

On automorphisms and endomorphisms of projective varieties

Michel Brion

Abstract

We first show that any connected algebraic group over a perfect field is the neutral component of the automorphism group scheme of some normal projective variety. Then we show that very few connected algebraic semigroups can be realized as endomorphisms of some projective variety X , by describing the structure of all connected subsemigroup schemes of $\text{End}(X)$.

Key words: automorphism group scheme, endomorphism semigroup scheme
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1 Introduction and statement of the results

By a result of Winkelmann (see [22]), every connected real Lie group G can be realized as the automorphism group of some complex Stein manifold X , which may be chosen complete, and hyperbolic in the sense of Kobayashi. Subsequently, Kan showed in [12] that we may further assume $\dim_{\mathbb{C}}(X) = \dim_{\mathbb{R}}(G)$.

We shall obtain a somewhat similar result for connected algebraic groups. We first introduce some notation and conventions, and recall general results on automorphism group schemes.

Throughout this article, we consider schemes and their morphisms over a fixed field k . Schemes are assumed to be separated; subschemes are locally closed unless mentioned otherwise. By a *point* of a scheme S , we mean a

Michel Brion
Institut Fourier, Université de Grenoble
B.P. 74, 38402 Saint-Martin d'Hères Cedex, France
e-mail: Michel.Brion@ujf-grenoble.fr

T -valued point $f : T \rightarrow S$ for some scheme T . A *variety* is a geometrically integral scheme of finite type.

We shall use [17] as a general reference for group schemes. We denote by e_G the neutral element of a group scheme G , and by G^o the neutral component. An *algebraic group* is a smooth group scheme of finite type.

Given a projective scheme X , the functor of automorphisms,

$$T \longmapsto \mathrm{Aut}_T(X \times T),$$

is represented by a group scheme, locally of finite type, that we denote by $\mathrm{Aut}(X)$. The Lie algebra of $\mathrm{Aut}(X)$ is identified with the Lie algebra of global vector fields, $\mathrm{Der}(\mathcal{O}_X)$ (these results hold more generally for projective schemes over an arbitrary base, see [13, p. 268]; they also hold for proper schemes of finite type over a field, see [16, Thm. 3.7]). In particular, the neutral component, $\mathrm{Aut}^o(X)$, is a group scheme of finite type; when k is perfect, the reduced subscheme, $\mathrm{Aut}^o(X)_{\mathrm{red}}$, is a connected algebraic group. As a consequence, $\mathrm{Aut}^o(X)$ is a connected algebraic group if $\mathrm{char}(k) = 0$, since every group scheme of finite type is reduced under that assumption. Yet $\mathrm{Aut}^o(X)$ is not necessarily reduced in prime characteristics (see e.g. the examples in [16, §4]).

We may now state our first result:

Theorem 1. *Let G be a connected algebraic group, and n its dimension.*

If $\mathrm{char}(k) = 0$, then there exists a smooth projective variety X such that $\mathrm{Aut}^o(X) \cong G$ and $\dim(X) = 2n$.

If $\mathrm{char}(k) > 0$ and k is perfect, then there exists a normal projective variety X such that $\mathrm{Aut}_{\mathrm{red}}^o(X) \cong G$ and $\dim(X) = 2n$ (resp. $\mathrm{Aut}^o(X) \cong G$ and $\dim(X) = 2n + 2$).

This result is proved in Section 2, first in the case where $\mathrm{char}(k) = 0$; then we adapt the arguments to the case of prime characteristics, which is technically more involved due to group schemes issues. We rely on fundamental results about the structure and actions of algebraic groups over an algebraically closed field, for which we refer to the recent exposition [4].

Theorem 1 leaves open many basic questions about automorphism group schemes. For instance, can one realize every connected algebraic group over an arbitrary field (or more generally, every connected group scheme of finite type) as the full automorphism group scheme of a normal projective variety? Also, very little seems to be known about the group of components, $\mathrm{Aut}(X)/\mathrm{Aut}^o(X)$, where X is a projective variety. In particular, the question of the finite generation of this group is open, already when X is a complex projective manifold.

As a consequence of Theorem 1, we obtain the following characterization of Lie algebras of vector fields:

Corollary 1. *Let \mathfrak{g} be a finite-dimensional Lie algebra over a field k of characteristic 0. Then the following conditions are equivalent:*

- (i) $\mathfrak{g} \cong \text{Der}(\mathcal{O}_X)$ for some proper scheme X of finite type.
- (ii) \mathfrak{g} is the Lie algebra of a linear algebraic group.

Under either condition, X may be chosen projective, smooth, and unirational of dimension $2n$, where $n := \dim(\mathfrak{g})$. If k is algebraically closed, then we may further choose X rational.

This result is proved in Subsection 2.3. The Lie algebras of linear algebraic groups over a field of characteristic 0 are called algebraic Lie algebras; they have been characterized by Chevalley in [6, 7]. More specifically, a finite-dimensional Lie algebra \mathfrak{g} is algebraic if and only if its image under the adjoint representation is an algebraic Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ (see [7, Chap. V, §5, Prop. 3]). Moreover, the algebraic Lie subalgebras of $\mathfrak{gl}(V)$, where V is a finite-dimensional vector space, are characterized in [6, Chap. II, §14]. Also, recall a result of Hochschild (see [11]): the isomorphism classes of algebraic Lie algebras are in bijective correspondence with the isomorphism classes of connected linear algebraic groups with unipotent center.

In characteristic $p > 0$, one should rather consider restricted Lie algebras, also known as p -Lie algebras. In this setting, characterizing Lie algebras of vector fields seems to be an open question. This is related to the question of characterizing automorphism group schemes, via the identification of restricted Lie algebras with infinitesimal group schemes of height ≤ 1 (see [17, Exp. VIIA, Thm. 7.4]).

Next, we turn to the monoid schemes of endomorphisms of projective varieties; we shall describe their connected subsemigroup schemes. For this, we recall basic results on schemes of morphisms.

Given two projective schemes X and Y , the functor of morphisms,

$$T \longmapsto \text{Hom}_T(X \times T, Y \times T) \cong \text{Hom}(X \times T, Y),$$

is represented by an open subscheme of the Hilbert scheme $\text{Hilb}(X \times Y)$, by assigning to each morphism its graph (see [13, p. 268], and [14, §1.10], [19, §4.6.6] for more details). We denote that open subscheme by $\text{Hom}(X, Y)$. The composition of morphisms yields a natural transformation of functors, and hence a morphism of schemes

$$\text{Hom}(X, Y) \times \text{Hom}(Y, Z) \longrightarrow \text{Hom}(X, Z), \quad (f, g) \longmapsto gf$$

where Z is another projective scheme.

As a consequence of these results, the functor of endomorphisms of a projective scheme X is represented by a scheme, $\text{End}(X)$; moreover, the composition of endomorphisms equips $\text{End}(X)$ with a structure of monoid scheme with neutral element being of course the identity, id_X . Each connected component of $\text{End}(X)$ is of finite type, and these components form a countable set. The automorphism group scheme $\text{Aut}(X)$ is open in $\text{End}(X)$ by [13, p. 267] (see also [14, Lem. I.1.10.1]). If X is a variety, then $\text{Aut}(X)$ is also

closed in $\text{End}(X)$, as follows from [5, Lem. 4.4.4]; thus, $\text{Aut}(X)$ is a union of connected components of $\text{End}(X)$. In particular, $\text{Aut}^o(X)$ is the connected component of id_X in $\text{End}(X)$.

As another consequence, given a morphism $f : X \rightarrow Y$ of projective schemes, the functor of sections of f is represented by a scheme that we shall denote by $\text{Sec}(f)$: the fiber at id_Y of the morphism

$$\lambda_f : \text{Hom}(Y, X) \longrightarrow \text{End}(Y), \quad g \longmapsto fg.$$

Every section of f is a closed immersion; moreover, $\text{Sec}(f)$ is identified with an open subscheme of $\text{Hilb}(X)$ by assigning to each section its image (see [13, p. 268] again; our notation differs from the one used there). Given a section $s \in \text{Sec}(f)(k)$, we may identify Y with the closed subscheme $Z := s(Y)$; then f is identified with a *retraction* of X onto that subscheme, i.e., to a morphism $r : X \rightarrow Z$ such that $ri = \text{id}_Z$, where $i : Z \rightarrow X$ denotes the inclusion. Moreover, the endomorphism $e := ir$ of X is *idempotent*, i.e., satisfies $e^2 = e$.

