DIFFUSIVE LIMIT OF THE BOLTZMANN EQUATION WITH FLUID INITIAL LAYER IN THE PERIODIC DOMAIN

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ABSTRACT. We justify the global-in-time diffusive limit of the Boltzmann equation inside a periodic domain $T^3$. We only assume that in the initial expansion (1.2) at $t = 0$, the kinetic parts are well-prepared, but the fluid parts could be general, i.e. the fluids parts are not required to satisfy the incompressibility and Boussinesq relations. For this case, the fluid initial layers are created and preserved in the periodic domain. Employing the method of time-averaging, we analyze the propagation of the initial layers, and thus generalize Y. Guo’s results in [15] in which both fluid and kinetic parts are required to be well-prepared.

1. Introduction

We study the diffusive limit of the Boltzmann equation in the periodic domain $T^D$, where $D \geq 2$ is the spatial dimension. We will work with the nondimensionalized form of the Boltzmann equation

\begin{equation}
\Ma \partial_t F^\epsilon + v \cdot \nabla_x F^\epsilon = \frac{1}{\epsilon} B(F^\epsilon, F^\epsilon),
\end{equation}

where $F^\epsilon(t, x, v) \geq 0$ is the number density of particles at $t \geq 0$, with velocity $v \in \mathbb{R}^D$ and position $x \in T^D$. The dimensionless numbers $\Ma$ and $\epsilon$ denote the Mach number and the Knudsen number respectively.

The fluid dynamical regimes are those where the mean free path is small compared to the macroscopic length scales, i.e. where the Knudsen number $\epsilon$ is small. In the current paper, we specifically consider the diffusive limit of the Boltzmann equation. By the von Kármán relation $\epsilon = \frac{\Ma}{\Re}$, where $\Re$ is the Reynold number, to derive the fluid equations with diffusion term, the Reynold number $\Re$ should be kept as order $O(1)$. This implies that the Mach number $\Ma$ in (1.1) should be taken as $\epsilon$, i.e. the time scale considered should be $O(\frac{1}{\epsilon})$. In other words, under this scaling (which we call the diffusive scaling), as $\epsilon \to 0$, the fluids models formally derived from the Boltzmann equation (1.1) should be incompressible, such as incompressible Navier-Stokes-Fourier equations, incompressible Stokes-Fourier equations, etc. (see [1].)

The justification of the fluid limits from the Boltzmann equations is a mathematically challenging problem which has been worked over two decades. Basically, the fluid limits can be justified in two contexts of the solutions to the Boltzmann equations: the classical solutions and weak solutions, more specifically, the renormalized DiPerna-Lions solutions. Among many works, we only list here for the first approach (classical solutions) [3, 7, 15], and for the second approach (DiPerna-Lions solutions) [2, 9, 10, 19, 24].

We concern the classical solutions approach in this paper. In this direction, we particularly mention that in [3], the incompressible Navier-Stokes equation was derived and justified from Boltzmann equation with general initial data but small amplitude on the whole space, by using the semigroup generated by the linearized Boltzmann operator.

In [15], using the nonlinear energy method developed in [12, 13], the diffusive expansion around the absolute Maxwellian was justified. This expansion is beyond Navier-Stokes limit in the sense made more precise in the following: the solutions of the re-scaled Boltzmann
equation (1.1) was expanded as

\[(1.2) \quad \mathcal{F}^\epsilon = \mu + \mu^{1/2} f^\epsilon = \mu + \mu^{1/2}\{\epsilon f_1 + \epsilon^2 f_2 + \ldots + \epsilon^{n-1} f_{n-1} + \epsilon^n f_n\},\]

where $\mu$ is the absolute Maxwellian

\[\mu(v) = \frac{1}{(2\pi)^{D/2}} \exp\left(-\frac{|v|^2}{2}\right).\]

Using the Hilbert expansion, all the terms $f_1, f_2, \ldots, f_{n-1}$ can be determined as the following way: for each $m \geq 1$, $f_m = P f_m + (I - P) f_m$, where $P f_m$ and $(I - P) f_m$ are the fluid [see (2.3)] and kinetic parts respectively. The fluid part for $m = 1$ is governed by the incompressible Navier-Stokes-Fourier equations and for $m \geq 2$, by the linearized incompressible Navier-Stokes-Fourier equations with source term from the known kinetic parts. The kinetic parts are also determined inductively in terms of the known fluid and kinetic parts.

In [15], the diffusive expansion was justified under the assumption that the initial expansion, i.e. $t = 0$ in (1.2), is well-prepared. More precisely, the fluid parts of the initial expansion satisfy the incompressibility and Boussinesq relations, and the kinetic parts of the initial expansion are also assumed to be a specific form which will be stated more precisely in Theorem 3.2-3.3.

This paper is devoted to remove the well-preparedness assumption on the fluid parts in the periodic domain $\mathbb{T}^D$. The initial fluid parts are not assumed to satisfy the incompressibility and Boussinesq relations. More precisely, the initial fluid parts are not assumed to be in the kernel of the acoustic operator $A$. The projections of the initial fluid parts on $\text{Ker}(A)$ and $\text{Ker}(A)\perp$ propagate in different time scales. The former is in the slow timescale and the latter is in the fast timescale. For this case, the fluid initial layers are created and preserved on the torus $\mathbb{T}^D$. In order to describe the initial layer, we introduce a slow time variable $\tau = \frac{t}{\epsilon}$ and consequently, the terms in the asymptotic expansion (1.2) depend on both $t$ and $\tau$. Employing the multiple timescale method, we formally determine the terms in the expansion (1.2). The leading order term $P f_1$ is governed by the fluid variable $U_1(t, \tau) = e^{\tau A} V_1(t)$, where $V_1$ satisfies a so-called averaged equation whose projection on $\text{Ker}(A)$ is the Navier-Stokes-Fourier equation and the projection on $\text{Ker}(A)\perp$ is non-local and parabolic. The higher order terms in the expansion have the similar structures and the corresponding equations are linear. The similar phenomenon also appear in the compressible-incompressible limits of the Navier-Stokes equations, Ekman layers of rotating fluids and many other problems [6, 11, 20, 23, 26]. We then estimate the error terms in the expansion using the nonlinear energy method of [12, 13, 15].

It should be mentioned that we still assume the kinetic part of the initial expansion is well-prepared as in [15], i.e. the initial kinetic parts are close (under a nonlinear norm) to a specific form so that the kinetic initial layers are not created. We will explain in more details in the end of Section 3. We also would like to mention that for the whole space $\mathbb{R}^D$, because of the dispersive property, the initial layer will be strongly decay, even the fluid initial data is general. We will treat this case in a forthcoming paper which also improves the result in [21].

The paper is organized as follows: the next section contains some preliminary material regarding the Boltzmann operators and notations used in the paper. We state the main theorems in Section 3 and determine formally the diffusive coefficients in Section 4. We calculate the averaged operators in Section 5, and finally prove the main theorems in Section 6. We collect some technical lemmas used in the proofs in the Appendix.
2. Preliminaries for the Boltzmann Equation

2.1. Collision Operators. Throughout this paper we consider two classes of collision operators. The first is the Boltzmann operator of the form

$$B(f, g) = \int_{\mathbb{R}^D} |v - u| \gamma B(\theta) \{ f(u + (v - u) \cdot \omega) g(v - (v - u) \cdot \omega) - f(u) g(v) \} \, du \, d\omega,$$

where $-\mathbb{D} < \gamma \leq 1$ and $B(\theta) \leq C|\cos \theta|$. Such collision operators cover both the hard-sphere interaction and inverse power law with an angular cutoff. The hard potential means $0 \leq \gamma \leq 1$, while soft potential means $-D < \gamma < 0$. The second is the Landau collision operator given by

$$B(f, g) = \nabla_v \cdot \int_{\mathbb{R}^D} \Phi(v - \nu') \{ f(\nu') \nabla_v g(v) - g(v) \nabla_v f(\nu') \} \, d\nu',$$

where $\Phi$ is the Landau kernel with Coulombic interaction:

$$\Phi(u) = \frac{1}{|u|} \left( I - \frac{u \otimes u}{|u|^2} \right).$$

We also define the linearized collision operator $L$ and the nonlinear collision operator $\Gamma$ as

$$L g = -\frac{1}{\sqrt{\mu}} \{ B(\mu, \sqrt{\mu} g) + B(\sqrt{\mu} g, \mu) \},$$

and

$$\Gamma(f, g) = \frac{1}{\sqrt{\mu}} B(\sqrt{\mu} f, \sqrt{\mu} g).$$

It is well-known that $\ker(L) = \text{Span}\{ 1, v, |v|^2 \}$, and the elements in $\ker(L)$ are called collision invariants. For more information on the Boltzmann and Landau collision operators, see the standard references [4, 5].

2.2. Notations. We use $\langle \cdot, \cdot \rangle$ to denote the $L^2$ inner product in $\mathbb{R}^D$ with corresponding $L^2$ norm $| \cdot |_2$, while we use $(\cdot, \cdot)$ to denote the $L^2$ inner product either in $\mathbb{T}^D \times \mathbb{R}^D$ or in $\mathbb{T}^D$ with corresponding $L^2$ norm $\| \cdot \|_2$. We use the standard notation $H^s$ to denote the Sobolev space $W^{s,2}$. We define $\partial_\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_D}^{\alpha_D}$ and $\partial_\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_D}^{\beta_D}$ where $\alpha = [\alpha_1, ..., \alpha_D]$ is related to the space derivatives, while $\beta = [\beta_1, ..., \beta_D]$ is related to the velocity derivatives. For the Boltzmann collision operator, we define the collision frequency

$$\nu(v) = \int_{\mathbb{R}^D} |v - u|^\gamma \mu(u) \, du$$

which behaves like $(1 + |v|^2)^{\frac{\gamma}{2}}$ as $|v| \to \infty$, and the corresponding weighted norm

$$|g|_\nu^2 = \int_{\mathbb{R}^D} g^2(v) \nu(v) \, dv, \quad \|g\|_\nu^2 = \int_{\mathbb{T}^D \times \mathbb{R}^D} g^2(x, v) \nu(v) \, dx.$$

For the Landau case, we given the following $\sigma$-norm as in [16]:

$$\|f\|_\sigma \equiv \|w(v)^{-\frac{1}{2}} f\|_2 + \|w(v)^{-\frac{1}{2}} \nabla_v f \cdot \frac{v}{|v|}\|_2 + \|w(v)^{-\frac{1}{2}} \nabla_v f \times \frac{v}{|v|}\|_2$$

where the weighted function $w(v)$ is

$$w(v) \equiv (1 + |v|^2)^{\frac{1}{2}}.$$

Finally, as in [15], we define the instant energy functional: for $N \geq 8$ and $l \geq 1,

$$(2.1) \quad \mathcal{E}_{N,l}(g) \sim \begin{cases} \sum_{|\alpha| \leq N} \| \partial_\alpha^a g \|^2 + \sum_{|\alpha| + |\beta| \leq N} \| w^l |\beta| \partial_\beta^a g \|^2 \quad \text{for hard potential case;} \\ \sum_{|\alpha| \leq N} \| \partial_\alpha^a g \|^2 + \sum_{|\alpha| + |\beta| \leq N} \| w^{l - |\beta|} |\beta| \partial_\beta^a g \|^2 \quad \text{for soft potential case;} \\ \sum_{|\alpha| \leq N} \| \partial_\alpha^a g \|^2 + \sum_{|\alpha| + |\beta| \leq N} \| w^{l - |\beta|} \partial_\beta^a g \|^2 \quad \text{for Landau case;} \end{cases}$$
and the corresponding dissipation rate: for $N \geq 8$ and $l \geq 1$,

$$
\mathcal{D}_{N,l}(g) = \begin{cases} \\
\sum_{|\alpha| \leq N} \{ \| \partial^\alpha P g \|^2 + \frac{1}{\epsilon^2} \| \partial^\alpha P_2 g \|^2 \} + \frac{1}{\epsilon^2} \sum_{|\alpha|+|\beta| \leq N} \| w^l \partial^\alpha \partial^\beta P_2 g \|^2, \\
\sum_{|\alpha| \leq N} \{ \| \partial^\alpha P g \|^2 + \frac{1}{\epsilon^2} \| \partial^\alpha P_2 g \|^2 \} + \frac{1}{\epsilon^2} \sum_{|\alpha|+|\beta| \leq N} \| w^l \partial^\alpha \partial^\beta P_2 g \|^2, \\
\sum_{|\alpha| \leq N} \{ \| \partial^\alpha g \|^2 + \frac{1}{\epsilon^2} \| \partial^\alpha P_2 g \|^2 \} + \frac{1}{\epsilon^2} \sum_{|\alpha|+|\beta| \leq N} \| w^l \partial^\alpha \partial^\beta P_2 g \|^2, \\
\end{cases}
$$

(2.2)

where $\mathbf{P}_2 = \mathbf{I} - \mathbf{P}$ and $\mathbf{P} g$ is the $\mathcal{L}^2_a$ projection of $g$ on the null space for $\mathcal{L}$ defined as

$$
\mathbf{P} g = \left\{ \rho_g(t,x) + v \cdot u_g(t,x) + \left( \frac{|v|^2}{2} - \frac{D}{2} \right) \theta_g(t,x) \right\} \sqrt{\mu}.
$$

More precisely,

$$
[\rho_g, u_g, \theta_g](t,x) = \int_{\mathbb{R}^b} [1, v, \left( \frac{|v|^2}{2} - 1 \right)] g(t,x,v) \sqrt{\mu} \, dv,
$$

(2.4)

We call $\mathbf{P} g$ the hydrodynamic part of $g$ and $\mathbf{P} g_2$ the kinetic part of $g$. In (2.1) we use the convention $f \sim g$ if there exists a constant $C > 0$ such that $\frac{1}{C} f \leq g \leq Cf$.

