a, S) + O(h^2)).$$

Here the constant C depends on S and h and so on, but will cancel in the numerator and denominator of (3.5) and the function f_1 is linear in the amplitude a . The first term, $\Pi_{h,E}(x, x)\Pi_{h,E}^{j,k}$, in the numerator of $\Omega_{x,E}$ is therefore

$$C^2 (A + h \cdot f_1(A, S)) \left(A \left[\frac{i}{h} S_{j,k} - \frac{1}{h^2} S_{j,0} S_{0,k} \right] + h \cdot f_1 \left(A \left[\frac{i}{h} S_{j,k} - \frac{1}{h^2} S_{j,0} S_{0,k} \right], S \right) \right) + O(h^2),$$

which becomes

$$C^2 A^2 \left(-\frac{1}{h^2} S_{j,0} S_{0,k} + \frac{1}{h} \left(i S_{j,k} - f_1(A, S) S_{j,0} S_{0,k} - \frac{1}{h} f_1(AS_{j,0} S_{0,k}, S) \right) + O(1) \right).$$

Similarly, the second term, $\Pi_{h,E}^{j,0}\Pi_{h,E}^{0,k}$, in the numerator of $\Omega_{x,E}$ is

$$-C^2 a^2 \left(\frac{1}{h^2} S_{j,0} S_{0,k} + \frac{1}{h} [S_{j,0} f_1(AS_{0,k}, S) + S_{0,k} f_1(AS_{j,0}, S)] + O(1) \right).$$

Note that the h^{-2} terms in the expansions of $\Pi_{h,E}(x, x)\Pi_{h,E}^{j,k}$ and $\Pi_{h,E}^{j,0}\Pi_{h,E}^{0,k}$ cancel. From expression (1.11), we see that the terms in f_1 that depend on at most 1 derivative of the amplitude will cancel between $\Pi_{h,E}(x, x)\Pi_{h,E}^{j,k}$, and $\Pi_{h,E}^{j,0}\Pi_{h,E}^{0,k}$. Hence, comparing the contributions of the single term in f_1 that involves two derivatives of the amplitude, the numerator of (3.5) becomes

$$\frac{C^2 A^2}{h} \left(i S_{j,k}(-i\beta) - \frac{S'_{j,0}(-i\beta) S'_{0,k}(-i\beta)}{S''_{0,0}(-i\beta)} + O(1) \right).$$

Finally, we use that the denominator in (3.5) is of the form $C^2 A^2 (1 + O(h))$ to complete the proof. \square

We may use (1.3) to obtain

$$(3.10) \quad S_i = -x_i \tan\left(\frac{t}{2}\right), \quad S'_i = -\frac{x_i}{2} \frac{1}{\cos^2\left(\frac{t}{2}\right)}$$

$$(3.11) \quad S_{i,j} = -\frac{\delta_{ij}}{\sin t}, \quad S'' = -\frac{|x|^2}{2} \frac{\sin(t/2)}{\cos^3(t/2)}.$$

Lemma 8 now gives

$$(\Omega_{x,E})_{j,k} = \frac{1}{h} \frac{\delta_{jk} - \hat{x}_j \hat{x}_k}{\sinh(\beta)} + O(1).$$

Combining (3.3) with

$$\sinh(\beta) = 2 \cosh\left(\frac{\beta}{2}\right) \sinh\left(\frac{\beta}{2}\right) = 2 \cosh\left(\frac{\beta}{2}\right) \sqrt{\cosh^2\left(\frac{\beta}{2}\right) - 1}$$

proves (3.1). Before going on to prove Theorem 1 in the allowed region, let us prove the following result, which we believe is of independent interest.