Conversely, every idempotent k -rational point of $\text{End}(X)$ can be written uniquely as $e = ir$, where $i : Y \rightarrow X$ is the inclusion of the image of e (which coincides with its fixed point subscheme), and $r : X \rightarrow Y$ is a retraction. When X is a variety, Y is a projective variety as well. We now analyze the connected component of e in $\text{End}(X)$:

Proposition 1. *Let X be a projective variety, $e \in \text{End}(X)(k)$ an idempotent, and C the connected component of e in $\text{End}(X)$. Write $e = ir$, where $i : Y \rightarrow X$ denotes the inclusion of a closed subvariety, and $r : X \rightarrow Y$ is a retraction.*

(i) *The morphism*

$$\rho_r : \text{Hom}(Y, X) \longrightarrow \text{End}(X), \quad f \longmapsto fr$$

restricts to an isomorphism from the connected component of i in $\text{Hom}(Y, X)$, to C . Moreover, C is a subsemigroup scheme of $\text{End}(X)$, and $f = fe$ for any $f \in C$.

(ii) *The morphism*

$$\lambda_i \rho_r : \text{End}(Y) \longrightarrow \text{End}(X), \quad f \longmapsto ifr$$

restricts to an isomorphism of semigroup schemes $\text{Aut}^o(Y) \xrightarrow{\cong} eC$. In particular, eC is a group scheme with neutral element e .

(iii) ρ_r *restricts to an isomorphism from the connected component of i in $\text{Sec}(r)$, to the subscheme $E(C)$ of idempotents in C . Moreover, $f_1 f_2 = f_1$ for all $f_1, f_2 \in E(C)$; in particular, $E(C)$ is a closed subsemigroup scheme of C .*

(iv) *The morphism*

$$\varphi : E(C) \times eC \longrightarrow C, \quad (f, g) \longmapsto fg$$

is an isomorphism of semigroup schemes, where the semigroup law on the left-hand side is given by $(f_1, g_1) \cdot (f_2, g_2) = (f_1, g_1 g_2)$.

This is proved in Subsection 3.1, by using a version of the rigidity lemma (see [5, §4.4]). As a straightforward consequence, the maximal connected subgroup schemes of $\text{End}(X)$ are exactly the $\lambda_i \rho_r(\text{Aut}^o(Y))$ with the above notation (this fact is easily checked directly).

As another consequence of Proposition 1, the endomorphism scheme of a projective variety can have everywhere nonreduced connected components, even in characteristic 0. Consider for example a ruled surface

$$r : X = \mathbb{P}(\mathcal{E}) \longrightarrow Y,$$

where Y is an elliptic curve and \mathcal{E} is a locally free sheaf on Y which belongs to a nonsplit exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

(such a sequence exists in view of the isomorphisms $\text{Ext}^1(\mathcal{O}_Y, \mathcal{O}_Y) \cong H^1(Y, \mathcal{O}_Y) \cong k$). Let $i : Y \rightarrow X$ be the section associated with the projection $\mathcal{E} \rightarrow \mathcal{O}_Y$. Then the image of i yields an isolated point of $\text{Hilb}(X)$ with Zariski tangent space of dimension 1 (see e.g. [19, Ex. 4.6.7]). Thus, the connected component of i in $\text{Sec}(r)$ is a nonreduced fat point. By Proposition 1 (iv), the connected component of $e := ir$ in $\text{End}(X)$ is isomorphic to the product of that fat point with $\text{Aut}^o(Y) \cong Y$, and hence is nonreduced everywhere. This explains a posteriori why we have to be so fussy with semigroup schemes.

A further consequence of Proposition 1 is the following:

Proposition 2. *Let X be a projective variety, S a connected subsemigroup scheme of $\text{End}(X)$, and $E(S) \subset S$ the closed subscheme of idempotents. Assume that S has a k -rational point.*

- (i) *$E(S)$ is a connected subsemigroup scheme of S , with semigroup law given by $f_1 f_2 = f_1$. Moreover, $E(S)$ has a k -rational point.*
- (ii) *For any $e \in E(S)(k)$, the closed subsemigroup scheme $eS \subset S$ is a group scheme. Moreover, the morphism*

$$\varphi : E(S) \times eS \longrightarrow S, \quad (f, g) \longmapsto fg$$

is an isomorphism of semigroup schemes.

- (iii) *Identifying S with $E(S) \times eS$ via φ , the projection $\pi : S \rightarrow E(S)$ is the unique retraction of semigroup schemes from S to $E(S)$. In particular, π is independent of the choice of the k -rational idempotent e .*

This structure result is proved in Subsection 3.2; a new ingredient is the fact that a subsemigroup scheme of a group scheme of finite type is a subgroup scheme (Lemma 10). The case where S has no k -rational point is discussed in Remark 5 at the end of Subsection 3.2.

Proposition 2 yields strong restrictions on the structure of connected subsemigroup schemes of $\text{End}(X)$, where X is a projective variety. For example,

if such a subsemigroup scheme is commutative or has a neutral element, then it is just a group scheme.

As another application of that proposition, we shall show that the dynamics of an endomorphism of X which belongs to some algebraic subsemigroup is very restricted. To formulate our result, we need the following:

Definition 1. Let X be a projective variety, and f a k -rational endomorphism of X .

We say that f is *bounded*, if f belongs to a subsemigroup of finite type of $\text{End}(X)$. Equivalently, the powers f^n , where $n \geq 1$, are all contained in a finite union of subvarieties of $\text{End}(X)$.

We say that a k -rational point $x \in X$ is *periodic*, if x is fixed by some f^n .

Proposition 3. *Let f be a bounded endomorphism of a projective variety X .*

- (i) *There exists a smallest closed algebraic subgroup $G \subset \text{End}(X)$ such that $f^n \in G$ for all $n \gg 0$. Moreover, G is commutative.*
- (ii) *When k is algebraically closed, f has a periodic point if and only if G is linear. If X is normal, this is equivalent to the assertion that some positive power f^n acts on the Albanese variety of X via an idempotent.*

This result is proved in Subsection 3.3. As a direct consequence, every bounded endomorphism of a normal projective variety X has a periodic point, whenever the Albanese variety of X is trivial (e.g., when X is unirational); we do not know if any such endomorphism has a fixed point. In characteristic 0, it is known that every endomorphism (not necessarily bounded) of a smooth projective unirational variety X has a fixed point: this follows from the Woods Hole formula (see [1, Thm. 2], [18, Exp. III, Cor. 6.12]) in view of the vanishing of $H^i(X, \mathcal{O}_X)$ for all $i \geq 1$, proved e.g. in [20, Lem. 1].

Also, it would be interesting to extend the above results to endomorphism schemes of complete varieties. In this setting, the rigidity lemma of [5, §4.4] still hold. Yet the representability of the functor of morphisms by a scheme is unclear: the Hilbert functor of a complete variety is generally not represented by a scheme (see e.g. [14, Ex. 5.5.1]), but no such example seems to be known for morphisms.

2 Proofs of Theorem 1 and of Corollary 1

2.1 Proof of Theorem 1 in characteristic 0

We begin by setting notation and recalling a standard result of Galois descent, for any perfect field k .

We fix an algebraic closure \bar{k} of k , and denote by Γ the Galois group of \bar{k}/k . For any scheme X , we denote by $X_{\bar{k}}$ the \bar{k} -scheme obtained from X by

the base change $\mathrm{Spec}(\bar{k}) \rightarrow \mathrm{Spec}(k)$. Then $X_{\bar{k}}$ is equipped with an action of Γ such that the structure map $X_{\bar{k}} \rightarrow \mathrm{Spec}(\bar{k})$ is equivariant; moreover, the natural morphism $X_{\bar{k}} \rightarrow X$ may be viewed as the quotient by this action. The assignment $Y \mapsto Y_{\bar{k}}$ defines a bijective correspondence between closed subschemes of X and Γ -stable closed subschemes of $X_{\bar{k}}$.

Next, recall Chevalley's structure theorem: every connected algebraic group G has a largest closed connected linear normal subgroup L , and the quotient G/L is an abelian variety (see e.g. [4, Thm. 1.1.1] when $k = \bar{k}$; the general case follows by the above result of Galois descent).

We shall also need the existence of a normal projective equivariant compactification of G , in the following sense:

Lemma 1. *There exists a normal projective variety Y equipped with an action of $G \times G$ and containing an open orbit isomorphic to G , where $G \times G$ acts on G by left and right multiplication.*

Proof. When k is algebraically closed, this statement is [4, Prop. 3.1.1 (iv)]. For an arbitrary k , we adapt the argument of [loc. cit.].