3. Statement of the Main Theorems

To state our main theorems precisely, we formally plug (1.2) with $f_j \equiv f_j(t, \frac{t}{\epsilon}, x, v)$ into (1.1),

$$
(\epsilon \partial_t + v \cdot \nabla_x \{ \epsilon f_1^\epsilon + \epsilon^2 f_2^\epsilon + \ldots + \epsilon^n f_n^\epsilon \}) = \frac{1}{\epsilon \sqrt{\mu}} \mathcal{B}(\mu + \mu^{1/2} \{ \epsilon f_1 + \epsilon^2 f_2 + \ldots + \epsilon^n f_n^\epsilon \}, \mu + \mu^{1/2} \{ \epsilon f_1 + \epsilon^2 f_2 + \ldots + \epsilon^n f_n^\epsilon \}),
$$

(3.1)

where $d_t = \partial_t + \frac{1}{\epsilon} \partial_r$. Collecting the terms on both sides of (3.1) in front of different powers of $\epsilon$, we obtain for $\epsilon^{-1}, \epsilon^0, \ldots, \epsilon^{n-2}$ in two different forms:

$$
\mathcal{L} f_1 = 0 \\
\partial_r f_1 + v \cdot \nabla_x f_1 + \mathcal{L} f_2 = \Gamma(f_1, f_1) \\
\partial_r f_2 + v \cdot \nabla_x f_2 + \mathcal{L} f_3 + \partial_t f_1 = \Gamma(f_1, f_2) + \Gamma(f_2, f_1) \\
\cdots \\
\partial_r f_{n-2} + v \cdot \nabla_x f_{n-2} + \mathcal{L} f_{n-1} + \partial_t f_{n-3} = \sum_{i+j=n-1 \atop i,j \geq 1} \Gamma(f_i, f_j)
$$

(3.2)

or

$$
\mathcal{L} f_1 = 0 \\
\partial_t f_1 + v \cdot \nabla_x f_1 + \mathcal{L} f_2 = \Gamma(f_1, f_1) \\
d_t f_1 + v \cdot \nabla_x f_2 + \mathcal{L} f_3 = \Gamma(f_1, f_2) + \Gamma(f_2, f_1) \\
\cdots \\
d_t f_{n-3} + v \cdot \nabla_x f_{n-2} + \mathcal{L} f_{n-1} = \sum_{i+j=n-1 \atop i,j \geq 1} \Gamma(f_i, f_j).
$$

(3.3)
Furthermore we can collect terms left in (3.1) with power $\epsilon^{n-1}$ or higher to get the equation for the equation of the remainder $f_n^\epsilon$:

\[
\begin{align*}
\epsilon^2 &\partial_t f_n^\epsilon + \epsilon \partial_r f_n^\epsilon + \epsilon v \cdot \nabla_x f_n^\epsilon + \mathcal{L} f_n^\epsilon = \{ - \partial_t f_{n-2} - \partial_r f_{n-1} - v \cdot \nabla_x f_{n-1} + \sum_{i+j=n} \Gamma(f_i, f_j) \} \\
&+ \epsilon \{ - \partial_t f_{n-1} + \sum_{i+j=n+1} \Gamma(f_i, f_j) \} + \epsilon^n \Gamma(f_n^\epsilon, f_n^\epsilon) \\
&+ \sum_{i=1}^{n-1} \epsilon^i \{ \Gamma(f_i^\epsilon, f_i) + \Gamma(f_i, f_n^\epsilon) \} + \sum_{i+j \geq n+2} \epsilon^{i+j-n} \Gamma(f_i, f_j).
\end{align*}
\]

or

\[
\begin{align*}
\epsilon^2 &\partial_t f_n^\epsilon + \epsilon v \cdot \nabla_x f_n^\epsilon + \mathcal{L} f_n^\epsilon = \{ - \partial_t f_{n-2} - v \cdot \nabla_x f_{n-1} + \sum_{i+j=n} \Gamma(f_i, f_j) \} \\
&+ \epsilon \{ - \partial_t f_{n-1} + \sum_{i+j=n+1} \Gamma(f_i, f_j) \} + \epsilon^n \Gamma(f_n^\epsilon, f_n^\epsilon) \\
&+ \sum_{i=1}^{n-1} \epsilon^i \{ \Gamma(f_i^\epsilon, f_i) + \Gamma(f_i, f_n^\epsilon) \} + \sum_{i+j \geq n+2} \epsilon^{i+j-n} \Gamma(f_i, f_j).
\end{align*}
\]

(3.4)

Now we state our main results. The first is about when the fluid part of the initial data is general, the coefficients $f_i(t, \tau, x, v)$ in a diffusive approximation (1.2) can be determined and estimated.

**Theorem 3.1.** Given $m$ vector-valued functions $[u_1^0(x), ..., u_m^0(x)]$ and $2m$ scalar functions $[\rho_1^0(x), ..., \rho_m^0(x)]$ and $[\theta_1^0(x), ..., \theta_m^0(x)]$ such that

\[
\| \frac{2}{H^2+2} \rho_1^0 - \frac{B}{H^2+2} \theta_1^0 \|_{H^2} + \| P_0 u_1^0 \|_{H^2} \leq M
\]

and

\[
\int_{\mathbb{T}^d} \rho_r^0(x) \, dx = \int_{\mathbb{T}^d} \theta_r^0(x) \, dx = 0, \quad \int_{\mathbb{T}^d} u_r^0(x) \, dx = 0,
\]

for $1 \leq r \leq m$. Then for $M$ sufficiently small, there exists unique functions $f_1(t, \tau, x, v)$ with zero-mean hydrodynamic fields satisfies (4.2), (4.5) and (6.1), and $f_m(t, \tau, x, v)$ satisfies (4.13), (4.14) or (4.20), (4.21) and (4.23) for $2 \leq r \leq m$. Moreover, for $1 \leq r \leq m$, and $\beta$, and all $l, s \geq 0$, there exists a polynomial $P_{r,\beta,l,s}$ with $P_{r,\beta,l,s}(0) = 0$ such that

\[
\sum_{|i| \leq s} \| u_i^\beta \partial_i^\beta f_r(t, \tau, x, v) \| \leq e^{-\lambda \rho_{r,0}^0} P_{r,\beta,l,s} \left( \sum_{1 \leq j \leq r} \| (\rho_j^0, u_j^0, \theta_j^0) \|_{H^{2s+4r-j}} \right).
\]

(3.5)

where $P_0$ is the Leray projection and $\lambda$ can be chosen as $\frac{1}{4} \min\{\eta, \kappa\}$ for $M$ sufficiently small.

Comparing to the Theorem 2.1 in [15], the main novelty of the above theorem is that it characterize the fluid initial layer in the diffusive expansion (1.2), and furthermore we only need the $H^2$ norm of the projection of the initial fluid part on Ker$(A)$ is sufficiently small, while the projection on Ker$(A)^\perp$ is allowed to be large.

For the estimate of the remainder $f_n^\epsilon$, our theorems are similar to those in [15]. To make the presentation self-contained, we now state the results and we will sketch the main ideas of the proofs later. We first study the first-order remainder $f^\epsilon \equiv f_1(t, \frac{\tau}{\epsilon}, x, v)$ which satisfies the nonlinear Boltzmann or Landau equation

\[
d_t f^\epsilon + \frac{1}{\epsilon} v \cdot \nabla_x f^\epsilon + \frac{1}{\epsilon^2} \mathcal{L} f^\epsilon = \frac{1}{\epsilon} \Gamma(f^\epsilon, f^\epsilon).
\]

(3.6)
Theorem 3.2. Let \( N \geq 8 \) and \( f^\epsilon(0,0,x,v) = f^\epsilon_0(x,v) \) satisfy the mass, momentum, and energy conservation laws

\[(f^\epsilon_0(x,v), [1,v,|v|^2]\sqrt{\mu}) = 0,\]

and \( F^\epsilon(0,0,x,v) = \mu(v) + \epsilon \sqrt{\mu} f^\epsilon_0(x,v) \geq 0 \). There exist an instant energy functional \( \mathcal{E}_{N,1}(f^\epsilon) \) such that if \( \mathcal{E}_{N,1}(f^\epsilon)(0) \) is sufficiently small, then

\[
\frac{d}{dt}\mathcal{E}_{N,1}(f^\epsilon)(t) + D_{N,1}(f^\epsilon) \leq 0.
\]

Moreover, there exists \( \lambda > 0 \) such that

(i) For the hard potential \( \gamma > 0 \),

\[
\mathcal{E}_{N,1}(f^\epsilon)(t) \leq C e^{-\lambda t} \mathcal{E}_{N,1}(f^\epsilon_0).
\]

(ii) For the soft potential \( -D < \gamma < 0 \) or the Landau kernel, if we further assume \( \mathcal{E}_{N+1,l+k}(f^\epsilon_0) \) is sufficiently small, then for \( k \geq 0 \)

\[
\mathcal{E}_{N,1}(f^\epsilon)(t) \leq C_k (1 + t)^{-k} \mathcal{E}_{N+1,l+k}(f^\epsilon_0).
\]

If in addition \( \mathcal{E}_{N+1,l+1/2}(f^\epsilon_0) \) is sufficiently small, then

\[
\mathcal{E}_{N,1}(f^\epsilon)(t) \leq C e^{-\lambda t} \mathcal{E}_{N,1}(f^\epsilon_0) + C \epsilon^2 \mathcal{E}_{N+1,l+1/2}(f^\epsilon_0).
\]

For the higher-order remainder \( f^\epsilon_n \) with \( n \geq 2 \), we have

Theorem 3.3. Given \( f_1, \ldots, f_n \) as constructed in Theorem 3.1, let \( N \geq 8 \) and

\[
\|(\rho^0_n, u^0_n, \theta^0_n)\|_N \equiv \sum_{1 \leq i \leq n} \|(\rho^0_n, u^0_n, \theta^0_n)\|_{H^{2N+12+4(n-j)}},
\]

and \( f^\epsilon_n(0,0,x,v) = f^\epsilon_n(t, \frac{1}{\epsilon}, x, v)|_{t=0} \) satisfy the mass, momentum, and energy conservation laws

\[(f^\epsilon_n(0,0,x,v), [1,v,|v|^2]\sqrt{\mu}) = 0,\]

and

\[
F^\epsilon(0,0,x,v) = \mu + \mu^{1/2}\{\epsilon f_1(0,0,x,v) + \ldots
\]

\[
+ \epsilon^{n-1} f_{n-1}(0,0,x,v) + \epsilon^n f^\epsilon_n(0,0,x,v)\} \geq 0.
\]

There exist an instant energy functional \( \mathcal{E}_{N,1}(f^\epsilon_n) \) and a positive polynomial with \( \mathcal{P}(0) = 0 \) such that if both \( \epsilon \)

\[
\mathcal{E}_{N,1}(f^\epsilon_n - f_n)(0)
\]

are sufficient small, then

\[
\mathcal{E}_{N,1}(f^\epsilon_n - f_n)(t) \leq \mathcal{P}\{(\rho^0_n, u^0_n, \theta^0_n)\|_{N}\} \mathcal{E}_{N,1}(f^\epsilon_n - f_n)(0) + \epsilon^2 \mathcal{P}\{(\rho^0_n, u^0_n, \theta^0_n)\|_{N}\}.
\]

Moreover, there exists \( \lambda > 0 \) such that

(i.) For the hard potential \( \gamma > 0 \),

\[
\mathcal{E}_{N,1}(f^\epsilon_f - f_n)(t) \leq C e^{-\lambda t}\left\{\mathcal{E}_{N,1}(f^\epsilon_n - f_n)(0) + \epsilon^2 \mathcal{P}\{(\rho^0_n, u^0_n, \theta^0_n)\|_{N}\}\right\}.
\]

(ii.) For the soft potential \( -D < \gamma < 0 \) or the Landau kernel, if we further assume \( \mathcal{E}_{N+1,l+k}(f^\epsilon_n - f_n)(0) \) is sufficiently small, then for \( k \geq 0 \)

\[
\mathcal{E}_{N,1}(f^\epsilon_n - f_n)(t) \leq C_k (1 + t)^{-k}\left\{\mathcal{E}_{N+1,l+k}(f^\epsilon_n - f_n)(0) + \epsilon^2 \mathcal{P}\{(\rho^0_n, u^0_n, \theta^0_n)\|_{N}\}\right\}.
\]
If in addition $E_{N+1,t+1/2}(f_n^1 - f_n)(t)$ is sufficiently small, then
\begin{equation}
E_{N,t}(f_n^1 - f_n)(t) \leq e^{-\lambda t} \left\{ e^{\mathcal{P}(\|\rho_n^0, u_n^0, \theta_n^0\|_N)} E_{N+1,t+k}(f_n^1 - f_n)(0) + e^{2\mathcal{P}(\|(\rho_n^0, u_n^0, \theta_n^0\|_N)} \right\} + C e^{2E_{N+1,t+1/2}(f_n^1 - f_n)(0)}.
\end{equation}

We remark that in the above theorem, the initial data is partially well-prepared in the following sense: from (3.9) and (3.10), the initial data considered in the theorem is not arbitrary but must have the form of (3.9). Note: all $f_j(0,0,x,v)'s$ for $1 \leq j \leq n$ have specific form as determined in Theorem 3.1, and the initial remainder term $f_n^1(0,0,x,v)$ must be close to $f_n(0,0,x,v)$ in the sense of (3.10). Thus, no Knudsen layer will be generated. However, for the fluid part of the initial data, $(\rho_j,u_j,\theta_j)(0,0,x,v)'s$ for $1 \leq j \leq n$ are not required to satisfy the incompressibility and Boussinesq relations. In this case, the fluid initial layer may be generated. This is the main concern of this paper. It will be harder to treat the general kinetic initial data for which the Knudsen layer may arise. This will be treated in our next paper.