Lemma 9 (Explicit Expression of $\Pi_{h,E}$ in the Forbidden Region). *Fix $|x|^2 > 2E$. Then, with β defined as in Lemma 7 and $1 \leq j, k \leq d$, we have*

$$(3.12) \quad \Pi_{h,E}(x, x) = (2\pi)^{-\frac{d+1}{2}} h^{-\frac{d-1}{2}} \frac{|x|^{1/2} e^{\frac{1}{h}(-|x|\sqrt{|x|^2-2E+E\beta})}}{E^{1/2} \cdot (|x|^2 - 2E)^{1/4} (\sinh \beta)^{d/2}} (1 + O(h))$$

where, as before $\hat{x}_k := \frac{x_k}{|x|}$.

Proof. Let us take $\epsilon = \beta$ in (1.4). The real part of the phase along the contour $[-\pi - i\beta, \pi - i\beta]$ is

$$\operatorname{Re} \left(\frac{i}{h} (S(t - i\beta, x, x) + (t - i\beta)E) \right) = \frac{|x|^2}{h} \operatorname{Im} \left(\tan \left(\frac{t - i\beta}{2} \right) \right) + \frac{\beta}{h} E,$$

which has a unique maximum when $t = 0$. We may therefore apply Lemma 3. Let us denote $a(t) := (i \sin t)^{-d/2}$. We have

$$\begin{aligned} \partial_{tt}|_{t=-i\beta} S(t, x, x) &= \frac{iE}{|x|} \sqrt{|x|^2 - 2E} \\ S(t, x, x) + tE|_{t=-i\beta} &= i \left(|x| \sqrt{|x|^2 - 2E} - \beta E \right) \\ a(-i\beta) &= (\sinh \beta)^{-d/2} = \left(\frac{|x|}{E} \sqrt{|x|^2 - 2E} \right)^{-d/2}. \end{aligned}$$

Thus,

$$(3.13) \quad \Pi_{h,E}(x, x) = (2\pi)^{-\frac{d+1}{2}} h^{-\frac{d-1}{2}} \frac{(\sinh \beta)^{-d/2} |x|^{1/2}}{E^{1/2} \cdot (|x|^2 - 2E)^{1/4}} e^{\frac{1}{h}(-|x|\sqrt{|x|^2-2E+E\beta})} (1 + O(h)),$$

which is precisely (3.12). This completes the proof of Lemma 9. \square

3.2. Proof of Proposition 3 in the Allowed Region. The goal of this section is to prove Equation (3.2), which is a consequence of the following Lemma.

Lemma 10 (Derivatives of $\Pi_{h,E}$ in the Allowed Region). *Suppose $0 < |x|^2 < 2E$. Then*

$$(3.14) \quad \Pi_{h,E}(x, x) = (2\pi h)^{-(d-1)} (2E - |x|^2)^{\frac{d}{2}-1} \omega_{d-1} (1 + O(h))$$

$$(3.15) \quad \partial_{x_i}|_{x=y} \Pi_{h,E}(x, y) = \partial_{y_i}|_{x=y} \Pi_{h,E}(x, y) = O(1/h^{d-1}) + O(1/h^{(d+1)/2})$$

$$(3.16) \quad \partial_{x_k} \partial_{y_j}|_{x=y} \Pi_{h,E}(x, y) = \delta_{kj} h^{-2} \cdot (2E - |x|^2) \cdot \frac{1}{d} \Pi_{h,E}(x, x) (1 + O(h))$$

Proof. Fix x with $0 < |x|^2 < 2E$. Equations (3.14)-(3.16) are obtained by applying stationary phase to the oscillatory integral representation (2.4) for $\Pi_{h,E}$ and its derivatives. To start, note that the second term in (2.4) is $O(h^\infty)$ since we may take k arbitrarily large. Also, by Lemma 7, we may apply stationary phase to the third term of (2.4) and find that its contribution is on the order of $h^{-\frac{d-1}{2}}$.