If $G = L$ is linear, then we may identify it to a closed subgroup of some GL_n , and hence of PGL_{n+1} . The latter group has an equivariant compactification by the projectivization, $\mathbb{P}(M_{n+1})$, of the space of matrices of size $(n+1) \times (n+1)$, where $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{n+1}$ acts via the action of $\mathrm{GL}_{n+1} \times \mathrm{GL}_{n+1}$ on M_{n+1} by left and right matrix multiplication. Thus, we may take for Y the normalization of the closure of L in $\mathbb{P}(M_{n+1})$.

In the general case, choose a normal projective equivariant compactification Z of L and let

$$Y := G \times^L Z \longrightarrow G/L$$

be the fiber bundle associated with the principal L -bundle $G \rightarrow G/L$ and with the L -variety Z , where L acts on the left. Then Y is a normal projective variety, since so is L and hence L has an ample $L \times L$ -linearized line bundle. Moreover, Y is equipped with a G -action having an open orbit, $G \times^L L \cong G$.

We now extend this G -action to an action of $G \times G$, where the open G -orbit is identified to the $G \times G$ -homogeneous space $(G \times G)/\mathrm{diag}(G)$, and the original G -action, to the action of $G \times e_G$. For this, consider the scheme-theoretic center $Z(G)$ (resp. $Z(L)$) of G (resp. L). Then $Z(L) = Z(G) \cap L$, since $Z(L)_{\bar{k}} = Z(G)_{\bar{k}} \cap L_{\bar{k}}$ in view of [4, Prop. 3.1.1 (ii)]. Moreover, $G_{\bar{k}} = Z(G)_{\bar{k}} L_{\bar{k}}$ by [loc. cit.]; hence the natural map $Z(G)/Z(L) \rightarrow G/L$ is an isomorphism of group schemes. It follows that G/L is isomorphic to

$$(Z(G) \times Z(G))/(Z(L) \times Z(L))\mathrm{diag}(Z(G)) \cong (G \times G)/(L \times L)\mathrm{diag}(Z(G)).$$

Moreover, the $L \times L$ -action on Z extends to an action of $(L \times L)\mathrm{diag}(Z(G))$, where $Z(G)$ acts trivially: indeed, $(L \times L)\mathrm{diag}(Z(G))$ is isomorphic to

$$(L \times L \times Z(G))/(L \times L) \cap \mathrm{diag}(Z(G)) = (L \times L \times Z(G))/\mathrm{diag}(Z(L)),$$

and the subgroup scheme $\text{diag}(Z(L)) \subset L \times L$ acts trivially on Z by construction. This yields an isomorphism

$$G \times^L Z \cong (G \times G) \times^{(L \times L)\text{diag}(Z(G))} Z,$$

which provides the desired action of $G \times G$. \square

From now on in this subsection, we assume that $\text{char}(k) = 0$. We shall construct the desired variety X from the equivariant compactification Y , by proving a succession of lemmas.

Denote by $\text{Aut}^G(Y)$ the subgroup scheme of $\text{Aut}(Y)$ consisting of automorphisms which commute with the left G -action. Then the right G -action on Y yields a homomorphism

$$\varphi : G \longrightarrow \text{Aut}^G(Y).$$

Lemma 2. *With the above notation, φ is an isomorphism.*

Proof. Note that $\text{Aut}^G(Y)$ stabilizes the open orbit for the left G -action, and this orbit is isomorphic to G . This yields a homomorphism $\text{Aut}^G(Y) \rightarrow \text{Aut}^G(G)$. Moreover, $\text{Aut}^G(G) \cong G$ via the action of G on itself by right multiplication, and the resulting homomorphism $\psi : \text{Aut}^G(Y) \rightarrow G$ is readily seen to be inverse of φ . \square

Lemma 3. *There exists a finite subset $F \subset G(\bar{k})$ which generates a dense subgroup of $G_{\bar{k}}$.*

Proof. We may assume that $k = \bar{k}$. If the statement holds for some closed normal subgroup H of G and for the quotient group G/H , then it clearly holds for G . Thus, we may assume that G is simple, in the sense that it has no proper closed connected normal subgroup. Then, by Chevalley's structure theorem, G is either a linear algebraic group or an abelian variety. In the latter case, there exists $g \in G(k)$ of infinite order, and every such point generates a dense subgroup of G (actually, every abelian variety, not necessarily simple, is topologically generated by some k -rational point, see [9, Thm. 9]). In the former case, G is either the additive group \mathbb{G}_a , the multiplicative group \mathbb{G}_m , or a connected semisimple group. Therefore, G is generated by finitely many copies of \mathbb{G}_a and \mathbb{G}_m , each of which is topologically generated by some k -rational point (specifically, by any nonzero $t \in k$ for \mathbb{G}_a , and by any $u \in k^*$ of infinite order for \mathbb{G}_m). \square

Choose $F \subset G(\bar{k})$ as in Lemma 3. We may further assume that F contains id_Y and is stable under the action of the Galois group Γ ; then $F = E_{\bar{k}}$ for a unique finite reduced subscheme $E \subset G$. We have

$$\text{Aut}^F(Y_{\bar{k}}) = \text{Aut}^{G(\bar{k})}(Y_{\bar{k}}) = \text{Aut}^{G_{\bar{k}}}(Y_{\bar{k}})$$

and the latter is isomorphic to $G_{\bar{k}}$ via φ , in view of Lemma 2. Thus, φ yields an isomorphism $G \cong \text{Aut}^E(Y)$.

Next, we identify G with a subgroup of $\text{Aut}(Y \times Y)$ via the closed embedding of group schemes

$$\iota : \text{Aut}(Y) \longrightarrow \text{Aut}(Y \times Y), \quad \varphi \longmapsto \varphi \times \varphi.$$

For any $f \in F$, let $\Gamma_f \subset Y_{\bar{k}} \times Y_{\bar{k}}$ be the graph of f ; in particular, Γ_{id_Y} is the diagonal, $\text{diag}(Y_{\bar{k}})$. Then there exists a unique closed reduced subscheme $Z \subset Y \times Y$ such that $Z_{\bar{k}} = \bigcup_{f \in F} \Gamma_f$. We may now state the following key observation:

Lemma 4. *With the above notation, we have*

$$\iota(G) = \text{Aut}^o(Y \times Y, Z),$$

where the right-hand side denotes the neutral component of the stabilizer of Z in $\text{Aut}(Y \times Y)$.

Proof. We may assume that $k = \bar{k}$, so that $Z = \bigcup_{f \in F} \Gamma_f$. Moreover, by connectedness, $\text{Aut}^o(Y \times Y, Z)$ is the neutral component of the intersection $\bigcap_{f \in F} \text{Aut}(Y \times Y, \Gamma_f)$. On the other hand, $\text{Aut}^o(Y \times Y, Z) \subset \text{Aut}^o(Y \times Y)$, and the latter is isomorphic to $\text{Aut}^o(Y) \times \text{Aut}^o(Y)$ via the natural homomorphism

$$\text{Aut}^o(Y) \times \text{Aut}^o(Y) \longrightarrow \text{Aut}^o(Y \times Y), \quad (\varphi, \psi) \longmapsto \varphi \times \psi$$

(see [4, Cor. 4.2.7]). Also, $\varphi \times \psi$ stabilizes a graph Γ_f if and only if $\psi f = f \varphi$. In particular, $\varphi \times \psi$ stabilizes $\text{diag}(Y) = \Gamma_{\text{id}_Y}$ iff $\psi = \varphi$, and $\varphi \times \varphi$ stabilizes Γ_f iff φ commutes with f . As a consequence, $\text{Aut}^o(Y \times Y, Z)$ is the neutral component of $\iota(\text{Aut}^F(Y))$. Since $\text{Aut}^F(Y) = G$ is connected, this yields the assertion. \square

Next, denote by X the normalization of the blow-up of $Y \times Y$ along Z . Then X is a normal projective variety equipped with a birational morphism

$$\pi : X \longrightarrow Y \times Y$$

which induces a homomorphism of group schemes

$$\pi^* : G \longrightarrow \text{Aut}(X),$$

since Z is stable under the action of G on $Y \times Y$.

Lemma 5. *Keep the above notation and assume that $n \geq 2$. Then π^* yields an isomorphism of algebraic groups $G \xrightarrow{\cong} \text{Aut}^o(X)$.*

Proof. It suffices to show the assertion after base change to $\text{Spec}(\bar{k})$; thus, we may assume again that $k = \bar{k}$.