4. The Diffusive Coefficients

Now we want to determine the coefficient $f_m(t,\tau,x,v)$ for $m \geq 1$ by an inductive process. We write
\begin{equation}
f_m = Pf_m + (I - P)f_m,
\end{equation}
and define the hydrodynamic filed of $g$ to be
\begin{equation}
U_m(t,\tau,x) = [\rho_m(t,\tau,x), u_m(t,\tau,x), \theta_m(t,\tau,x)].
\end{equation}

By the first equality of (3.2), i.e. $L f_1 = 0$, we know that
\begin{equation}
(I - P)f_1 = 0,
\end{equation}
which implies that $f_1$ is hydrodynamic, i.e.
\begin{equation}
f_1 = \{\rho_1 + v \cdot u_1 + (\frac{|v|^2 - D}{2})\theta_1\} \sqrt{\mu} = U_1(t,\tau,x) \cdot (1,v,\frac{|v|^2 - D}{2}) \sqrt{\mu}.
\end{equation}

By the second equality of (3.2), and the identities (for the proof, see [1].)
\begin{align*}
\Gamma(Pg,Pg) = L(\frac{Pg}{\sqrt{\mu}})^2,
\Gamma(Pg_1,Pg_2) + \Gamma(Pg_2,Pg_1) = L(\frac{Pg_1,Pg_2}{\sqrt{\mu}}),
\end{align*}
we can deduce that
\begin{equation}
L(f_2 - \frac{f_1^2}{\sqrt{\mu}}) = -(\partial_\tau + v \cdot \nabla_x)f_1
\end{equation}
\begin{equation}
= -(\partial_\tau U_1 + AU_1 \cdot (1,v,\frac{|v|^2 - D}{2}) \sqrt{\mu}
- (A(v) : \nabla_x u_1 + B(v) \cdot \nabla_x \theta_1),
\end{equation}
where $A$ is the acoustic operator defined as
\begin{equation}
A(\rho, u, \theta) = (\text{div}u, \nabla_x (\rho + \theta), \frac{2}{D} \text{div}u).
\end{equation}
and $A(v), B(v)$ are Burnett functions defined as
\begin{equation}
A(v) = (v \otimes v - \frac{|v|^2}{D} I) \sqrt{\mu},
B(v) = \frac{|v|^2 - (D+2)}{2} v \sqrt{\mu},
\end{equation}
which lay in ker $L^\perp$. The solvability of (4.3) implies that
\begin{equation}
\partial_\tau U_1 + AU_1 = 0,
\end{equation}
the solution of which is
\begin{equation}
U_1(t,\tau) = e^{-\tau A} V_1(t),
\end{equation}
where $V_1(t) = U_1(t,0)$ is to be determined and the initial value of $V_1(t)$ is
\begin{equation}
V_1(0,x) = (\rho_1^0(x), u_1^0(x), \theta_1^0(x)).
\end{equation}
Simple calculations show that
\[
\{\mathbf{I} - \mathbf{P}\} \left( \frac{\mathbf{f}_2}{\sqrt{\mu}} \right) = \frac{1}{2} u_1 \otimes u_1 : A(v) + B(v) \cdot u_1 \theta_1 + \frac{1}{2} C(v) \theta_1^2,
\]
where \( C(v) = \left( \frac{|v|^4}{4} - \frac{D_4 + 2|v|^2 + \frac{D_4 + 2D}{4} \sqrt{\mu}}{2} \right) \). Then it follows from (4.3) we can solve
\[
\{\mathbf{I} - \mathbf{P}\} f_2 = -\hat{A}(v) : \nabla_x u_1 - \hat{B}(v) \cdot \nabla_x \theta_1 + \frac{1}{2} u_1 \otimes u_1 : A(v) + B(v) \cdot u_1 \theta_1 + \frac{1}{2} C(v) \theta_1^2,
\]
where \( \hat{A}(v) \) and \( \hat{B}(v) \) are uniquely determined and satisfy
\[
\mathcal{L} \hat{A}(v) = A(v), \quad \mathcal{L} \hat{B}(v) = B(v).
\]
To determine \( U_1 \) we need to solve \( V_1(t) \). For the purpose, we notice the third equation of (3.2) and get that
\[
\mathcal{L} f_3 = \Gamma(f_1, f_2) + \Gamma(f_2, f_1) - \partial_t f_2 - v \cdot \nabla_x f_2 - \partial_t f_1
\]
\[
= \Gamma(f_1, f_2) + \Gamma(f_2, f_1) - \partial_t \{\mathbf{I} - \mathbf{P}\} f_2 - \{\mathbf{I} - \mathbf{P}\} f_2 - \{\mathbf{I} - \mathbf{P}\} \{\mathbf{I} - \mathbf{P}\} f_2
\]
\[
- A(v) : \nabla_x u_2 - B(v) \cdot \nabla_x \theta_2 - \mathbf{P} \{\mathbf{v} \cdot \nabla_x \{\mathbf{I} - \mathbf{P}\} f_2\}
\]
\[
- (\partial_t U_1 + \partial_t U_2 + A U_2) \cdot (\sqrt{\mu}, v \sqrt{\mu}, (|v|^2 - D) \sqrt{\mu}/2),
\]
the solvability of which implies that
\[
(\partial_t U_1 + \partial_t U_2 + A U_2) \cdot (\sqrt{\mu}, v \sqrt{\mu}, (|v|^2 - D) \sqrt{\mu}/2) + \mathbf{P} \{\mathbf{v} \cdot \nabla_x \{\mathbf{I} - \mathbf{P}\} f_2\} = 0.
\]
From the formula (2.4), we can compute the term
\[
\mathbf{P} \{\mathbf{v} \cdot \nabla_x \{\mathbf{I} - \mathbf{P}\} f_2\}
\]
\[
= - \left[ 0, \eta \text{div}(\nabla u_1 + (\nabla u_1)^T - \frac{2}{\mu} \text{div} u_1 I), \frac{D}{2} + \frac{2D}{4} \kappa \Delta \theta_1 \right].
\]
(4.8)
\[
= - \left( 0, \text{div}(u_1 \otimes u_1) - \nabla (|u_1|^2), \frac{D}{2} + \frac{2D}{4} \text{div}(u_1 \theta_1) \right)
\]
\[
\cdot (\sqrt{\mu}, v \sqrt{\mu}, (|v|^2 - D) \sqrt{\mu}/2)
\]
\[
= - D U_1 + Q(U_1, U_1) \cdot (\sqrt{\mu}, v \sqrt{\mu}, (|v|^2 - D) \sqrt{\mu}/2),
\]
where \( \eta \) and \( \kappa \) are constants given by (1.51) in [2]. Then (4.7) is equivalent to
\[
\partial_t U_2 + A U_2 = -\partial_t U_1 - Q(U_1, U_1) + D U_1.
\]
This equation can be solved as
\[
e^{\tau A} U_2(t, \tau) - U_2(t, 0)
\]
\[
= - \tau \partial_t V_1(t) - \int_0^\tau e^{sA} Q(e^{-sA} V_1, e^{-sA} V_1) ds + \int_0^\tau e^{sA} D e^{-sA} V_1 ds.
\]
(4.9)
We divide (4.9) by \( \tau \) and take time average in the following sense yields
\[
\lim_{\tau \to \infty} \frac{e^{\tau A} U_2(t, \tau) - U_2(t, 0)}{\tau}
\]
\[
= - \partial_t V_1(t) - \frac{1}{\tau} \int_0^\tau e^{sA} Q(e^{-sA} V_1, e^{-sA} V_1) ds
\]
\[
+ \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau e^{sA} D e^{-sA} V_1 ds.
\]
(4.10)
The left-hand side of (4.10) is an almost periodic function of \( \tau \) which yields the averaged equation
\[
\partial_t V_1 + \overline{Q}(V_1, V_1) = \overline{D} V_1,
\]
(4.11)
where the operators $\overline{Q}$ and $\overline{D}$ are formally defined by

\[
\overline{Q}(V, V) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau e^{sA}Q(e^{-sA}V, e^{-sA}V)ds,
\]

\[
\overline{D}V = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau e^{-sA}D e^{-sA}V ds.
\]

(4.12)

We will compute the average operators $\overline{Q}$ and $\overline{D}$ in details in the next section. By solving the average equation (4.11), we can obtain $V_i(t)$, and put it in (4.5), then $U_i(t, \tau)$ is completely determined. Also $\{I - P\}_2$ is known by (4.6).

Inductively, if we have determined $f_1(t, \tau, x, v), ..., f_m(t, \tau, x, v)$ for $m \geq 2$ then we can represent

\[
\{I - P\} f_m = \mathcal{L}^{-1}\{ - \partial_{\tau}f_m - v \cdot \nabla_x f_m - \partial_t f_m + \sum_{i+j=m} \Gamma(f_i, f_j) \}.
\]

(4.13)

On the other hand, to determine $P f_m$, we can follow the following steps:

- From (3.2)$_{m+1}$:

\[
\partial_\tau f_{m+1} + v \cdot \nabla_x f_{m+1} + \partial_t f_{m+1} + \mathcal{L} f_{m+1} = \sum_{i+j=m+1} \Gamma(f_i, f_j),
\]

the solvability condition of which implies that

\[
\partial_\tau U_{m+1} + AU_{m+1} = \{\text{terms in } U_i, \ 1 \leq i \leq m\};
\]

- By time-averaging, we can deduce the averaged equation of $U_m$ which reads

\[
\partial_t V_m + [\overline{Q}(V_1, V_m) + \overline{Q}(V_m, V_1)] - \overline{D} V_m = \{\text{known terms}\},
\]

which is a linear equation and solvable, then we can obtain $U_m$ which can be expressed

\[
U_m(t, \tau) = e^{-\tau A}V_m(t) + e^{-\tau A}W(\tau, V_1, ..., V_{m-1}),
\]

(4.14)

where $W$ is some explicit function, and furthermore $\{I - P\} f_{m+1}$ by (4.13)$_{m+1}$;

- Expressing $\{I - P\} f_{m+2}$ by (4.13)$_{m+2}$ which linear depends on the unknown term $U_{m+1}$. To solve $U_{m+1}$, we go back to step 1.

Follow the above induction process, we can solve $f_m$ step by step. However, it seems to be inconvenient to estimate $U_m$ since we need to deal with the $t$-derivative and $\tau$-derivative. To overcome this difficult, we can employ the expansion (3.3). We now construct $f_{m+1}(t, \frac{\tau}{\mu}, x, v)$ for $m \geq 1$ by induction if we assume $f_1, ..., f_m$ are known. The first step is to derive the microscopic part $\{I - P\} f_{m+1}$. By (3.3) we know

\[
d_t f_{m+1} + v \cdot \nabla_x f_m + \mathcal{L} f_{m+1} = \sum_{i+j=m+1} \Gamma(f_i, f_j),
\]

which implies

\[
\mathcal{L}\{I - P\} f_{m+1} = -d_t f_{m+1} - v \cdot \nabla_x f_m + \sum_{i+j=m+1} \Gamma(f_i, f_j)
\]

\[
= -[d_t U_{m+1} + AU_m](\sqrt{\mu}, v \sqrt{\mu}, (u^2 - D)\sqrt{\mu})
\]

\[
- A(v) \cdot \nabla_x u_m - B(v) \cdot \nabla_x \theta_m - \{v \cdot \nabla_x \{I - P\} f_m\}
\]

\[
- d_t \{I - P\} f_{m+1} + \Gamma(f_1, \{I - P\} f_m) + \Gamma\{\{I - P\} f_m, f_1\} + \Gamma(f_1, P f_m)
\]

\[
+ \Gamma(P f_m, f_1) + \sum_{i+j=m+1} \Gamma(f_i, f_j) - \{I - P\} \{v \cdot \nabla_x \{I - P\} f_m\}.
\]