As we prove below, the first integral gives the leading contribution to $\Pi_{h,E}(x, x)$, which is on the order of $h^{-(d-1)}$. To see this, we will apply stationary phase. Let us compute the critical set for the phase function $\widehat{S}(t, x, p) - y \cdot p + Et$. To emphasize the relation between the phase factor $\widehat{S}(t, x, p)$ and the classical path of energy E ending in time t at x with initial momentum p , let us write $x = x_f$ and $p = p_i$, where the subscripts f and i stand for initial and terminal positions and momenta. We have the relations:

$$(3.17) \quad x_i = \frac{x_f - p_i \sin t}{\cos t}, \quad p_f = \frac{p_i - x_f \sin t}{\cos t}.$$

The t -critical point equation is

$$E + \partial_t \widehat{S}(t, x_f, p_i) = E - \frac{(x_i(t, x_f, p_i))^2 + p_i^2}{2} = 0$$

since \widehat{S} satisfies the Hamilton-Jacobi equation associated to H_h . The p_i critical point equation is:

$$y = \partial_{p_i} \widehat{S}(t, x_f, p_i) = x_i.$$

On the diagonal, we have $y = x_f$ so that this relation is $x_i = x_f$. Therefore, the critical manifold for $\widehat{S}(t, x, x) - y \cdot p + Et$ is

$$W_{x,E} = \left\{ (t, p) \in [0, 2\pi) \times T_x^* \mathbb{R}^d \mid \pi \Phi^t(x, p) = x \quad \text{and} \quad \frac{|p|^2 + |x|^2}{2} = E \right\},$$

where $\pi : T^* \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the projection to the base and Φ^t is the Hamilton flow for the classical harmonic oscillator $\frac{1}{2}(|x|^2 + |p|^2)$. To apply the method of stationary phase, we must be sure that $\Omega_{x,E}$ is non-degenerate.

Lemma 11. *For $\delta > 0$ sufficiently small, $W_{x,E}$ restricted to the support of χ_δ is*

$$(3.18) \quad W_{x,E} \cap \text{supp}(\chi_\delta(t)) = \{t = 0\} \times S^* \sqrt{2E - |x|^2} = \{0\} \times \left\{ p \in T_x^* \mathbb{R}^d \mid \frac{|x|^2 + |p|^2}{2} = E \right\},$$

which is a non-degenerate critical manifold. Moreover, the Morse index of $\widehat{S}(t, x, p) - y \cdot p + Et$ along $W_{x,E}$ is 1.

Proof. From the relations (3.17), we see that if $x \neq 0$, then for δ sufficiently small, the only value of t that is in the support of χ_δ for which we may simultaneously solve

$$p_f(t, x_f, p_i) = p_i \quad \text{and} \quad x_f = x_i(t, x_f, p_i) \quad \text{and} \quad \frac{|x|^2 + |p|^2}{2} = E$$

is $t = 0$. This proves (3.18). To check that the critical manifold is non-degenerate, let us compute the normal Hessian of $\widehat{S}(t, x, p) - y \cdot p + Et$ along $W_{x,E}$. Note that the fiber of the

normal bundle to $W_{x,E} \cap \text{supp}(\chi_\delta(t))$ is spanned by ∂_t and ∂_r , where $r = |p|$ so that ∂_r is the radial vector field. We have

$$\begin{aligned}\partial_{tt} \left(\widehat{S}(t, x_f, p_i) - y \cdot p_i + tE \right) &= x_f \cdot p_i \\ \partial_{tr} \left(\widehat{S}(t, x, p) - y \cdot p + tE \right) &= -|p_i| \\ \partial_{rr} \left(\widehat{S}(t, x, p) - y \cdot p + tE \right) &= 0.\end{aligned}$$

Hence, the normal Hessian is

$$\begin{pmatrix} x_f \cdot p_i & -|p_i| \\ -|p_i| & 0 \end{pmatrix}.$$

The determinant is $-|p_i|^2 = |x_f|^2 - 2E$, which is non-zero as long as $|x_f| < 2E$. Hence, $W_{x,E}$ is non-degenerate. Moreover, one easily verifies that the normal Hessian always has one positive and one negative eigenvalue. The Morse index of $\widehat{S}(t, x_f, p_i) - y \cdot p_i + Et$ along $W_{x,E}$ is therefore equal to 1. This completes the proof of Lemma 11. \square