The morphism π is proper and birational, and $Y \times Y$ is normal (since normality is preserved under separable field extension). Thus, $\pi_*(\mathcal{O}_X) =$

$\mathcal{O}_{Y \times Y}$ by Zariski's Main Theorem. It follows that π induces a homomorphism of algebraic groups

$$\pi_* : \text{Aut}^o(X) \rightarrow \text{Aut}^o(Y \times Y)$$

(see e.g. [4, Cor. 4.2.6]). In particular, $\text{Aut}^o(X)$ preserves the fibers of π , and hence stabilizes the exceptional divisor E of that morphism; as a consequence, the image of π_* stabilizes $\pi(E)$. By connectedness, this image stabilizes every irreducible component of $\pi(E)$; but these components are exactly the graphs Γ_f , where $f \in F$ (since the codimension of any such graph in $Y \times Y$ is $\dim(Y) = n \geq 2$). Thus, the image of π_* is contained in $\iota(G)$: we may view π_* as a homomorphism $\text{Aut}^o(X) \rightarrow G$. Since π is birational, both maps π_* , π^* are injective and the composition $\pi_*\pi^*$ is the identity. It follows that π_* is inverse to π^* . \square

We may now complete the proof of Theorem 1 when $n \geq 2$. Let X be as in Lemma 5; then X admits an equivariant desingularization, i.e., there exists a smooth projective variety X' equipped with an action of G and with a G -equivariant birational morphism

$$f : X' \rightarrow X$$

(see [15, Prop. 3.9.1, Thm. 3.36]). We check that the resulting homomorphism of algebraic groups

$$f^* : G \rightarrow \text{Aut}^o(X')$$

is an isomorphism. For this, we may assume that $k = \bar{k}$; then again, [4, Cor. 4.2.6] yields a homomorphism of algebraic groups

$$f_* : \text{Aut}^o(X') \rightarrow \text{Aut}^o(X) = G$$

which is easily seen to be inverse of f^* .

On the other hand, if $n = 1$, then G is either an elliptic curve, or \mathbb{G}_a , or a k -form of \mathbb{G}_m . We now construct a smooth projective surface X such that $\text{Aut}^o(X) \cong G$, via case-by-case elementary arguments.

When G is an elliptic curve, we have $G \cong \text{Aut}^o(G)$ via the action of G by translations on itself. It follows that $G \cong \text{Aut}^o(G \times C)$, where C is any smooth projective curve of genus ≥ 2 .

When $G = \mathbb{G}_a$, we view G as the group of automorphisms of the projective line \mathbb{P}^1 that fix the point ∞ and the tangent line at that point, $T_\infty(\mathbb{P}^1)$. Choose $x \in \mathbb{P}^1(k)$ such that $0, x, \infty$ are all distinct, and let X be the smooth projective surface obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at the three points $(\infty, 0)$, (∞, x) , and (∞, ∞) . Arguing as in the proof of Lemma 5, one checks that $\text{Aut}^o(X)$ is isomorphic to the neutral component of the stabilizer of these three points, in $\text{Aut}^o(\mathbb{P}^1 \times \mathbb{P}^1) \cong \text{PGL}_2 \times \text{PGL}_2$. This identifies $\text{Aut}^o(X)$ with the stabilizer of ∞ in PGL_2 , i.e., with the automorphism group Aff_1 of the affine line, acting on the first copy of \mathbb{P}^1 . Thus, $\text{Aut}^o(X)$ acts on each

exceptional line via the natural action of Aff_1 on $\mathbb{P}(T_\infty(\mathbb{P}^1) \oplus k)$, with an obvious notation; this action factors through an action of $\mathbb{G}_m = \text{Aff}_1/\mathbb{G}_a$, isomorphic to the \mathbb{G}_m -action on \mathbb{P}^1 by multiplication. Let X' be the smooth projective surface obtained by blowing up X at a k -rational point of some exceptional line, distinct from 0 and ∞ ; then $\text{Aut}^o(X') \cong \mathbb{G}_a$.

Finally, when G is a k -form of \mathbb{G}_m , we consider the smooth projective curve C that contains G as a dense open subset; then C is a k -form of the projective line \mathbb{P}^1 on which \mathbb{G}_m acts by multiplication. Thus, the complement $P := C \setminus G$ is a point of degree 2 on C (a k -form of $\{0, \infty\}$); moreover, G is identified with the stabilizer of P in $\text{Aut}(C)$. Let X be the smooth projective surface obtained by blowing up $C \times C$ at $(P \times P) \cup (P \times e_G)$, where the neutral element e_G is viewed as a k -point of C . Arguing as in the proof of Lemma 5 again, one checks that $\text{Aut}^o(X)$ is isomorphic to the neutral component of the stabilizer of $(P \times P) \cup (P \times e_G)$ in $\text{Aut}^o(C \times C) \cong \text{Aut}^o(C) \times \text{Aut}^o(C)$, i.e., to G acting on the first copy of C . \square

Remark 1. One may ask for analogues of Theorem 1 for automorphism groups of compact complex spaces. Given any such space X , the group of biholomorphisms, $\text{Aut}(X)$, has the structure of a complex Lie group acting biholomorphically on X (see [8]). If X is Kähler, or more generally in Fujiki's class \mathcal{C} , then the neutral component $\text{Aut}^0(X) =: G$ has a meromorphic structure, i.e., a compactification G^* such that the multiplication $G \times G \rightarrow G$ extends to a meromorphic map $G^* \times G^* \rightarrow G^*$ which is holomorphic on $(G \times G^*) \cup (G^* \times G)$; moreover, G is Kähler and acts biholomorphically and meromorphically on X (see [10, Th. 5.5, Cor. 5.7]).

Conversely, every connected meromorphic Kähler group of dimension n is the connected automorphism group of some compact Kähler manifold of dimension $2n$; indeed, the above arguments adapt readily to that setting. But it seems to be unknown whether any connected complex Lie group can be realized as the connected automorphism group of some compact complex manifold.

2.2 Proof of Theorem 1 in prime characteristic

In this subsection, the base field k is assumed to be perfect, of characteristic $p > 0$. Let Y be an equivariant compactification of G as in Lemma 1. Consider the closed subgroup scheme $\text{Aut}^G(Y) \subset \text{Aut}(Y)$, defined as the centralizer of G acting on the left; then the G -action on the right still yields a homomorphism of group schemes $\varphi : G \rightarrow \text{Aut}^G(Y)$.

As in Lemma 2, φ is an isomorphism. To check this claim, note that φ induces an isomorphism $G(\bar{k}) \xrightarrow{\cong} \text{Aut}^G(Y)(\bar{k})$ by the argument of that lemma. Moreover, the induced homomorphism of Lie algebras

$$\text{Lie}(\varphi) : \text{Lie}(G) \longrightarrow \text{Lie } \text{Aut}^G(Y)$$

is an isomorphism as well: indeed, $\text{Lie}(\varphi)$ is identified with the natural map

$$\psi : \text{Lie}(G) \longrightarrow \text{Der}^G(\mathcal{O}_Y),$$

where $\text{Der}^G(\mathcal{O}_Y)$ denotes the Lie algebra of left G -invariant derivations of \mathcal{O}_Y . Furthermore, the restriction to the open dense subset G of Y yields an injective map

$$\eta : \text{Der}^G(\mathcal{O}_Y) \longrightarrow \text{Der}^G(\mathcal{O}_G) \cong \text{Lie}(G)$$

such that $\eta\psi = \text{id}$; thus, η is the inverse of ψ . It follows that $\text{Aut}^G(Y)$ is reduced; this completes the proof of the claim.