(4.15)
Noticing that
\[ \Gamma(f_1, P_{f_m}) + \Gamma(P_{f_m}, f_1) = \mathcal{L}(\frac{H_{f_m}}{\sqrt{\mu}}) \]
we can deduce
\[ \{ I - P \} f_{m+1} = -\hat{A}(v) : \nabla_x u_m - \hat{B}(v) \cdot \nabla_x \theta_m + B(v) \cdot (u_1 \theta_m + u_m \theta_1) \]
\[ + \frac{1}{2}(u_1 \otimes u_m + u_m \otimes u_1) : A(v) + \frac{1}{2} C(v) \theta_1 \theta_m \]
\[ (4.16) \]
For the hydrodynamic part, by (3.3) again, we have
\[ \left\{ \begin{array}{l}
  d_t \rho_m + \nabla_x \cdot u_{m+1} = 0; \\
  d_t u_m + \nabla_x (\rho_{m+1} + \theta_{m+1}) + \text{div}(u_1 \otimes u_m + u_m \otimes u_1) \\
  - \frac{2}{5} \nabla u_1 \cdot u_m - \eta \text{div} (\nabla u_m + (\nabla u_m)^T - \frac{2}{5} \text{div} u_m I) = R^u_m; \\
  d_t \theta_r + \nabla_x \cdot u_{m+1} + \frac{\theta_{m+1} + 2}{5} \text{div}(\theta u_m + \theta_1 u_1) - \frac{\mu + 2}{5} \kappa \Delta \theta_m = R^\theta_m,
\end{array} \right. \]
\[ (4.18) \]
where
\[ R^u_m = - \left( v \cdot \nabla_x \mathcal{L}^{-1} \left\{ - d_t \{ I - P \} f_{m-1} + \Gamma(f_1, \{ I - P \} f_m) + \Gamma(\{ I - P \} f_m, f_1) \right. \right. \]
\[ + \sum_{i+j=m-1} \Gamma(f_i, f_j) - \{ I - P \} \{ v \cdot \nabla_x (I - P) f_m \} \left. \right\}, v \sqrt{\mu} \right), \]
\[ (4.19) \]
\[ R^\theta_m = - \left( v \cdot \nabla_x \mathcal{L}^{-1} \left\{ - d_t \{ I - P \} f_{m-1} + \Gamma(f_1, \{ I - P \} f_m) + \Gamma(\{ I - P \} f_m, f_1) \right. \right. \]
\[ + \sum_{i+j=m-1} \Gamma(f_i, f_j) - \{ I - P \} \{ v \cdot \nabla_x (I - P) f_m \} \left. \right\}, (\frac{|v|^2}{2} - 1) \sqrt{\mu} \right). \]
By the first equation of (4.18), we can solve that
\[ (I - P_0) u_{m+1} = \Delta_x^{-1} \nabla d_t \rho_m \]
where \( I - P_0 = \Delta^{-1} \nabla \text{div} \). Also by the second equation of (4.18) we can solve
\[ \rho_{m+1} + \theta_{m+1} = \Delta_x^{-1} \nabla \left\{ - d_t u_m - \text{div}(u_1 \otimes u_m + u_m \otimes u_1) + \frac{2}{5} \nabla (u_1 \cdot u_m) \right. \]
\[ + \eta \text{div} (\nabla u_m + (\nabla u_m)^T - \frac{2}{5} \text{div} u_m I) + R^u_m \right\}. \]
That is to say, we have obtained \( \Pi^U U_{m+1} \), and it remains to determine \( \Pi^\perp U_{m+1} \), see the definition of \( \Pi \) and \( \Pi^\perp \) in (5.3). By the \((m+2)\)-th order solvability condition, we can deduce that
\[ d_t U_{m+1} + AU_{m+2} + Q(U_{m+1}, U_1) + Q(U_1, U_{m+1}) = DU_{m+1} + R_{m+1}, \]
\[ (4.22) \]
where \( R_{m+1} = (0, R_m^\nu, R_m^\rho) \) is defined by (4.19). Taking the projection in \( \ker \mathcal{A} \) yields
(4.23) \[ d_t \Pi U_{m+1} + \Pi [Q(U_{m+1}, U_1) + Q(U_1, U_{m+1})] = \Pi D U_{m+1} + \Pi R_{m+1}, \]
which is a linear Navier-Stokes-Fourier system and we remark here that \( \Pi^\perp U_{r+1} \) and \( R_{r+1} \) are known since \( (I - \Pi) f_{r+1} \) is known.

5. AVERAGED OPERATORS

In this section, we compute the averaged dissipation operator \( \overline{D} \) and the quadratic operator \( \overline{Q} \) which are defined by (4.12). Our calculations basically use the method developed in [23, 6, 17, 18]. We first collect some properties about the acoustic operator \( \mathcal{A} \) defined in (4.4). We define Hilbert spaces
\[
\mathbb{H} = \left\{ U = (\rho, u, \theta) \in L^2(dx; \mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}) \right\},
\]
(5.1)
\[
\mathbb{V} = \left\{ U \in \mathbb{H} : \int_{\mathbb{T}^2} |\nabla_x U|^2 \, dx < \infty \right\},
\]
endowed with inner product
(5.2)
\[
(U_1, U_2) = \int_{\mathbb{T}^2} \left( \rho_1 \bar{v}_2 + u_1 \cdot \bar{u}_2 + \frac{\partial}{\partial x} \theta_1 \bar{\theta}_2 \right) \, dx,
\]
where \( \bar{f} \) means the complex conjugate of \( f \). The domain of the acoustic operator \( \mathcal{A} \) is \( \mathbb{V} \), and it is easy to check that \( \mathcal{A} \) is skew-symmetric on \( \mathbb{V} \), i.e.
\[
(\mathcal{A} U_1, U_2) = -(U_1, \mathcal{A} U_2) \quad \text{and} \quad (\mathcal{A} U, U) = 0.
\]
The null space of \( \mathcal{A} \) and its orthogonal with respect to this inner product are given by
\[
\ker(\mathcal{A}) = \{ (\rho, u, \theta) \in \mathbb{V} : \rho + \theta = 0, \, \text{div} u = 0 \},
\]
\[
\ker(\mathcal{A})^\perp = \{ (\rho, u, \theta) \in \mathbb{V} : \rho = \frac{\partial}{\partial x} \theta, \, u = \nabla \phi, \, \phi \in H^1(dx) \}.
\]

For any \( U = (\rho, u, \theta) \in \mathbb{H} \), we can define \( \Pi \) and \( \Pi^\perp \) the projections to \( \ker(\mathcal{A}) \) and \( \ker(\mathcal{A})^\perp \) respectively
(5.3)
\[
\Pi U = \begin{pmatrix}
\frac{2}{B+2} \rho - \frac{\partial}{\partial x} \theta \\
P_0 u \\
\frac{2}{B+2} \rho + \frac{\partial}{\partial x} \theta
\end{pmatrix} \equiv \begin{pmatrix}
-\vartheta \\
w \\
\theta
\end{pmatrix}, \quad \Pi^\perp U = \begin{pmatrix}
\frac{\partial}{\partial x} \rho + \theta \\
Q u \\
\frac{2}{B+2} \rho + \theta
\end{pmatrix} \equiv \begin{pmatrix}
\pi \\
\Omega \\
\frac{2}{B} \pi
\end{pmatrix}.
\]
Thus every \( U \in \mathbb{H} \) can be decomposed uniquely as
\[
U = \Pi U + \Pi^\perp U.
\]

Now we compute the basis \( \{ \Phi_k \}_{k \in \mathbb{Z}} \) of space \( \ker \mathcal{A}^\perp \). The spectral decomposition of acoustic operator \( \mathcal{A} \) can be characterized in terms of the eigenfunctions of Laplacian. Let \( \phi_\nu \) be an eigenfunction of the Laplacian satisfying
\[
-\Delta_x \phi_\nu = \nu^2 \phi_\nu \text{ on } \mathbb{T}^2 \text{ for some } \nu > 0.
\]
Then we take \( (\phi_\nu, c_0 \nabla_x \phi_\nu, \frac{2}{B} \phi_\nu)^T \in \ker \mathcal{A}^\perp \), it is easy to verify
\[
\mathcal{A} \begin{pmatrix}
\phi_\nu \\
c_0 \nabla_x \phi_\nu \\
\frac{2}{B} \phi_\nu
\end{pmatrix} = \begin{pmatrix}
c_0 \Delta \phi_\nu \\
\frac{D+2}{B} \nabla_x \phi_\nu \\
\frac{2c_0}{B} \Delta \phi_\nu
\end{pmatrix} = \begin{pmatrix}
-\nu^2 \phi_\nu \\
\frac{D+2}{B} \nu^2 \phi_\nu \\
\frac{2c_0}{B} \nu^2 \phi_\nu
\end{pmatrix} = \lambda_\nu \begin{pmatrix}
\phi_\nu \\
c_0 \nabla_x \phi_\nu \\
\frac{2}{B} \phi_\nu
\end{pmatrix}.
\]
We thereby see that if \( \nu^2 \) is a eigenvalue of \( -\Delta_x \) with eigenfunction \( \phi_\nu \), then \( \mp \sqrt{-\nu^2} \sqrt{\frac{D+2}{B}} \) is a conjugate pair of eigenvalue given above. On the other hand, it’s well-known that the
Fourier mode \( e^{\sqrt{-1}k \cdot x} \) solves \(-\Delta \phi_\nu = \nu^2 \phi_\nu \) on \( \mathbb{T}^D \) with \( \nu = |k| \). Finally we can construct the eigenfunction for \( \mathcal{A} \) in the re-normalization form

\[
\Phi_\alpha^k(x) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{D}{D+2}} \alpha \text{sg}(k) \frac{k}{|k|} \right) e^{\sqrt{-1}k \cdot x}
\]

with corresponding eigenvalue \( \sqrt{-1} \lambda_\alpha^k = \sqrt{-1} \alpha \text{sg}(k) |k| \sqrt{\frac{D+2}{D}} \) and here \( \alpha \in \{1, -1\} \), and the notation \( \text{sg}(k) \) stands for a generalized sign function on \( \mathbb{R}^D \setminus \{0\} \): its value is 1 if and only if the first nonzero component of \( k \) is positive, and \(-1\) elsewhere (this notation was first used by Masmoudi in [23]). Now we can represent

\[
\Pi^\perp U = \sum_{\alpha,k} \hat{U}_\alpha^k \Phi_\alpha^k,
\]

where \( \hat{U}_\alpha^k \) can be determined by \( U \) in terms of \( \hat{\rho}(k), \hat{u}(k), \hat{\theta}(k) \) which are

\[
\rho = \sum_k \hat{\rho}(k) e^{\sqrt{-1}k \cdot x}, \quad u = \sum_k \hat{u}(k) e^{\sqrt{-1}k \cdot x}, \quad \theta = \sum_k \hat{\theta}(k) e^{\sqrt{-1}k \cdot x}.
\]

In fact,

\[
\Pi^\perp U = \sum_{\alpha,k} (\hat{U}_\alpha^k \Phi_\alpha^k) \Phi_\alpha^k.
\]

With these preparation we can compute the averaged dissipation operator now. Recall the definition of \( \overline{D} \) (4.12). For any \( \psi \in \ker \mathcal{A} \), since the exponential operator \( e^{s\mathcal{A}} \) doesn’t affect \( \ker \mathcal{A} \), we have

\[
(\overline{D} U|\psi) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (e^{-s\mathcal{A}} D e^{-s\mathcal{A}} U|\psi) ds
\]

\[
= \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (D e^{-s\mathcal{A}} U|\psi) ds
\]

\[
= \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (D \Pi U|\psi) ds + \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (D e^{-s\mathcal{A}} \Pi^\perp U|\psi) ds
\]

\[
=(D \Pi U|\psi) = (\Pi D \Pi U|\psi).
\]

Since by Riemann-Lebesgue lemma (A.2), we have

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (D e^{-s\mathcal{A}} \Pi^\perp U|\psi) ds = \sum_{\alpha,k} \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (D e^{-s\sqrt{-1} \lambda_\alpha^k \Phi_\alpha^k}|\psi) ds = 0.
\]

So we deduce the projection of the averaged dissipation operator on \( \ker \mathcal{A} \) is given by

\[
\Pi \overline{D} U = \Pi D \Pi U = \begin{pmatrix} -\frac{D}{D+2} \kappa \Delta \vartheta \\ \eta P_0 \Delta w \\ \frac{D}{D+2} \kappa \Delta \vartheta \end{pmatrix} = \begin{pmatrix} -\kappa \Delta \vartheta \\ \eta \Delta w \\ \kappa \Delta \vartheta \end{pmatrix}.
\]

Next we compute the projection of \( \overline{D} \) on \( \ker \mathcal{A}^\perp \):

\[
\Pi^\perp \overline{D} U = \sum_{\alpha,k} (\overline{D} U|\Phi_\alpha^k) \Phi_\alpha^k,
\]
where

\[
(\overline{DU}|\Phi_k^\alpha) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (e^{sA}De^{-sA}U|\Phi_k^\alpha) \, ds
\]

\[
= \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (De^{-sA}\Pi U|e^{-s\sqrt{-\Delta}}\Phi_k^\alpha) \, ds
\]

\[
+ \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (De^{-sA}\sum_{\beta,l} \widehat{U}^\beta_l \Phi^\beta_l|e^{-s\sqrt{-\Delta}}\Phi_k^\alpha) \, ds
\]

(5.4)

\[
\sum_{\beta,l} \widehat{U}^\beta_l \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (De^{-sA}\Phi^\beta_l|e^{-s\sqrt{-\Delta}}\Phi_k^\alpha) \, ds
\]

\[
= \sum_{\beta,l} \widehat{U}^\beta_l \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau e^{-s\sqrt{-\Delta}(\lambda_k^\alpha - \lambda_l^\beta)}(D\Phi^\beta_l|\Phi^\alpha_k) \, ds \neq 0
\]

if and only if

\[
\lambda_k^\alpha = \lambda_l^\beta \iff \alpha \text{sgn}(k)|k| = \beta \text{sgn}(l)|l|.
\]

Straightforward calculations imply that \((D\Phi^\beta_l|\Phi^\alpha_k)\) is non-zero only when \(k = l\). In fact, we can compute

\[
D\Phi^\beta_l(x) = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 \\
-\eta \beta \text{sgn}(l)(2 - \frac{2}{\pi})|ll| \\
-2k^2 \sqrt{\frac{D}{\pi} + 2} vl^2
\end{pmatrix} e^{\sqrt{-\Delta}x}.
\]

Finally we deduce from (5.4) that

\[
\Pi^\perp \overline{DU} = -\sum_{\alpha,k} \frac{1}{2} \left[ \eta(2 - \frac{2}{\pi}) + \frac{2k^2}{\pi} \right] |k|^2 \widehat{U}^\alpha_k \Phi^\alpha_k \equiv -\overline{\eta}|k|^2 \Delta \Pi^\perp U.
\]

Note that \(\overline{\eta} > 0\) for \(D \geq 2\).