Returning to the proof of Equations (3.14)-(3.16), we apply the stationary phase method (§1.4) to the first term in (2.4). Writing r for the radial coordinate on \mathbb{R}^d , we have

$$d\mu_{W_{x,E}} = \frac{dt \wedge dx}{\left| \det \text{Hess}^\perp \widehat{S} \right|^{1/2} \cdot dt \wedge dr} \Big|_{W_{x,E}}.$$

Write $d\omega$ for the uniform measure on the sphere of radius $\sqrt{2E - |x|^2}$ normalized to have volume 1 and ω_{d-1} of the volume of unit sphere in \mathbb{R}^d . We may thus express $dt \wedge dx$ as $\omega_{d-1} dt \wedge r^{d-1} dr \wedge d\omega$. We find that

$$d\mu_{W_{x,E}} = (2E - |x|^2)^{\frac{d}{2}-1} \omega_{d-1} \cdot d\omega.$$

Observe that the amplitude $\frac{\chi_\delta(t)}{\cos t^{-d/2}} \widetilde{\chi}_R(p)$ in the first integral of (2.4) is identically equal to 1 on $W_{x,E}$. Hence its integral over with respect to $d\mu_{W_{x,E}}$ is

$$(2E - |x|^2)^{\frac{d}{2}-1} \omega_{d-1}.$$

Noting that $\left(\widehat{S}(t, x, p) - y \cdot p + tE \right) |_{M_0} = 0$ we obtain from Lemma 2 that $\Pi_{h,E}(x, x)$ may written as

$$(3.19) \quad (2\pi h)^{-(d-1)} (2E - |x|^2)^{\frac{d}{2}-1} \omega_{d-1} (1 + O(h)).$$

This confirms Equation (3.14).

In order to compute the asymptotics of $\partial_{x_k}|_{x=y} \Pi_{h,E}(x, y)$, we argue in a similar fashion. First, we differentiate under the integral in all three terms of (2.4). Just as before, the second term has order $O(h^\infty)$, and the third term is $O(1/h^{\frac{d+1}{2}})$. The first term, whose leading order term $O(1/h^d)$ vanishes, will at most give a $O(1/h^{d-1})$ contribution. To prove this, note that when $t = 0$,

$$(3.20) \quad \partial_{x_k}|_{x=y} \left(\widehat{S}(t, x, p) - y \cdot p + Et \right) = p_k,$$

whose integral over $W_{x,E}$ vanishes. Thus, applying Lemma 2, we find that

$$\partial_{x_k}|_{x=y}\Pi_{h,E}(x,y) = O(\Pi_{h,E}(x,x)).$$

This proves Equation (3.15). It remains to study $\partial_{x_k}\partial_{y_j}|_{x=y}\Pi_{h,E}(x,y)$. Like before, we differentiate under the integral sign in (2.4). The main contribution comes from the first term. To evaluate it, note that when $t = 0$,

$$\partial_{x_k}\partial_{y_j}|_{x=y}e^{\frac{i}{h}(\widehat{S}(t,x,p)-y\cdot p+Et)} = \delta_{kj}(h^{-2}p_k \cdot p_j)e^{\frac{i}{h}(\widehat{S}(t,x,p)-y\cdot p+Et)}.$$

We therefore have

$$\int_{W_{x,E}} \partial_{x_k}\partial_{y_j}|_{x=y}e^{\frac{i}{h}(\widehat{S}(t,x,p)-y\cdot p+Et)}d\mu_{W_{x,E}} = h^{-2}\delta_{jk}\frac{(2E - |x|^2)^{\frac{d}{2}}}{d}.$$

Applying Lemma 2 proves (3.16) and completes the proof of Proposition 3 in the allowed region. \square

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208, USA

E-mail address, B. Hanin: bhanin@math.northwestern.edu

E-mail address, S. Zelditch: zelditch@math.northwestern.edu

E-mail address, P. Zhou: pengzhou@math.northwestern.edu