Next, *Lemma 3 fails in positive characteristics*, already for \mathbb{G}_a since every finite subset of \bar{k} generates a finite additive group; that lemma also fails for \mathbb{G}_m when \bar{k} is the algebraic closure of a finite field. Yet we have the following replacement:

Lemma 6. *With the above notation, there exists a finite subset F of $G(\bar{k})$ such that $\text{Aut}^{G(\bar{k})}(Y_{\bar{k}}) = \text{Aut}^{F,o}(Y_{\bar{k}})$, where the right-hand side denotes the neutral component of $\text{Aut}^F(Y_{\bar{k}})$.*

Proof. We may assume that $k = \bar{k}$; then $\text{Aut}^G(Y) = \text{Aut}^{G(k)}(Y)$. The subgroup schemes $\text{Aut}^{E,o}(Y)$, where E runs over the finite subsets of $G(k)$, form a family of closed subschemes of $\text{Aut}^o(Y)$. Thus, there exists a minimal such subgroup scheme, say, $\text{Aut}^{F,o}(Y)$. For any $g \in G(k)$, the subgroup scheme $\text{Aut}^{F \cup \{g\},o}(Y)$ is contained in $\text{Aut}^{F,o}(Y)$; thus, equality holds by minimality. In other words, $\text{Aut}^{F,o}(Y)$ centralizes g ; hence F satisfies the assertion. \square

It follows from Lemmas 2 and 6 that $\text{Aut}^{F,o}(Y_{\bar{k}}) \cong G_{\bar{k}}$ for some finite subset $F \subset G(\bar{k})$; we may assume again that F contains id_Y and is stable under the action of the Galois group Γ . Thus, $G \cong \text{Aut}^E(Y)$, where $E \subset G$ denotes the finite reduced subscheme such that $E_{\bar{k}} = F$.

Next, *Lemma 4 still holds* with the same proof, in view of [4, Cor. 4.2.7]. In other words, we may again identify G with the connected stabilizer in $\text{Aut}(Y \times Y)$ of the unique closed reduced subscheme $Z \subset Y \times Y$ such that $Z_{\bar{k}} = \bigcup_{f \in F} \Gamma_f$.

Consider again the morphism $\pi : X \rightarrow Y \times Y$ obtained as the normalization of the blow-up of Z . Then X is a normal projective variety, and π induces a homomorphism of group schemes

$$\pi^* : G \longrightarrow \text{Aut}^o(X).$$

Now the statement of Lemma 5 adapts as follows:

Lemma 7. *Keep the above notation and assume that $n \geq 2$. Then π^* yields an isomorphism of algebraic groups $G \xrightarrow{\cong} \text{Aut}^o(X)_{\text{red}}$.*

Proof. Using the fact that normality is preserved under separable field extension, we may assume that $k = \bar{k}$. By [4, Cor. 4.2.6] again, we have a homomorphism of group schemes

$$\pi_* : \mathrm{Aut}^o(X) \longrightarrow \mathrm{Aut}^o(Y \times Y)$$

and hence a homomorphism of algebraic groups

$$\pi_{*,\mathrm{red}} : \mathrm{Aut}^o(X)_{\mathrm{red}} \longrightarrow \mathrm{Aut}^o(Y \times Y)_{\mathrm{red}}.$$

Arguing as in the proof of Lemma 5, one checks that $\pi_{*,\mathrm{red}}$ maps $\mathrm{Aut}^o(X)_{\mathrm{red}}$ onto $\iota(G)$, and is injective on k -rational points. Also, the homomorphism of Lie algebras $\mathrm{Lie}(\pi_{*,\mathrm{red}})$ is injective, as it extends to a homomorphism

$$\mathrm{Lie}(\pi_*) : \mathrm{Lie} \mathrm{Aut}^o(X) = \mathrm{Der}(\mathcal{O}_X) \longrightarrow \mathrm{Der}(\mathcal{O}_{Y \times Y}) = \mathrm{Lie} \mathrm{Aut}^o(Y \times Y)$$

which is injective, since π is birational. Thus, we obtain an isomorphism $\pi_{*,\mathrm{red}} : \mathrm{Aut}^o(X)_{\mathrm{red}} \xrightarrow{\cong} \iota(G)$ which is the inverse of π^* . \square

To realize G as a connected automorphism group scheme, we now prove:

Lemma 8. *With the above notation, the homomorphism of Lie algebras*

$$\mathrm{Lie}(\pi^*) : \mathrm{Lie}(G) \longrightarrow \mathrm{Der}(\mathcal{O}_X)$$

is an isomorphism if $n \geq 2$ and $n - 1$ is not a multiple of p .

Proof. We may assume again that $k = \bar{k}$. Since π is birational, both maps $\mathrm{Lie}(\pi^*)$ and $\mathrm{Lie}(\pi_*)$ are injective and the composition $\mathrm{Lie}(\pi_*) \mathrm{Lie}(\pi^*)$ is the identity of $\mathrm{Lie}(G)$. Thus, it suffices to show that the image of $\mathrm{Lie}(\pi_*)$ is contained in $\mathrm{Lie}(G)$. For this, we use the natural action of $\mathrm{Der}(\mathcal{O}_X)$ on the jacobian ideal of π , defined as follows. Consider the sheaf Ω_X^1 of Kähler differentials on X . Recall that $\Omega_X^1 \cong \mathcal{I}_{\mathrm{diag}(X)} / \mathcal{I}_{\mathrm{diag}(X)}^2$ with an obvious notation; thus, Ω_X^1 is equipped with an $\mathrm{Aut}(X)$ -linearization (see [17, Exp. I, §6] for background on linearized sheaves, also called equivariant). Likewise, $\Omega_{Y \times Y}^1$ is equipped with an $\mathrm{Aut}(Y \times Y)$ -linearization, and hence with an $\mathrm{Aut}^o(X)$ -linearization via the homomorphism π_* . Moreover, the natural map $\pi^*(\Omega_{Y \times Y}^1) \rightarrow \Omega_X^1$ is a morphism of $\mathrm{Aut}^o(X)$ -linearized sheaves, since it arises from the inclusion $\pi^{-1}(\mathcal{I}_{\mathrm{diag}(Y \times Y)}) \subset \mathcal{I}_{\mathrm{diag}(X)}$. This yields a morphism of $\mathrm{Aut}^o(X)$ -linearized sheaves

$$\pi^*(\Omega_{Y \times Y}^{2n}) \longrightarrow \Omega_X^{2n}.$$

Since the composition

$$\Omega_X^{2n} \times \mathrm{Hom}(\Omega_X^{2n}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

is also a morphism of linearized sheaves, we obtain a morphism of linearized sheaves

$$\mathrm{Hom}(\Omega_X^{2n}, \pi^*(\Omega_{Y \times Y}^{2n})) \longrightarrow \mathcal{O}_X$$

with image the jacobian ideal \mathcal{I}_π . Thus, \mathcal{I}_π is equipped with an $\text{Aut}^\circ(X)$ -linearization. In particular, for any open subset U of X , the Lie algebra $\text{Der}(\mathcal{O}_X)$ acts on $\mathcal{O}(U)$ by derivations that stabilize $\Gamma(U, \mathcal{I}_\pi)$.

We now take $U = \pi^{-1}(V)$, where V denotes the open subset of $Y \times Y$ consisting of those smooth points that belong to at most one of the graphs Γ_f . Then the restriction

$$\pi_U : U \longrightarrow V$$

is the blow-up of the smooth variety V along a closed subscheme W , the disjoint union of smooth subvarieties of codimension n . Thus, $\mathcal{I}_{\pi_U} = \mathcal{O}_U(-(n-1)E)$, where E denotes the exceptional divisor of π_U . Hence we obtain an injective map

$$\text{Der}(\mathcal{O}_X) = \text{Der}(\mathcal{O}_X, \mathcal{I}_\pi) \longrightarrow \text{Der}(\mathcal{O}_U, \mathcal{O}_U(-(n-1)E)),$$

with an obvious notation. Since $n-1$ is not a multiple of p , we have

$$\text{Der}(\mathcal{O}_U, \mathcal{O}_U(-(n-1)E)) = \text{Der}(\mathcal{O}_U, \mathcal{O}_U(-E)).$$

(Indeed, if $D \in \text{Der}(\mathcal{O}_U, \mathcal{O}_U(-(n-1)E))$ and z is a local generator of $\mathcal{O}_U(-E)$ at $x \in X$, then $z^{n-1}\mathcal{O}_{X,x}$ contains $D(z^{n-1}) = (n-1)z^{n-2}D(z)$, and hence $D(z) \in z\mathcal{O}_{X,x}$). Also, the natural map

$$\text{Der}(\mathcal{O}_U) \longrightarrow \text{Der}(\pi_{U,*}(\mathcal{O}_U)) = \text{Der}(\mathcal{O}_V)$$

is injective and sends $\text{Der}(\mathcal{O}_U, \mathcal{O}_U(-E))$ to $\text{Der}(\mathcal{O}_V, \pi_{U,*}(\mathcal{O}_U(-E)))$. Moreover, $\pi_{U,*}(\mathcal{O}_U(-E))$ is the ideal sheaf of W , and hence is stable under $\text{Der}(\mathcal{O}_X)$ acting via the composition

$$\text{Der}(\mathcal{O}_X) \longrightarrow \text{Der}(\pi_*(\mathcal{O}_X)) = \text{Der}(\mathcal{O}_{Y \times Y}) \longrightarrow \text{Der}(\mathcal{O}_V).$$