By a similar approach, we can compute \(\overline{Q}(U,U) = \Pi \overline{Q}(U,U) + \Pi^\perp \overline{Q}(U,U)\). For \(\Pi \overline{Q}(U,U)\), for any \(\psi \in \ker A\), by the definition (4.12), one has

\[
(\overline{Q}(U,U)|\psi) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (e^{sA}Q(e^{-sA}U,e^{-sA}U)|\psi) \, ds = I_1 + I_2 + I_3,
\]

corresponding to the following decomposition

(5.5)

\[
Q(e^{-sA}U,e^{-sA}U) = Q(\Pi U,\Pi U) + Q(e^{-sA}\Pi^\perp U,e^{-sA}\Pi^\perp U)
\]

\[
+ Q(\Pi U,e^{-sA}\Pi^\perp U) + Q(e^{-sA}\Pi^\perp U,\Pi U).
\]

By Riemann-Lebesgue Lemma A.2, we claim that

\[
I_3 = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \left( Q(\Pi U,e^{-sA}\Pi^\perp U) + Q(e^{-sA}\Pi^\perp U,\Pi U) | \psi) \right) ds = 0.
\]

Recalling (5.3), we further write

(5.6)

\[
\Pi U = \begin{pmatrix}
-\theta \\
w
\end{pmatrix} = \begin{pmatrix}
-\sum_{l} \hat{Q}_l e^{\sqrt{-\Delta}x} \\
\sum_{l} \hat{w}_l e^{\sqrt{-\Delta}x} \\
\sum_{l} \hat{\theta}_l e^{\sqrt{-\Delta}x}
\end{pmatrix},
\]
where \( l \cdot \hat{w}_l = 0 \). Thus recalling the definition of \( Q \) in (4.8), we deduce that

\[
\Pi Q(\Pi U, \Pi U) = \begin{pmatrix} -\frac{D}{B+2} \frac{D+2}{B} \text{div}(w \vartheta) \\ P_0 \text{div}(w \otimes w) - P_0 \nabla (\frac{1}{B} |w|^2) \end{pmatrix} = \begin{pmatrix} -\text{div}(w \vartheta) \\ P_0 \text{div}(w \otimes w) \end{pmatrix}.
\]

For \( I_2 \), noticing

\[
e^{-s A} \Pi^\perp U = \sum_{\alpha, k} \hat{U}_k^\alpha e^{-s A} \Phi_k^\alpha = \sum_{\alpha, k} \hat{U}_k^\alpha e^{-s \sqrt{-1} \lambda_k^\alpha \frac{l}{\sqrt{2}}} \begin{pmatrix} \sqrt{\frac{D}{B+2}} \\ \alpha \text{sg}(k) \frac{|k|}{|l|} \frac{2}{B+2} \frac{D+2}{B} \text{div}(w \vartheta) \end{pmatrix} e^{\sqrt{-1} k \cdot x}.
\]

Thus by the definition (4.8), we can compute the \( i \)-component \( (i = 1, 2, 3) \) of \( Q(e^{-s A} \Pi^\perp U, e^{-s A} \Pi^\perp U) \) as following: (we use \( U_i \) denote the \( i \)-th component of \( U \))

\[
Q(e^{-s A} \Pi^\perp U, e^{-s A} \Pi^\perp U)_1 = 0;
\]

\[
Q(e^{-s A} \Pi^\perp U, e^{-s A} \Pi^\perp U)_2
= \sum \sqrt{-1} \hat{U}_k^\alpha \hat{U}_l^\beta \text{sg}(k) \text{sg}(l) e^{-s \sqrt{-1} (\lambda_k^\alpha + \lambda_l^\beta) e^{\sqrt{-1} (k+l) \cdot x}}
\times \left[ \frac{1}{4} \left( \frac{k \cdot (k+l)}{|k||l|} + \frac{l \cdot (k+l)}{|k||l|} \right) - \frac{k \cdot l}{2B|k||l|}(l + k) \right];
\]

\[
Q(e^{-s A} \Pi^\perp U, e^{-s A} \Pi^\perp U)_3
= \sum \sqrt{-1} \hat{U}_k^\alpha \hat{U}_l^\beta e^{-s \sqrt{-1} (\lambda_k^\alpha + \lambda_l^\beta) e^{\sqrt{-1} (k+l) \cdot x}}
\times \frac{1}{4} \sqrt{\frac{D+2}{B}} \left( \text{sg}(k) \frac{|k|}{|l|} + \text{sg}(l) \frac{|l|}{|k|} \right) \cdot (k + l).
\]

By Riemann-Lebesgue Lemma, the only non-trivial contribution is the case \( \lambda_k^\alpha + \lambda_l^\beta = 0 \), i.e. \( \text{sg}(k) \frac{|k|}{|l|} + \text{sg}(l) \frac{|l|}{|k|} = 0 \), which implies \( |k| = |l| \). Furthermore, we deduce that

\[
\left( \text{sg}(k) \frac{k}{|l|} + \text{sg}(l) \frac{l}{|k|} \right) \cdot (k + l) = 0,
\]

and

\[
\frac{1}{4} \left( \frac{k \cdot (k+l)}{|k||l|} + \frac{l \cdot (k+l)}{|k||l|} \right) - \frac{k \cdot l}{2B|k||l|}(l + k) = \left( \frac{1}{4} + \frac{|k|-2|l|}{4B|k||l|} \right)(l + k).
\]

These imply that \( I_2 \) is in \( \ker^\perp \mathcal{A} \) after time averaging. Thus we conclude that

\[
\Pi Q(U, U) = \Pi Q(\Pi U, \Pi U) = \begin{pmatrix} -\text{div}(w \vartheta) \\ P_0 \text{div}(w \otimes w) \end{pmatrix}.
\]

To finish \( \Pi^\perp Q(U, U) \), as (5.7), we can compute each component of \( I_3 \) as following

\[
[Q(\Pi U, e^{-s A} \Pi^\perp U) + Q(e^{-s A} \Pi^\perp U, \Pi U)]_1 = 0;
\]

\[
[Q(\Pi U, e^{-s A} \Pi^\perp U) + Q(e^{-s A} \Pi^\perp U, \Pi U)]_2
= \sum \sqrt{-1} \hat{U}_k^\alpha \text{sg}(k) e^{-s \sqrt{-1} \lambda_k^\alpha e^{\sqrt{-1} (k+l) \cdot x}}
\times \frac{1}{4} \sqrt{\frac{D+2}{B}} \left( \hat{w}_l \cdot (k + l) \frac{k}{|l|} + \frac{k \cdot l}{|k||l|} \hat{w}_l \right) - \frac{B}{B} \frac{k \cdot l}{B|k||l|}(k + l) \right];
\]

\[
[Q(\Pi U, e^{-s A} \Pi^\perp U) + Q(e^{-s A} \Pi^\perp U, \Pi U)]_3
= \sum \sqrt{-1} \hat{U}_k^\alpha \text{sg}(k) e^{-s \sqrt{-1} \lambda_k^\alpha e^{\sqrt{-1} (k+l) \cdot x}}
\times \frac{B+2}{B} \frac{1}{2} \left( \text{sg}(k) \frac{k \cdot (k+l)}{|k|} \hat{w}_l + \hat{w}_l \cdot (k + l) \frac{2}{B} \frac{D+2}{B} \sqrt{\frac{D}{B+2}} \right).
\]
Then we can compute
\[
\Pi^\perp \mathcal{Q}(U,U) = \sum_{\gamma,m} (\mathcal{Q}(U,U)|\Phi_m^\gamma)\Phi_m^\gamma \equiv J_1 + J_2 + J_3,
\]
according to (5.5). By Riemann-Lebesgue, we can easily check \( J_1 = 0 \). And we notice that
\[
(5.10) \quad J_2 = \sum_{\gamma,m} \lim_{r \to \infty} \frac{1}{r} \int_0^r \left( Q(e^{-sA}\Pi^\perp U, e^{-sA}\Pi^\perp U)|e^{-s\sqrt{\gamma} \lambda_m^\gamma} \Phi_m^\gamma} \right) ds.
\]
By noticing the computation (5.7), we remark that \( J_2 \neq 0 \) if and only if \( \lambda_k^\alpha + \lambda_l^\beta = \lambda_m^\gamma \) and \( k + l = m \) that implies \( k \parallel l \parallel m \). We further compute
\[
J_2 = \sum_{\gamma,m} \Phi_{\gamma,m} \sum_{\alpha \beta \gamma,m} \left\{ \sqrt{-1} \hat{U}_k^\alpha \hat{U}_l^\beta \alpha \beta \gamma \gamma \Phi_{\gamma,m} (k) \beta \gamma \Phi_{\gamma,m} (l) e^{-s\sqrt{\gamma} (\lambda_k^\alpha + \lambda_l^\beta)} \times \\
\left[ \frac{1}{2} \left( \frac{k(k+1)}{|k||l|} + \frac{l(k+l)}{|k||l|} \right) - \frac{l-k}{2\beta_0} \right] \frac{1}{\sqrt{2}} \gamma \gamma \Phi_{\gamma,m} (m) \right\} \\
= \text{div} \left( \sum_{\gamma,m} \Omega_{\gamma,m} \Phi_{\gamma,m} \right) \equiv Q_{3r}(\Pi^\perp U, \Pi^\perp U),
\]
where
\[
\Omega_{\gamma,m} = \frac{1}{\sqrt{2}} \beta + \beta_0 \sum_{\gamma,m} \sum_{\alpha \beta \gamma,m} \hat{U}_k^\alpha \hat{U}_l^\beta \gamma \gamma \Phi_{\gamma,m} (m) \gamma \gamma \Phi_{\gamma,m} (m).
\]
Here we have used the fact
\[
\alpha \beta \gamma \gamma (k) \beta \gamma \Phi_{\gamma,m} (l) \left[ \frac{1}{2} \left( \frac{k(k+1)}{|k||l|} + \frac{l(k+l)}{|k||l|} \right) - \frac{l-k}{2\beta_0} \right] \gamma \gamma \Phi_{\gamma,m} (m) \gamma \gamma \Phi_{\gamma,m} (m) = \frac{\beta - 1}{2\beta_0} \gamma \gamma \Phi_{\gamma,m} (m) \gamma \gamma \Phi_{\gamma,m} (m),
\]
\[
\left( \alpha \beta \gamma \gamma (k) \beta \gamma \Phi_{\gamma,m} (l) \right) \cdot (k + l) = 2 \gamma \gamma \Phi_{\gamma,m} (m) \gamma \gamma \Phi_{\gamma,m} (m),
\]
under the construction of \( \lambda_k^\alpha + \lambda_l^\beta = \lambda_m^\gamma \) and \( k + l = m \).

**Remark 5.1.** Since the value of \( \alpha \beta \gamma \gamma (k) \beta \gamma \Phi_{\gamma,m} (l) \gamma \gamma \Phi_{\gamma,m} (m) \) can only be taken \( \pm 1 \), also by the facts \( \alpha \beta \gamma \gamma (k) \beta \gamma \Phi_{\gamma,m} (l) \gamma \gamma \Phi_{\gamma,m} (m) \) and \( k + l = m \) which imply \( k \parallel l \parallel m \). On the other hand, we have
\[
\left\{ \begin{array}{l}
|k|^2 + 2k \cdot l + |l|^2 = |m|^2, \\
|k|^2 + 2\alpha \beta \gamma \gamma (k) \beta \gamma \Phi_{\gamma,m} (l) \beta \gamma \Phi_{\gamma,m} (l) k \cdot l + |l|^2 = |m|^2,
\end{array} \right.
\]
that is to say \( k \cdot l = \alpha \beta \gamma \gamma (k) \beta \gamma \Phi_{\gamma,m} (l) \gamma \gamma \Phi_{\gamma,m} (l) \gamma \gamma \Phi_{\gamma,m} (l) \) which tells us \( \alpha = \beta \). Similarly we can deduce \( \beta = \gamma \) by the facts \( k = m - l \) and \( \lambda_k^\alpha = \lambda_m^\gamma - \lambda_l^\beta \).
Similarly, by noticing (5.9), we can also remark that $J_3 \neq 0$ if and only if $\lambda_k^\alpha = \lambda_1^\beta$ and $k + l = m$. We further compute that

$$J_3 = \sum_{\gamma, m} \sum_{k + l = m \atop |k| = |l|} \left\{ \frac{1}{\sqrt{2}} \tilde{U}_k^\alpha \text{asg}(k) \sqrt{-1} \left[ (\tilde{w}_l \cdot (k + l) \frac{k}{|k|} + \frac{k}{|k|} \tilde{w}_l) - \frac{1}{\sqrt{2}} \frac{k}{|k|} \tilde{w}_l (k + l) \right] \right. \times \frac{1}{\sqrt{2}} \gamma \text{sg}(m) \left. \frac{m}{|m|} + \frac{\sqrt{2}^2 + 2 \gamma \text{sg}(k)}{\sqrt{2}^2 + 2} \frac{\text{asg}(k)}{|k|} \tilde{\vartheta}_l + \tilde{w}_l \cdot (k + l) \frac{2}{\sqrt{2}^2 + 2} \sqrt{\frac{2}{\sqrt{2}^2 + 2}} \right\} \Phi_{\gamma, m}$$

$$= \sum_{\gamma, m} \sum_{k + l = m \atop |k| = |l|} \left\{ \frac{1}{\sqrt{2}} \tilde{U}_k^\alpha \sqrt{-1} \left[ (\frac{m}{|m|} + \frac{\sqrt{2}^2 + 2 \gamma \text{sg}(k)}{\sqrt{2}^2 + 2} \frac{\text{asg}(k)}{|k|} \tilde{\vartheta}_l + \tilde{w}_l \cdot (k + l) \frac{2}{\sqrt{2}^2 + 2} \sqrt{\frac{2}{\sqrt{2}^2 + 2}} \right] \Phi_{\gamma, m} \right\}$$

$$= \text{div} \left( \sum_{\gamma, m} \Lambda_{\gamma, m} \Phi_{\gamma, m} \right) \equiv Q_{2r}(\Pi U, \Pi^+ U).$$

Here we have used $l \cdot \tilde{w}_l = 0$.