It follows that the image of $\text{Lie}(\pi_*)$ stabilizes the ideal sheaf of the closure of W in $Y \times Y$, i.e., of the union of the graphs Γ_f . In view of Lemma 4, we conclude that $\text{Lie}(\pi_*)$ sends $\text{Der}(\mathcal{O}_X)$ to $\text{Lie}(G)$. \square

Lemmas 7 and 8 yield an isomorphism $G \cong \text{Aut}^\circ(X)$ when $n \geq 2$ and p does not divide $n-1$. Next, when $n \geq 2$ and p divides $n-1$, we choose a smooth projective curve C of genus $g \geq 2$, and consider $Y' := Y \times C$. This is a normal projective variety of dimension $n+1$, equipped with an action of $G \times G$. Moreover, we have isomorphisms

$$\text{Aut}^\circ(Y) \xrightarrow{\cong} \text{Aut}^\circ(Y) \times \text{Aut}^\circ(C) \xrightarrow{\cong} \text{Aut}^\circ(Y'), \quad \varphi \longmapsto \varphi \times \text{id}_C$$

(where the second isomorphism follows again from [4, Cor. 4.2.6]); this identifies $G \cong \text{Aut}^{G,\circ}(Y)$ with $\text{Aut}^{G,\circ}(Y')$. We may thus replace everywhere Y with Y' in the above arguments, to obtain a normal projective variety X' of dimension $2n+2$ such that $\text{Aut}^0(X') \cong G$.

Finally, if $n = 1$ then G is again an elliptic curve, or \mathbb{G}_a , or a k -form of \mathbb{G}_m (since every form of \mathbb{G}_a over a perfect field is trivial). It follows that $G \cong \text{Aut}^o(X)$ for some smooth projective surface X , constructed as at the end of Subsection 2.1. \square

Remark 2. If G is linear, then there exists a normal projective *unirational* variety X such that $\text{Aut}^o(X)_{\text{red}} \cong G$ and $\dim(X) = 2n$. Indeed, G itself is unirational (see [17, Exp. XIV, Cor. 6.10]), and hence so is the variety X considered in the above proof when $n \geq 2$; on the other hand, when $n = 1$, the above proof yields a smooth projective rational surface X such that $\text{Aut}^o(X) \cong G$. If in addition k is algebraically closed, then G is rational; hence we may further choose X rational.

Conversely, if X is a normal projective variety having a trivial Albanese variety (e.g., X is unirational), then $\text{Aut}^o(X)$ is linear. Indeed, the Albanese variety of $\text{Aut}^o(X)_{\text{red}}$ is trivial in view of [2, Thm. 2]. Thus, $\text{Aut}^o(X)_{\text{red}}$ is affine by Chevalley's structure theorem. It follows that $\text{Aut}^o(X)$ is affine, or equivalently linear.

Returning to a connected linear algebraic group G , the above proof adapts to show that there exists a normal projective unirational variety X such that $\text{Aut}^o(X) \cong G$: in the argument after Lemma 8, it suffices to replace the curve C with a normal projective rational variety Z such that $\text{Aut}^o(Z)$ is trivial and $\dim(Z) \geq 2$ is not a multiple of p . Such a variety may be obtained by blowing up \mathbb{P}^2 at 4 points in general position when $p \geq 3$; if $p = 2$, then we blow up \mathbb{P}^3 along a smooth curve which is neither rational, nor contained in a plane.

Remark 3. It is tempting to generalize the above proof to the setting of an arbitrary base field k . Yet this raises many technical difficulties; for instance, Chevalley's structure theorem fails over any imperfect field (see [17, Exp. XVII, App. III, Prop. 5.1], and [21] for a remedy). Also, normal varieties need not be geometrically normal, and hence the differential argument of Lemma 8 also fails in that setting.

2.3 Proof of Corollary 1

(i) \Rightarrow (ii) Let $G := \text{Aut}^o(X)$. Recall from [16, Lem. 3.4, Thm. 3.7] that G is a connected algebraic group with Lie algebra \mathfrak{g} . Also, recall that

$$G_{\bar{k}} = Z(G)_{\bar{k}} L_{\bar{k}},$$

where $Z(G)$ denotes the center of G , and L the largest closed connected normal linear subgroup of G . As a consequence, $\mathfrak{g}_{\bar{k}} = \text{Lie}(Z(G))_{\bar{k}} + \text{Lie}(L)_{\bar{k}}$. It follows that $\mathfrak{g} = \text{Lie}(Z(G)) + \text{Lie}(L)$, and hence we may choose a subspace $V \subset \text{Lie}(Z(G))$ such that

$$\mathfrak{g} = V \oplus \text{Lie}(L)$$

as vector spaces. This decomposition also holds as Lie algebras, since $[V, V] = 0 = [V, \text{Lie}(L)]$. Hence $\mathfrak{g} = \text{Lie}(U \times L)$, where U is the (commutative, connected) unipotent algebraic group with Lie algebra V .

(ii) \Rightarrow (i) Let G be a connected linear algebraic group such that $\mathfrak{g} = \text{Lie}(G)$. By Theorem 1 and Remark 2, there exists a smooth projective unirational variety X of dimension $2n$ such that $G \cong \text{Aut}^o(X)$; when k is algebraically closed, we may further choose X rational. Then of course $\mathfrak{g} \cong \text{Der}(\mathcal{O}_X)$. \square

3 Proofs of the statements about endomorphisms

3.1 Proof of Proposition 1

(i) Since C is connected and has a k -rational point, it is geometrically connected in view of [17, Exp. VIB, Lem. 2.1.2]. Likewise, the connected component of i in $\text{Hom}(Y, X)$ is geometrically connected. To show the first assertion, we may thus assume that k is algebraically closed. But then that assertion follows from [5, Prop. 4.4.2, Rem. 4.4.3].

The scheme-theoretic image of $C \times C$ under the morphism

$$\text{End}(X) \times \text{End}(X) \longrightarrow \text{End}(X), \quad (f, g) \longmapsto gf$$

is connected and contains $e^2 = e$; thus, this image is contained in C . Therefore, C is a subsemigroup scheme of $\text{End}(X)$. Also, every $g \in \text{Hom}(Y, X)$ satisfies $gre = grir = gr$. Thus, $f = fe$ for any $f \in C$.

(ii) Since $(if_1r)(if_2r) = if_1f_2r$ for all $f_1, f_2 \in \text{End}(Y)$, we see that $\lambda_i\rho_r$ is a homomorphism of semigroup schemes which sends id_Y to e . Also, $eifr = irifir = ifr$ for all $f \in \text{End}(Y)$, so that $\lambda_i\rho_r$ sends $\text{End}(Y)$ to $e\text{End}(X)$. Since Y is a projective variety, $\text{Aut}^o(Y)$ is the connected component of i in $\text{End}(Y)$, and hence is sent by $\lambda_i\rho_r$ to $C \cap e\text{End}(X) = eC$.

To show that $\lambda_i\rho_r$ is an isomorphism, note that $eC = eCe = irCir$ by (i). Moreover, the morphism

$$\lambda_r\rho_i : \text{End}(X) \longrightarrow \text{End}(Y), \quad f \longmapsto rfi$$

sends e to id_Y , and hence C to $\text{Aut}^o(Y)$. Finally, $\lambda_r\rho_i(\lambda_i\rho_r(f)) = r(ifr)i = f$ for all $f \in \text{End}(Y)$, and $\lambda_i\rho_r(\lambda_r\rho_i(f)) = i(rfi)r = efe = f$ for all $f \in eC$. Thus, $\lambda_r\rho_i$ is the desired inverse.

(iii) Let $f \in C$ such that $f^2 = f$. Then $fef = f$ by (i), and hence $ef \in eC$ is idempotent. But eC is a group scheme by (ii); thus, $ef = e$. Write $f = gr$, where g is a point of the connected component of i in $\text{Hom}(Y, X)$. Then $egr = e$ and hence $rgr = r$, so that $rg = \text{id}_Y$. Conversely, if $g \in \text{Sec}(r)$, then

gr is idempotent as already noted. This shows the first assertion. For the second assertion, just note that $(g_1r)(g_2r) = g_1r$ for all $g_1, g_2 \in \text{Sec}(r)$.