6. Proof of the main theorems

6.1. **Proof of Theorem 3.1.** We use induction argument over $r$ as in [15]. We first consider the case $r = 1$. Noticing (4.5), we split $U(t, \tau)$ as following

$$U_1(t, \tau) = e^{-r A} \Pi V_1(t) + e^{-r A} \Pi^\perp V_1(t) = \Pi V_1(t) + e^{-r A} \Pi^\perp V_1(t).$$

From the computation of averaged operators in the last section, taking the projection of (4.11) into the spaces $\text{ker} \ A$ and noticing (5.6) and (5.8), then we can get the equation for $\Pi V_1(t)$:

$$\begin{cases} \partial_t w_1 + w \cdot \nabla_x w_1 + \nabla_x P_1 = \eta \Delta w_1, \\
\partial_t \vartheta_1 + w_1 \cdot \nabla_x \vartheta_1 = \kappa \Delta \vartheta_1, \quad \text{div} w = 0, \end{cases}$$

with the initial data

$$w_1, \vartheta_1,(0, x) = (P_0 w_1^0, \frac{\rho_0}{\sqrt{2}^2 + 2} P_1^0)(x).$$

The equations (6.1) with the initial data (6.2) are the classical incompressible Navier-Stokes-Fourier equations with $\int_{\mathbb{T}^d} w_1 \, dx = \int_{\mathbb{T}^d} \vartheta_1 \, dx = 0$. A classical result [27] says that there is a unique solution $(w_1, \vartheta_1)$ and $\lambda > 0$ such that

$$\| w_1 \|_{H^2} + \| \vartheta_1 \|_{H^2} \leq C e^{-\lambda t} \left\{ \| \frac{\rho_0}{\sqrt{2}^2 + 2} - \frac{\rho_1^0}{\sqrt{2}^2 + 2} \|_{H^2} + \| P_0 w_1^0 \|_{H^2} \right\},$$

for $\| \frac{\rho_0}{\sqrt{2}^2 + 2} - \frac{\rho_1^0}{\sqrt{2}^2 + 2} \|_{H^2} + \| P_0 w_1^0 \|_{H^2}$ sufficiently small. We claim that for $s \geq 2$, there exists a polynomial $\mathcal{P}_s(0) = 0$ such that

$$\| w_1 \|_{H^s} + \| \vartheta_1 \|_{H^s} \leq C e^{-\lambda t} \mathcal{P}_s \left( \| \frac{\rho_0}{\sqrt{2}^2 + 2} - \frac{\rho_1^0}{\sqrt{2}^2 + 2} \|_{H^s} + \| P_0 w_1^0 \|_{H^s} \right).$$

For higher-order spatial derivatives $\partial^\alpha$ with $|\alpha| = s \geq 3$, we notice that

$$\begin{cases} \partial_t \partial^\alpha w_1 + w_1 \cdot \nabla_x \partial^\alpha w_1 + \nabla_x \partial^\alpha P_1 - \eta \Delta \partial^\alpha w_1 = - \sum_{|\alpha_1 + \alpha_2 = \alpha | \alpha_1 | \geq 1} C_{\alpha}^\alpha w_1 \nabla_x \partial^\alpha w_1, \\
\partial_t \partial^\alpha \vartheta_1 + w_1 \cdot \nabla_x \partial^\alpha \vartheta_1 - \kappa \Delta \partial^\alpha \vartheta_1 = - \sum_{|\alpha_1 + \alpha_2 = \alpha | \alpha_1 | \geq 1} C_{\alpha}^\alpha \partial^\alpha w_1 \nabla_x \partial^\alpha \vartheta_1. \end{cases}$$
Separating the case of $|\alpha_1| = 1$ and $|\alpha_1| = s$ from the case of $|\alpha_1| \leq s - 1$, $|\alpha_2| \leq s - 2$, we estimate the $L^2$-norm of the right-hand side by
\[
\{\|w_1\|_{H^2} + \zeta\} \sum_{|\alpha|=s+1} \{\|\partial^\alpha w_1\| + \|\partial^\alpha \partial_1\|\} + C\zeta\{\|w_1\|_{H^{s+1}} + \|\partial_1\|_{H^{s+1}}\}^2,
\]
for any small $\zeta > 0$ by an interpolation in the Sobolev space. We can also use a standard energy estimate, i.e. multiplying (6.4) by $\partial^\alpha w_1$ and $\partial^\alpha \partial_1$ respectively to get
\[
\frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=s} \{\|\partial^\alpha w_1\|^2 + \|\partial^\alpha \partial_1\|^2\} + \sum_{|\alpha|=s} \{\eta\|\nabla_\partial \partial^\alpha w_1\|^2 + \kappa\|\nabla_\partial \partial^\alpha \partial_1\|^2\} \leq \{C\|w_1\|_{H^2} + \zeta\} \sum_{|\alpha|=s+1} \{\|\partial^\alpha w_1\| + \|\partial^\alpha \partial_1\|\}^2 + C\zeta\{\|w_1\|_{H^{s+1}} + \|\partial_1\|_{H^{s+1}}\}^4.
\]
For $C\|w_1\|_{H^2} + \zeta$ sufficiently small, we deduce from the Poincaré inequality and induction over the last order term that
\[
\sum_{|\alpha| \leq s} \{\|\partial^\alpha w_1\| + \|\partial^\alpha \partial_1\|\} \leq e^{-\lambda t} \sum_{|\alpha| \leq s} \{\|\partial^\alpha w_1(0)\| + \|\partial^\alpha \partial_1(0)\|\}
\]
\[
+ \left( \int_0^t e^{-\lambda(t-s)} e^{-4\lambda s} \{P_{s-1}(\|w_1(s)\|_{H^{s+1}} + \|\partial_1(s)\|_{H^{s+1}})\}^4 ds \right)^{\frac{1}{2}}
\]
\[
\leq Ce^{-\lambda t} P_s(\|w_1^0\|_{H^s} + \|\partial_1^0\|_{H^s}).
\]
We thus conclude our claim (6.3). We now turn to the general space-time derivatives $\partial_1 = \partial^\alpha_1 \partial^\alpha_2 \ldots \partial^\alpha_d$. Notice that
\[
\Delta P_1 = -\nabla \cdot \{w_1 \cdot \nabla w_1\},
\]
we thus use equation (6.1) repeatedly to deduce for the temporal derivatives to obtain
\[
\sum_{|\alpha| \leq s} \{\|\partial_1 w_1\| + \|\partial_1 \partial_1\|\} \leq e^{-\lambda t} P_{2s}(\|w_1^0\|_{H^{2s}} + \|\partial_1^0\|_{H^{2s}}).
\]
Here we remark that we need twice as many as spatial derivatives now for the initial data. For the part $e^{-\tau A} \Pi^\perp V_1(t)$ in ker$^\perp A$, we write $V_1(t, \tau) = e^{-\tau A} V_1(t)$, then it is easy to check
\[(6.5) \quad d_t \Pi^\perp V_1 + \frac{1}{\epsilon} A \Pi^\perp V_1 + Q_{2s}(\Pi^\perp V_1, \Pi^\perp V_1) + Q_{3r}(\Pi^\perp V_1, \Pi^\perp V_1) = \bar{\eta} \Delta \Pi^\perp V_1^\epsilon.
\]
Firstly, taking $L^2$ inner product with $\Pi^\perp V_1^\epsilon$, by Lemma A.3 we deduce that
\[
\frac{1}{2} \frac{d}{dt} \|\Pi^\perp V_1^\epsilon\|^2 + \eta\|\nabla \Pi^\perp V_1^\epsilon\|^2 \leq 0,
\]
also by Poincaré inequality, we can get the exponential decay
\[
\|\Pi^\perp V_1^\epsilon\| \leq Ce^{-\lambda t}\|\Pi^\perp V_1^\epsilon(0)\|.
\]
Analogously, taking $H^s$ ($s > 0$) inner product of the averaged system (6.5), we obtain that
\[
\frac{1}{2} \frac{d}{dt} \|\Pi^\perp V_1^\epsilon\|_{H^s} + \eta\|\nabla \Pi^\perp V_1^\epsilon\|_{H^s}^2 + (Q_{2s}(\Pi^\perp V_1^\epsilon, \Pi^\perp V_1^\epsilon), \Pi^\perp V_1^\epsilon)_{H^s} \leq 0,
\]
By the third estimate in Lemma A.3 and Young inequality and the embedding $H^1 \hookrightarrow B^{1/2}$ yields
\[
|(Q_{3r}(\Pi^\perp V_1^\epsilon, \Pi^\perp V_1^\epsilon), \Pi^\perp V_1^\epsilon)_{H^s}| \leq C\|\Pi^\perp V_1^\epsilon\|_{H^{s+1}} \|\Pi^\perp V_1^\epsilon\|_{B^{1/2}} \|\Pi^\perp V_1^\epsilon\|_{H^s}
\]
\[
\leq \frac{\bar{\eta}}{2}\|\Pi^\perp V_1^\epsilon\|_{H^{s+1}} + C\|\Pi^\perp V_1^\epsilon\|_{H^s}^2 \|\Pi^\perp V_1^\epsilon\|_{H^s}^2.
\]
Thus we further get
\[
\frac{1}{2} \frac{d}{dt} \|\Pi^\perp V_1^\epsilon\|_{H^s}^2 + \frac{\bar{\eta}}{2}\|\nabla \Pi^\perp V_1^\epsilon\|_{H^s}^2 \leq C\|\Pi^\perp V_1^\epsilon\|_{H^1}^2 \|\Pi^\perp V_1^\epsilon\|_{H^s}^2.
\]
By Poincaré inequality and Gronwall’s Lemma, we finally arrive
\[
\|\Pi^\perp V_1^\epsilon\|_{H^s}^2 \leq Ce^{-C \int_0^t \|\nabla \Pi^\perp V_1^\epsilon\|_{H^s}^2 ds} \|\Pi^\perp V_1^\epsilon(0)\|_{H^s}^2 e^{-\lambda t} \leq Ce^{-\lambda t}\|\Pi^\perp V_1^\epsilon(0)\|_{H^s}^2.
\]
For the general space-time derivatives $\partial_t$, observing that, for time derivatives
\[ d_t^i Q(U, V) = \sum_{\gamma_1 + \gamma_2 = \gamma} C_{\gamma_1}^i Q(d_t^{\gamma_1} U, d_t^{\gamma_2} V), \]

Multiplying (6.5) by $\partial_t \Pi^t V^c_t$ and employing the interpolation result in
\[ \sum_{|t| \leq s} \| \partial_t \Pi^t V^c_t \| \leq C e^{-\lambda t} \sum_{|t| \leq s} \| \partial_t \Pi^t V^c_t(0) \| \leq C e^{-\lambda t} \sum_{|t| \leq s} \| \partial_t \Pi^t U_t(0) \|. \]

We remark that for the initial data $\partial_t \Pi^t U_t(0)$ should be determined by (4.22). As so far, we have finish the proof of our Theorem 3.1 for $r = 1$.