(iv) We have with an obvious notation $\varphi(f_1, g_1)\varphi(f_2, g_2) = f_1g_1f_2g_2 = f_1g_1ef_2g_2$ by (i). Since $ef_2 = e$ by (iv), it follows that $\varphi(f_1, g_1)\varphi(f_2, g_2) = f_1g_1eg_2 = f_1g_1g_2$. Thus, φ is a homomorphism of semigroup schemes.

We now construct the inverse of φ . Let $f \in C$; then $ef \in eC$ has a unique inverse, $(ef)^{-1}$, in eC . Moreover, $f = fe = f(ef)^{-1}ef$ and $f(ef)^{-1}$ is idempotent, since

$$f(ef)^{-1}f(ef)^{-1} = f(ef)^{-1}ef(ef)^{-1} = f(ef)^{-1}e = f(ef)^{-1}.$$

We may thus define a morphism

$$\psi : C \longrightarrow E(C) \times eC, \quad f \longmapsto (f(ef)^{-1}, ef).$$

Then $\varphi\psi(f) = f(ef)^{-1}ef = fe = f$ for all $f \in C$, and $\psi\varphi(f, g) = (fg(efg)^{-1}, efg) = (fgg^{-1}, eg) = (f, g)$ for all $f \in E(C)$ and $g \in eC$. Thus, ψ is the desired inverse. \square

3.2 Proof of Proposition 2

(i) Consider the connected component C of $\text{End}(X)$ that contains S . Then C is of finite type, and hence so is S . Choose a k -rational point f of S and denote by $\langle f \rangle$ the smallest closed subscheme of S containing all the powers f^n , where $n \geq 1$. Then $\langle f \rangle$ is a reduced commutative subsemigroup scheme of S . By the main result of [3], it follows that $\langle f \rangle$ has an idempotent k -rational point. In particular, $E(S)$ has a k -rational point.

Since $E(S) \subset E(C)$, we have $f_1f_2 = f_1$ for any $f_1, f_2 \in E(S)$, by Proposition 1. It remains to show that $E(S)$ is connected; this will follow from (ii) in view of the connectedness of S .

(ii) By Proposition 1 again, φ yields an isomorphism $E(C) \times eC \xrightarrow{\cong} C$. Moreover, $fe = f$ for all $f \in C$, and $eC = eCe$ is a group scheme. Thus, $eS = eSe$ is a submonoid scheme of eC , and hence a closed subgroup scheme by Lemma 9 below. In other words, ef is invertible in eS for any $f \in S$. One may now check as in the proof of Proposition 1 (iv) that the morphism

$$\psi : S \longrightarrow E(S) \times eS, \quad f \longmapsto (f(ef)^{-1}, ef)$$

yields an isomorphism of semigroup schemes, with inverse φ .

(iii) Since φ is an isomorphism of semigroup schemes, π is a homomorphism of such schemes. Moreover, $\pi(f) = f(ef)^{-1}$ for all $f \in S$, since ψ is the inverse of φ . If $f \in E(S)$, then $f = f^2 = fef$ and hence $\pi(f) = fef(ef)^{-1} = fe = f$. Thus, π is a retraction.

Let $\rho : S \rightarrow E(S)$ be a retraction of semigroup schemes. For any $f \in S$, we have $\rho(f) = \rho(f(ef)^{-1}ef) = \rho(f(ef)^{-1})\rho(ef)$. Moreover, $\rho(f(ef)^{-1}) = f(ef)^{-1}$, since $f(ef)^{-1} \in E(S)$; also, $\rho(ef) = \rho(ef)\rho((ef)^{-1}) = \rho(e) = e$. Hence $\rho(f) = f(ef)^{-1}e = f(ef)^{-1} = \pi(f)$. \square

Lemma 9. *Let G be a group scheme of finite type, and $S \subset G$ a subsemigroup scheme. Then S is a closed subgroup scheme.*

Proof. We have to prove that S is closed, and stable under the automorphism $g \mapsto g^{-1}$ of G . It suffices to check these assertions after base extension to any larger field; hence we may assume that k is algebraically closed.

Arguing as at the beginning of the proof of Proposition 2 (i), we see that S has an idempotent k -rational point; hence S contains the neutral element, e_G . In other words, S is a submonoid scheme of G . By Lemma 10 below, there exists an open subgroup scheme $G(S) \subset S$ which represents the invertibles in S . In particular, $G(S)_{\text{red}}$ is the unit group of the algebraic monoid S_{red} . Since that monoid has a unique idempotent, it is an algebraic group by [5, Prop. 2.2.5]. In other words, we have $G(S)_{\text{red}} = S_{\text{red}}$. As $G(S)$ is open in S , it follows that $G(S) = S$. Thus, S is a subgroup scheme of G , and hence is closed by [17, Exp. VIA, Cor. 0.5.2]. \square

To complete the proof, it remains to show the following result of independent interest:

Lemma 10. *Let M be a monoid scheme of finite type. Then the group functor of invertibles of M is represented by a group scheme $G(M)$, open in M .*

Proof. We first adapt the proof of the corresponding statement for (reduced) algebraic monoids (see [5, Thm. 2.2.4]). Denote for simplicity the composition law of M by $(x, y) \mapsto xy$, and the neutral element by 1. Consider the closed subscheme $G \subset M \times M$ defined in set-theoretic notation by

$$G = \{(x, y) \in M \times M \mid xy = yx = 1\}.$$

Then G is a subgroup scheme of the monoid scheme $M \times M^{\text{op}}$, where M^{op} denotes the opposite monoid, that is, the scheme M equipped with the composition law $(x, y) \mapsto yx$. Moreover, the first projection

$$p : G \longrightarrow M$$

is a homomorphism of monoid schemes, which sends the T -valued points of G isomorphically to the T -valued invertible points of M for any scheme T . It follows that the group scheme G represents the group functor of invertibles in M .

To complete the proof, it suffices to check that p is an open immersion; for this, we may again assume that k is algebraically closed. Clearly, p is universally injective; we now show that it is étale. Since that condition defines an

open subscheme of G , stable under the action of $G(k)$ by left multiplication, we only need to check that p is étale at the neutral element 1 of G . For this, the argument of [loc. cit.] does not adapt readily, and we shall rather consider the formal completion of M at 1,

$$N := \mathrm{Spf}(\widehat{\mathcal{O}}_{M,1}).$$

Then N is a formal scheme having a unique point; moreover, N has a structure of formal monoid scheme, defined as follows. The composition law $\mu : M \times M \rightarrow M$ sends $(1, 1)$ to 1, and hence yields a homomorphism of local rings $\mu^\# : \mathcal{O}_{M,1} \rightarrow \mathcal{O}_{M \times M, (1,1)}$. In turn, $\mu^\#$ yields a homomorphism of completed local rings

$$\Delta : \widehat{\mathcal{O}}_{M,1} \longrightarrow \widehat{\mathcal{O}}_{M \times M, (1,1)} = \widehat{\mathcal{O}}_{M,1} \widehat{\otimes} \widehat{\mathcal{O}}_{M,1}.$$

We also have the homomorphism

$$\varepsilon : \widehat{\mathcal{O}}_{M,1} \longrightarrow k$$

associated with 1. One readily checks that Δ and ε satisfy conditions (i) (co-associativity) and (ii) (co-unit) of [17, Exp. VIIB, 2.1]; hence they define a formal monoid scheme structure on N . In view of [loc. cit., 2.7. Prop.], it follows that N is in fact a group scheme. As a consequence, p is an isomorphism after localization and completion at 1; in other words, p is étale at 1. \square

Remark 4. Proposition 2 gives back part of the description of all algebraic semigroup structures on a projective variety X , obtained in [5, Thm. 4.3.1].

Specifically, every such structure $\mu : X \times X \rightarrow X$, $(x, y) \mapsto xy$ yields a homomorphism of semigroup schemes $\lambda : X \rightarrow \mathrm{End}(X)$, $x \mapsto (y \mapsto xy)$ (the “left regular representation”). Thus, $S := \lambda(X)$ is a closed subsemigroup scheme of $\mathrm{End}(X)$. Choose an idempotent $e \in X(k)$. In view of Proposition 2, we have $\lambda(x)\lambda(e) = \lambda(x)$ for all $x \in X$; moreover, $\lambda(e)\lambda(x)$ is invertible in $\lambda(e)S$. It follows that $xy = ye$ for all $x, y \in X$. Moreover, for any $x \in X$, there exists $y \in eX$ such that $xy = ey = e$ for all $x \in X$. In particular, $(ex)(ey) = (ey)(ex) = e$, and hence eXe is an algebraic group.

These results are the main ingredients in the proof of [5, Thm. 4.3.1]. They are deduced there from the classical rigidity lemma, while the proof of Proposition 2 relies on a generalization of that lemma.

Remark 5. If k is not algebraically closed, then connected semigroup schemes of endomorphisms may well have no k -rational point. For example, let X be a projective variety having no k -rational point; then the subsemigroup scheme $S \subset \mathrm{End}(X)$ consisting of constant endomorphisms (i.e., of those endomorphisms that factor through the inclusion of a closed point in X) is isomorphic to X itself, equipped with the composition law $(x, y) \mapsto y$. Thus, S has no k -rational point either.

Yet Proposition 2 can be extended to any geometrically connected subsemigroup scheme $S \subset \text{End}(X)$, not necessarily having a k -rational point. Specifically, $E(S)$ is a nonempty, geometrically connected subsemigroup scheme, with semigroup law given by $f_1 f_2 = f_1$. Moreover, there exists a unique retraction of semigroup schemes

$$\pi : S \longrightarrow E(S);$$

it assigns to any point $f \in S$, the unique idempotent $e \in E(S)$ such that $ef = f$. Finally, the above morphism π defines a structure of $E(S)$ -monoid scheme on S , with composition law induced by that of S , and with neutral section the inclusion

$$\iota : E(S) \longrightarrow S.$$

In fact, this monoid scheme is a group scheme: consider indeed the closed subscheme $T \subset S \times S$ defined in set-theoretic notation by

$$T = \{(x, y) \in S \times S \mid xy = yx, x^2y = x, xy^2 = y\},$$

and the morphism

$$\rho : T \longrightarrow S, \quad (x, y) \longmapsto xy.$$

Then one may check that ρ is a retraction from T to $E(S)$, with section

$$\varepsilon : E(S) \longrightarrow T, \quad x \longmapsto (x, x).$$

Moreover, T is a group scheme over $E(S)$ via ρ , with composition law given by $(x, y)(x', y') := (xx', y'y)$, neutral section ε , and inverse given by $(x, y)^{-1} := (y, x)$. Also, the first projection

$$p_1 : T \longrightarrow S, \quad (x, y) \longrightarrow x$$

is an isomorphism which identifies ρ with π ; furthermore, p_1 is an isomorphism of monoid schemes. This yields the desired group scheme structure.

When k is algebraically closed, all these assertions are easily deduced from the structure of S obtained in Proposition 2; the case of an arbitrary field follows by descent.

3.3 Proof of Proposition 3

(i) can be deduced from the results of [3]; we provide a self-contained proof by adapting some of the arguments from [loc. cit.].

As in the beginning of the proof of Proposition 2, we denote by $\langle f \rangle$ the smallest closed subscheme of $\text{End}(X)$ containing all the powers f^n , where $n \geq 1$. In view of the boundedness assumption, $\langle f \rangle$ is an algebraic subsemi-

group; clearly, it is also commutative. The subsemigroups $\langle f^m \rangle$, where $m \geq 1$, form a family of closed subschemes of $\langle f \rangle$; hence there exists a minimal such subsemigroup, $\langle f^{n_0} \rangle$. Since $\langle f^m \rangle \cap \langle f^n \rangle \supset \langle f^{mn} \rangle$, we see that $\langle f^{n_0} \rangle$ is the smallest such subsemigroup.

The connected components of $\langle f^{n_0} \rangle$ form a finite set F , equipped with a semigroup structure such that the natural map $\varphi : \langle f^{n_0} \rangle \rightarrow F$ is a homomorphism of semigroups. In particular, the finite semigroup F is generated by $\varphi(f^{n_0})$. It follows readily that F has a unique idempotent, say $\varphi(f^{n_0 n})$. Then the fiber $\varphi^{-1}\varphi(f^{n_0 n})$ is a closed connected subsemigroup of $\langle f^{n_0} \rangle$, and contains $\langle f^{n_0 n} \rangle$. By the minimality assumption, we must have $\langle f^{n_0 n} \rangle = \varphi^{-1}\varphi(f^{n_0 n}) = \langle f^{n_0} \rangle$. As a consequence, $\langle f^{n_0} \rangle$ is connected.

Also, recall that $\langle f^{n_0} \rangle$ is commutative. In view of Proposition 2, it follows that this algebraic semigroup is in fact a group. In particular, $\langle f^{n_0} \rangle$ contains a unique idempotent, say e . Therefore, e is also the unique idempotent of $\langle f \rangle$: indeed, if $g \in \langle f \rangle$ is idempotent, then $g = g^{n_0} \in \langle f^{n_0} \rangle$, and hence $g = e$.

Thus, $e\langle f \rangle = \langle ef \rangle$ is a closed submonoid of $\langle f \rangle$ with neutral element e and no other idempotent. In view of [5, Prop. 2.2.5], it follows that $e\langle f \rangle$ is a group, say, G . Moreover, $f^{n_0} = ef^{n_0} \in e\langle f \rangle$, and hence $f^n \in G$ for all $n \geq n_0$. On the other hand, if H is a closed subgroup of $\text{End}(X)$ and n_1 is a positive integer such that $f^n \in H$ for all $n \geq n_1$, then H contains $\langle f^{n_1} \rangle$, and hence $\langle f^{n_0} \rangle$ by minimality. In particular, the neutral element of H is e . Let g denote the inverse of f^{n_1} in H ; then H contains $f^{n_1+1}g = ef$, and hence $G \subset H$. Thus, G satisfies the assertion.

(ii) Assume that $f^n(x) = x$ for some $n \geq 1$ and some $x \in X(k)$. Replacing n with a large multiple, we may assume that $f^n \in G$. Let $Y := e(X)$, where e is the neutral element of G as above, and let $y := e(x)$. Then Y is a closed subvariety of X , stable by f and hence by G ; moreover, G acts on Y by automorphisms. Also, $y \in Y$ is fixed by f^n . Since $f^n = (ef)^n$, it follows that the $(ef)^m(y)$, where $m \geq 1$, form a finite set. As the positive powers of ef are dense in G , the G -orbit of y must be finite. Thus, y is fixed by the neutral component G^o . In view of [4, Prop. 2.1.6], it follows that G^o is linear; hence so is G .

Conversely, if G is linear, then G^o is a connected linear commutative algebraic group, and hence fixes some point $y \in Y(k)$ by Borel's fixed point theorem. Then y is periodic for f .

Next, assume that X is normal; then so is Y by Lemma 11 below. In view of [2, Thm. 2], it follows that G^o acts on the Albanese variety $A(Y)$ via a finite quotient of its own Albanese variety, $A(G^o)$. In particular, G is linear if and only if G^o acts trivially on $A(Y)$. Also, note that $A(Y)$ is isomorphic to a summand of the abelian variety $A(X)$: the image of the idempotent $A(e)$ induced by e . If G^o acts trivially on $A(Y)$, then it acts on $A(X)$ via $A(e)$, since $G^o = G^o e$. Thus, some positive power f^n acts on $A(X)$ via $A(e)$ as well. Conversely, if f^n acts on $A(X)$ via some idempotent g , then we may assume that $f^n \in G^o$ by taking a large power. Thus, $f^n = ef^n = f^n e$ and hence $g = A(e)g = gA(e)$; in other words, g acts on $A(X)$ as an idempotent of the

summand $A(Y)$. On the other hand, $g = A(f^n)$ yields an automorphism of $A(Y)$; it follows that $g = A(e)$.

Lemma 11. *Let X be a normal variety, and $r : X \rightarrow Y$ a retraction. Then Y is a normal variety as well.*

Proof. Consider the normalization map, $\nu : \tilde{Y} \rightarrow Y$. By the universal property of ν , there exists a unique morphism $\tilde{r} : X \rightarrow \tilde{Y}$ such that $r = \nu\tilde{r}$. Since r has a section, so has ν . As \tilde{Y} is a variety and ν is finite, it follows that ν is an isomorphism. \square

Remark 6. With the notation of the proof of (i), the group G is the closure of the subgroup generated by ef . Hence G is *monothetic* in the sense of [9], which obtains a complete description of this class of algebraic groups. Examples of monothetic algebraic groups include all the semiabelian varieties, except when k is the algebraic closure of a finite field (then the monothetic algebraic groups are exactly the finite cyclic groups).

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