Assume that $f_1, \ldots, f_r$ have been constructed and satisfied the estimates in our theorem. In fact, $f_{r+1}$ can be constructed by the process in the last section. Our task now is to verify that those estimates (3.5) are valid for the $f_{r+1}$. The first step we estimate the microscopic part $\{I - P\} f_{r+1}$ for each potential cases. We only write down the process for the soft potential cases since the other case is analogous and more simper. By the first line of (4.15) and Lemma A.1, if $\beta = 0$, we have
\[
\frac{1}{2} \| w^{|\gamma|} \partial_t \{I - P\} f_{r+1} \|_{L^2} \leq \left( w^{2|\gamma|} \partial_t L \{I - P\} f_{r+1}, \partial_t \{I - P\} f_{r+1} \right)
\leq \left( w^{2|\gamma| - \frac{7}{2}} (-d_t \partial_t f_{r-1} - \partial_t \nabla_x f_r), w^{2|\gamma| - \frac{7}{2}} \partial_t \{I - P\} f_{r+1} \right)
\leq \left( \| w^{2|\gamma| - \frac{7}{2}} \partial_t \{I - P\} f_{r+1} \| + \| w^{2|\gamma| - \frac{7}{2} + 1} \partial_t \{I - P\} f_{r+1} \| \right)
\times \| w^{|\gamma| + \frac{7}{2}} \partial_t \{I - P\} f_{r+1} \| + \sum_{i+j=r} \left\{ \| w^{|\gamma|} \partial_t f_i \| \| w^{|\gamma|} \partial_t f_j \| \right\}
\| w^{|\gamma|} \partial_t \{I - P\} f_{r+1} \| \nu,
\]

By the induction hypothesis we obtain
\[
(6.6) \quad \| w^{|\gamma|} \partial_t \{I - P\} f_{r+1} \| \nu \leq e^{-\lambda t} \rho_{r+1, l,s} \left( \sum_{1 \leq j \leq r} \| (\rho^0_j, u^0_j, 0^0_j) \|_{H^{2+4(\nu-j)}} \right).
\]

For $\beta > 0$, we get
\[
\frac{1}{2} \| w^{(|\beta|)} \partial_t^\beta \{I - P\} f_{r+1} \|^2 - C_{|\beta|} \| \partial_t \{I - P\} f_{r+1} \|^2
\leq \left( w^{2(|\beta|)} \partial_t^\beta L \{I - P\} f_{r+1}, \partial_t^\beta \{I - P\} f_{r+1} \right)
\leq \left( w^{(|\beta|)} - \frac{7}{2} (-d_t \partial_t^\beta f_{r-1} - \partial_t^\beta \nabla_x f_r), w^{(|\beta|) + \frac{7}{2}} \partial_t^\beta \{I - P\} f_{r+1} \right)
\leq \left( \| w^{(|\beta|) - \frac{7}{2}} \partial_t^\beta \{I - P\} f_{r+1} \| + \| w^{(|\beta|) - \frac{7}{2} + 1} \partial_t^\beta \{I - P\} f_{r+1} \| \right)
\times \| w^{(|\beta|) + \frac{7}{2}} \partial_t^\beta \{I - P\} f_{r+1} \| + \sum_{i+j=r} \left\{ \| w^{|\gamma|} \partial_t^\beta f_i \| \| w^{|\gamma|} \partial_t^\beta f_j \| \right\}
\| w^{(|\beta|) \nu} \partial_t^\beta \{I - P\} f_{r+1} \| \nu,
\]

by the induction hypothesis again and (6.6), we deduce that
\[
\| w^{(|\beta|) \nu} \partial_t^\beta \{I - P\} f_{r+1} \| \nu \leq e^{-\lambda t} \rho_{r+1, l,s} \left( \sum_{1 \leq j \leq r} \| (\rho^0_j, u^0_j, 0^0_j) \|_{H^{2+4(\nu-j)}} \right),
\]

since we can choose $l$ freely.
Finally we need to estimate the remaining $\Pi_{U_{r+1}}$. By the standard energy method, we deduce from the linear Navier-Stokes-Fourier system (4.23) that

$$\frac{1}{2} \frac{d}{dt} \| \partial_t (w_{r+1}, \vartheta_{r+1}) \|^2 + \frac{\min \{ n, \kappa \}}{2} \| \nabla_x \partial_t (w_{r+1}, \vartheta_{r+1}) \|^2 \leq \left( \partial_t (\Pi DU_{m+1} + \Pi R_{m+1}), \partial_t (w_{r+1}, \vartheta_{r+1}) \right)$$

$$+ \left( \| \partial_t \nabla_x (u_1, \theta_1) \|^2 + \| \Pi U_{r+1} \|^2 \right) \| \partial_t (w_{r+1}, \vartheta_{r+1}) \|^2,$$

Here we have used $\frac{\kappa}{2} \| \nabla_x \partial_t w_{r+1} \|^2$ and $\frac{\kappa}{4} \| \nabla_x \partial_t \vartheta_{r+1} \|^2$ to absorb lower order terms of $\partial_t (w_{r+1}, \vartheta_{r+1})$. Notice that by integration by parts

$$\left( \partial_t \text{div} \left[ (w_{r+1} \theta_1, \vartheta_{r+1} u_1) \right], \partial_t (w_{r+1}, \vartheta_{r+1}) \right)$$

$$= - \left( \partial_t \left[ (w_{r+1} \theta_1, \vartheta_{r+1} u_1) \right], \text{div} \partial_t (w_{r+1}, \vartheta_{r+1}) \right)$$

$$\leq \frac{\kappa}{4} \| \nabla_x \partial_t w_{r+1} \|^2 + \frac{\kappa}{4} \| \nabla_x \partial_t \vartheta_{r+1} \|^2 + C \sum_{i \leq s} \| \partial_t (u_1, \theta_1) \|^2 \| \partial_t (w_{r+1}, \vartheta_{r+1}) \|^2.$$

For the estimate of $\left( \partial_t (\Pi DU_{m+1} + \Pi R_{m+1}), \partial_t (w_{r+1}, \vartheta_{r+1}) \right)$, we deduce via an integration again as well as our induction hypothesis that

$$\left( \partial_t (\Pi DU_{m+1} + \Pi R_{m+1}), \partial_t (w_{r+1}, \vartheta_{r+1}) \right) \leq \frac{\min \{ n, \kappa \}}{4} \| \nabla_x \Pi U_{r+1} \|^2 + e^{-2\lambda t} \mathcal{P}^2_{r+1, l, s} \left( \sum_{1 \leq j \leq r} \| (\rho^0_j, u_j^0, \theta_j^0) \|_{H^{2(s+1)+4(r-j)}} \right).$$

We remark that there are $(s+1)$-order derivatives present for from the right-hand of (3.5) for $\{ I - P \} f_{r+1}$ and $\Pi U_{r+1}$. Then we arrived at

$$\frac{1}{2} \frac{d}{dt} \| \partial_t \Pi U_{r+1} \|^2 + \frac{\min \{ n, \kappa \}}{2} \| \nabla_x \partial_t \Pi U_{r+1} \|^2$$

$$\leq C e^{-2\lambda t} \mathcal{P}^2_{r+1, l, s} \left( \sum_{1 \leq j \leq r} \| (\rho^0_j, u_j^0, \theta_j^0) \|_{H^{2(s+1)+4(r-j)}} \right)$$

$$+ C e^{-2\lambda t} \mathcal{P}^2_{r+1, l, s} \left( \sum_{1 \leq j \leq r} \| (\rho^0_j, u_j^0, \theta_j^0) \|_{H^{2(s+1)+4(r-j)}} \right) \| \partial_t \Pi U_{r+1} \|^2.$$

We thus conclude our Theorem 3.1 by Gronwall’s Lemma.

6.2. Proof of Theorem 3.2. In this subsection, we don’t intend to give a detailed proof process of Theorem 3.2 since these are almost the same as Part 6 and 7 in [15]. We will write down some modification to drop the demand of smallness of $\varepsilon$ as in Lemma 6.1, 6.2 and 7.1 of [15]. In fact, for the pure spatial energy estimate for $\partial^\alpha f^\varepsilon$, we borrow the following lemma from [15]:

**Lemma 6.1.** ([15], Lemma 7.1) Assume that $f^\varepsilon$ is a solution of (3.6) and satisfies (3.7), then for any energy functional $\mathcal{E}_{N,0}^{1/2}(f^\varepsilon)$, it holds that

$$\frac{d}{dt} \left\{ C_1 \sum_{|\alpha| \leq N} \| \partial^\alpha f^\varepsilon \|^2 - \varepsilon \delta G_f(t) \right\} + \delta \sum_{|\alpha| \leq N} \left\{ \frac{1}{\alpha!} \| \partial^\alpha \{ I - P \} f^\varepsilon \|_D^2 + \| \partial^\alpha P f^\varepsilon \|_D^2 \right\}$$

$$\leq \left\{ \mathcal{E}_{N,0}^{1/2}(f^\varepsilon) + \mathcal{E}_{N,0}(f^\varepsilon) \right\} \mathcal{D}_{N,0}(f^\varepsilon),$$
where $G_f(t)$ is defined as

$$
- \sum_{|\alpha| \leq N-1} \int_{T^3} \left\{ \langle \partial^\alpha (I - P) f \rangle, \zeta_0 \rangle \cdot \nabla_x \partial^\alpha c - \langle \partial^\alpha (I - P) f \rangle, \zeta_{ij} \rangle \cdot \partial_j \partial^\alpha b^i \right\} dx 
$$

(6.7)

$$
- \sum_{|\alpha| \leq N-1} \int_{T^3} \left\{ \langle \partial^\alpha (I - P) f \rangle, \zeta_0 \rangle \cdot \nabla_x \partial^\alpha a^i - \partial^\alpha b^i \cdot \nabla_x \partial^\alpha a^i \right\} dx,
$$

and $\zeta_0(v)$, $\zeta_{ij}(v)$ and $\zeta_c(v)$ are some fixed linear combinations of the basis $\{\sqrt{\mu}, v_i \sqrt{\mu}, v_i v_j \sqrt{\mu}, v_i |v| \sqrt{\mu} \}$.

For the mixed derivatives energy estimates as well as weight function, we just want to point out that it is unnecessary to demand the smallness of $\epsilon$ in Guo’s proof. For example, for the soft potential cases, in fact when we deal with the term

$$
\sum_{|\beta_1|=1} \left( \frac{1}{2} \partial_{\beta_1} v \cdot \nabla_x \partial^\alpha_{\beta_2-\beta_1} (I - P) \langle f \rangle, w^{2(|\beta|-|\beta_1|)|\gamma|} \partial^\alpha_{\beta_2} (I - P) \langle f \rangle \right)
$$

$$
\leq \frac{1}{\epsilon \sqrt{2}} \left\| w^{(l-|\beta_1|)|\gamma|} \partial^\alpha_{\beta_2} (I - P) \langle f \rangle \right\|_\nu^2 + C \sum_{|\beta_1|=1} \left\| \sum_{|\beta_1|=1} \left\| w^{(l-|\beta_1|)|\gamma|} \partial^\alpha_{\beta_2+\beta_1} (I - P) \langle f \rangle \right\|_\nu^2,
$$

since $|\beta_1| < |\beta|$, the second term above can be estimated in term of pure $x$-derivatives $\partial^\alpha (I - P) f^{\nu}$ with weight via an induction starting from $|\beta| = N$, $N - 1, ..., 1$ as done in Guo’s pioneer work [14]. So we need to add a weighted energy estimates about the pure spatial derivatives to control the term $\| w^{l|\gamma|} \partial^\alpha_{\beta} (I - P) f^{\nu} \|^2$. To do this, we take $\partial^\alpha$ to (3.6), and multiply the result function by $w^{2l|\gamma|} \partial^\alpha (I - P) f^{\nu}$ and integrate over $T^3 \times \mathbb{R}^3$, then sum over $|\alpha| \leq N$ to obtain

$$
\frac{d}{dt} \left\{ \sum_{|\alpha| \leq N} \left\| w^{l|\gamma|} \partial^\alpha (I - P) f^{\nu} \right\|_D^2 \right\} + \delta \sum_{|\alpha| \leq N} \left\{ \frac{1}{\epsilon \sqrt{2}} \left\| w^{l|\gamma|} \partial^\alpha (I - P) f^{\nu} \right\|_D \right\}
$$

$$
\leq \left\{ \varepsilon_{N,I}^{1/2} (f^\nu) + \varepsilon_{N,I} (f^\nu) \right\} \mathcal{D}_{N,I}(f^\nu) + \frac{C}{\epsilon \sqrt{2}} \left\| (I - P) f^{\nu} \right\|_\nu + C \sum_{|\alpha| \leq N} \left\| P f^{\nu} \right\|_\nu.
$$

The hard potential and Landau kernel cases are similar and more simply. Finally we can unite above estimates and Lemma 6.1 and the weighted mixed energy estimates via a suitable linear combination to get

$$
\frac{d}{dt} \left\{ K_1 \left\{ C_1 \sum_{|\alpha| \leq N} \left\| \partial^\alpha f^{\nu} \right\|^2 - \epsilon \delta G_f(t) \right\} + K_2 \sum_{|\alpha| \leq N} \left\| w^{l|\gamma|} \partial^\alpha (I - P) f^{\nu} \right\|_D \right\}
$$

$$
+ \sum_{|\alpha|+|\beta| \leq N \atop |\beta| \geq 1} \left\| w^{l|\gamma|} \partial^\alpha_{\beta} (I - P) f^{\nu} \right\|_D + \mathcal{D}_{N,I}(f^\nu) \leq \left\{ \varepsilon_{N,I}^{1/2} (f^\nu) + \varepsilon_{N,I} (f^\nu) \right\} \mathcal{D}_{N,I}(f^\nu),
$$

for the hard potential cases. And

$$
\frac{d}{dt} \left\{ K_1 \left\{ C_1 \sum_{|\alpha| \leq N} \left\| \partial^\alpha f^{\nu} \right\|^2 - \epsilon \delta G_f(t) \right\} + K_2 \sum_{|\alpha| \leq N} \left\| w^{l|\gamma|} \partial^\alpha (I - P) f^{\nu} \right\|_D \right\}
$$

$$
+ \sum_{|\alpha|+|\beta| \leq N \atop |\beta| \geq 1} \left\| w^{l-|\beta|} \partial^\alpha_{\beta} (I - P) f^{\nu} \right\|_D + \mathcal{D}_{N,I}(f^\nu) \leq \left\{ \varepsilon_{N,I}^{1/2} (f^\nu) + \varepsilon_{N,I} (f^\nu) \right\} \mathcal{D}_{N,I}(f^\nu),
$$

for the hard potential cases. And
We can get the equation for the unknown $g$, which comes from further construction of $(1.1)$ (Lemma 8.1, Lemma 6.2. We establish the similar lemma as follow:

$$ f_{6.3.} $$

We are now in a position to establish the estimate of the weighted mixed-derivatives of corresponding to each kernel case. Thus we deduce (3.8). For the decay rates are totally same as [15], so we omit them and complete our proof of Theorem 3.2.

6.3. **Proof of Theorem 3.3.** In this subsection, we will establish Theorem 3.3. Our argument closely follows the proof of section 8 of [15]. Noticing the remainder equation (3.4) for $f_n^\epsilon(t, \frac{t}{\epsilon}, x, v)$, there are zeroth-order terms in $\epsilon$, $-d_t f_{n-2} - v \cdot \nabla_x f_{n-1}$ and a first-order term in $\epsilon$, $-d_t f_{n-1}$. As in [15], to overcome these singularities, we introduce a new unknown $g_n^\epsilon$ which comes from further construction of the $(n + 1)$th and $(n + 2)$th coefficients $f_{n+1}(t, \frac{t}{\epsilon}, x, v)$ and $f_{n+2}(t, \frac{t}{\epsilon}, x, v)$ by letting $m = n + 1$ and $m = n + 2$ in Theorem 3.1 with zero initial conditions

$$ \rho_{n+1}^0 \equiv \rho_{n+2}^0 \equiv u_{n+1}^0 \equiv u_{n+2}^0 \equiv \theta_{n+1}^0 \equiv \theta_{n+2}^0. $$

We can get the equation for the unknown $g^\epsilon(t, \frac{t}{\epsilon}, x, v) = f_n^\epsilon - f_n - \epsilon f_{n+1} - \epsilon^2 f_{n+2}$:

$$ d_t g^\epsilon + \frac{v}{\epsilon} \cdot \nabla_x g^\epsilon + \frac{1}{\epsilon^2} L g^\epsilon = \epsilon^{n-2} \Gamma (g^\epsilon, g^\epsilon) + \sum_{i=1}^{n+4} \epsilon^{-i-2} \{ \Gamma (g^\epsilon, f_i) + \Gamma (f_i, g^\epsilon) \} + \sum_{i+j \geq n+3} \epsilon^{i+j-n-2} \Gamma (f_i, f_j) - \epsilon \{ d_t f_{n+1} + v \cdot \nabla_x f_{n+2} \} - \epsilon^2 d_t f_{n+2} \equiv h^\epsilon (g^\epsilon) + h^\epsilon (f) \equiv h^\epsilon.

Our purpose is to estimate $g^\epsilon$ instead of $f_n^\epsilon$. Note that $(g^\epsilon, [1, v, |v|^2], \sqrt{\mu}) = 0$. We can establish the similar lemma as follow:

**Lemma 6.2.** *(Lemma 8.1 [15]) Assume that $F^\epsilon(t, \frac{t}{\epsilon}, x, v) = \mu + \sqrt{\mu} \{ \sum_{r=1}^{n-1} \epsilon^r f_r + \epsilon^r f_n^\epsilon \}$ is the solution of the Boltzmann or Landau equation (1.1). Then there exists constant $\lambda > 0$ and some polynomial $P_{\epsilon}(0) = 0$, for any $\zeta > 0$, such that

$$ \frac{\partial}{\partial t} \left\{ C_1 \sum_{|\alpha| \leq N} \| \partial^\alpha g^\epsilon \|^2 - \epsilon \delta G_g(t) \right\} + \hat{\delta} \sum_{|\alpha| \leq N} \left\{ \frac{1}{\epsilon^2} \| \partial^\alpha (I - P) g^\epsilon \|^2 \right\} \leq e^{-\lambda t} P_{\epsilon}(\| (\rho_n^0, u_n^0, \theta_n^0) \|) \left\{ \| (\rho_n^0, u_n^0, \theta_n^0) \| \right\} \left\{ \epsilon^2 + E_{N,1}(g^\epsilon) \right\} + \left\{ E_{N,1}^1(g^\epsilon) + E_{N,1}(g^\epsilon) + \epsilon P_{\epsilon}(\| (\rho_n^0, u_n^0, \theta_n^0) \|) \right\} D_{N,1}(g^\epsilon), $$

where $G_g(t)$ is similarly defined as (6.7) corresponding to $g^\epsilon$ and $\epsilon$ are sufficiently small.

We are now in a position to establish the estimate of the weighted mixed-derivatives of the remaining $\partial^\beta (I - P) g^\epsilon$ with suitable weight function. We will divide our proof in two different cases.
6.3.1. Hard potential case. We take $\partial^2_\beta \{I-P\}g^\epsilon$ to the equation (6.8) to obtain,
\begin{equation}
\begin{aligned}
dt \partial^2_\beta \{I-P\}g^\epsilon + \frac{z}{\epsilon} \cdot \nabla_x \partial^2_\beta \{I-P\}g^\epsilon + \frac{1}{\epsilon} \partial^2_\beta L\{I-P\}g^\epsilon \\
+ dt \partial^0_\betaPg^\epsilon + \frac{z}{\epsilon} \cdot \nabla_x \partial^0_\beta Pg^\epsilon + \frac{1}{\epsilon} \sum_{\beta_i > 0} C^0_{\beta_i} \partial_{\beta_i} \nu \cdot \nabla_x \partial^0_{\beta-\beta_1} = \partial^2_\beta h^\epsilon.
\end{aligned}
\end{equation}

Taking inner product with $w^2 \partial^2_\beta \{I-P\}g^\epsilon$, we obey the same argument as in the proof of Theorem 3.2. It suffices to estimate the nonlinear term ($\partial^2_\beta h^\epsilon, w^2 \partial^0_\beta \{I-P\}g^\epsilon$). For the second line in (6.8), i.e. $h^\epsilon(f)$ can be bounded by
\begin{equation}
(\partial^2_\beta h^\epsilon(f), w^2 \partial^0_\beta \{I-P\}g^\epsilon) \leq \epsilon e^{-\lambda_\beta t} \rho_\beta \left( \left\| \left( \rho^0_n, u^0_n, \theta^0_n \right) \right\|_N \right) \| w^2 \partial^0_\beta \{I-P\}g^\epsilon \|_\nu \\
\leq \epsilon^2 e^{-\lambda_\beta t} \rho_\beta \left( \left\| \left( \rho^0_n, u^0_n, \theta^0_n \right) \right\|_N \right) D^1_{N,t}(g^\epsilon).
\end{equation}

For the third line of (6.8), i.e. $h^\epsilon(g^\epsilon)$, by Lemma A.1, and noticing the fact $\| w^2 \partial^0_\beta Pg^\epsilon \|_\nu \sim \| w^2 \partial^0_\beta Pg^\epsilon \|$, which can be bounded by $E_{N,t}(Pg^\epsilon)$, then we have
\begin{equation}
(\partial^2_\beta h^\epsilon(g^\epsilon), w^2 \partial^0_\beta \{I-P\}g^\epsilon) \leq \epsilon^{1/2} E_{N,t}^{1/2}(g^\epsilon) \| w^2 \partial^0_\beta g^\epsilon \|_\nu \| w^2 \partial^0_\beta \{I-P\}g^\epsilon \|_\nu \\
+ \sum_{i=1}^{n+2} \epsilon^{i-1} \left\{ E_{N,t}^{1/2} \left( \left\| w^2 \partial^{i}_\beta \{I-P\}g^\epsilon \right\|_\nu \right) \right\} \left\{ \frac{1}{\epsilon} \| w^2 \partial^0_\beta \{I-P\}g^\epsilon \|_\nu \right\}
\end{equation}

Collecting above estimate and combining with Lemma 6.2 by a suitable linear combination, we can arrive at, for any $\delta > 0$,
\begin{equation}
\frac{d}{dt} \left\{ K_1 \left( C_1 \sum_{|\alpha| \leq N} \| \partial^\alpha g^\epsilon \|^2 - \epsilon \delta G_g(t) \right) + \sum_{|\alpha|+|\beta| \leq N} \| w^2 \partial^\beta_\beta \{I-P\}g^\epsilon \|^2 \right\} + D_{N,t}(g^\epsilon)
\end{equation}

By firstly choosing $\zeta$ small enough, then $\epsilon$ small enough, and also $E_{N,t}(g^\epsilon) \leq M$ sufficiently small, we further get
\begin{equation}
\frac{d}{dt} E_{N,t}(g^\epsilon) + D_{N,t}(g^\epsilon) \leq C e^{-\lambda \beta t} \left( \left\| \left( \rho^0_n, u^0_n, \theta^0_n \right) \right\|_N \right) \left\{ \epsilon^2 + E_{N,t}(g^\epsilon) \right\}.
\end{equation}

As in [15], by multiplying the integrator $e^{-\lambda \beta t} \left( \left\| \left( \rho^0_n, u^0_n, \theta^0_n \right) \right\|_N \right)$ $E_{N,t}(g^\epsilon)$ and then integrating over time yields
\begin{equation}
E_{N,t}(g^\epsilon) \leq C e^{\lambda \beta t} \left( \left\| \left( \rho^0_n, u^0_n, \theta^0_n \right) \right\|_N \right) \left\{ E_{N,t}(g^\epsilon)(0) + C \epsilon^2 \right\} \leq \frac{M}{2},
\end{equation}
by choosing $\epsilon$ sufficiently small. On the other hand, by Theorem 3.1, we enjoy

$$E_{N,I}(f_{n+1} + \epsilon^2 f_{n+2}) \leq C\epsilon^2 e^{-\lambda t} \mathcal{P}\left(\| (\rho^0_n, u^0_n, \theta^0_n) \|_N \right).$$

Consequently, we infer (3.11) for the hard potential case from (6.12) and (6.13) since

$$f_n^\epsilon - f_n = g^\epsilon - \{ \epsilon f_{n+1} + \epsilon^2 f_{n+2} \}.$$

For the decay (3.12), by noticing for the hard potential case, we have the fact that $\mathcal{D}_{N,I} \geq \lambda' E_{N,I}$. By (6.11) and (6.12) also the Gronwall inequality yield (3.12).

### 6.3.2. Soft potential and Landau cases.

We take inner product with $w^{2(l-[\beta])|\gamma|} \partial^\alpha_\beta \{ I - P \} g^\epsilon$ to (6.9) for the soft potential case and $w^{2(l-[\beta])} \partial^\alpha_\beta \{ I - P \} g^\epsilon$ for the Landau case. It also suffices to estimate the nonlinear term $h^\epsilon$. Firstly, for the soft potential case, one has

$$\langle \partial^\alpha_\beta h^\epsilon (f), w^{2(l-[\beta])|\gamma|} \partial^\alpha_\beta \{ I - P \} g^\epsilon \rangle \leq \epsilon e^{-\lambda t} \mathcal{P}_\zeta \left(\| (\rho^0_n, u^0_n, \theta^0_n) \|_N \right) \| w^{l-[\beta]|\gamma|} \partial^\alpha_\beta \{ I - P \} g^\epsilon \|_\nu,$$

while for the Landau case

$$\langle \partial^\alpha_\beta h^\epsilon (f), w^{2(l-[\beta])} \partial^\alpha_\beta \{ I - P \} g^\epsilon \rangle \leq \epsilon e^{-\lambda t} \mathcal{P}_\zeta \left(\| (\rho^0_n, u^0_n, \theta^0_n) \|_N \right) \| w^{l-[\beta]} \partial^\alpha_\beta \{ I - P \} g^\epsilon \|_\sigma,$$

which are both bounded by $\epsilon^2 e^{-\lambda t} \mathcal{P}_\zeta \left(\| (\rho^0_n, u^0_n, \theta^0_n) \|_N \right) \mathcal{D}_{N,I}^{1/2}(g^\epsilon)$. One the other hand, we have similarly as (6.10) that

$$\langle \partial^\alpha_\beta h^\epsilon (g^\epsilon), w^{2(l-[\beta])|\gamma|} \partial^\alpha_\beta \{ I - P \} g^\epsilon \rangle \leq \mathcal{E}_{N,I}^{1/2}(g^\epsilon) \| w^{l-[\beta]|\gamma|} \partial^\alpha_\beta \{ I - P \} g^\epsilon \|_\nu,$$

while for the Landau case

$$\langle \partial^\alpha_\beta h^\epsilon (g^\epsilon), w^{2(l-[\beta])} \partial^\alpha_\beta \{ I - P \} g^\epsilon \rangle \leq \mathcal{E}_{N,I}^{1/2}(g^\epsilon) \| w^{l-[\beta]} \partial^\alpha_\beta \{ I - P \} g^\epsilon \|_\sigma,$$

by observing that $\| w^{l-[\beta]|\gamma|} \partial^\alpha_\beta g^\epsilon \|_\nu \sim \| w^{l-[\beta]} \partial^\alpha_\beta g^\epsilon \|_\sigma$, thus can be bounded by $\mathcal{E}_{N,I}^{1/2}(g^\epsilon)$. Hence, as in the proof for the hard potential case, the inner products above are bounded by

$$\epsilon \mathcal{E}_{N,I}^{1/2}(g^\epsilon) \mathcal{D}_{N,I}(g^\epsilon) + e^{-\lambda t} \mathcal{P}_\zeta \left(\| (\rho^0_n, u^0_n, \theta^0_n) \|_N \right) \mathcal{E}_{N,I}(g^\epsilon) + \{ \zeta + \epsilon \mathcal{P}_\zeta \left(\| (\rho^0_n, u^0_n, \theta^0_n) \|_N \right) \} \mathcal{D}_{N,I}(g^\epsilon),$$

for any $\zeta > 0$. Repeating the same argument as in the hard potential case, we conclude (3.11) for soft potential case and Landau case. For the derivation of (3.13) and (3.14), we refer to [15] in a totally same way and we omit it for the sake of simplicity.

### APPENDIX A. SOME LEMMAS

We collect some lemmas in this appendix. We first summarize some estimates for the collision operators $\mathcal{L}$ and $\Gamma$. 
Lemma A.1. [15] There exist $C_{|\beta|} > 0$, such that for hard potential case with $\gamma \geq 0$, 
\[
\left( w^{2l} \partial_\beta^2 \mathcal{L} g, \partial_\beta^2 g \right) \geq \frac{1}{2} \| w^l \partial_\beta g \|_\nu^2 - C_{|\beta|} \| \partial_\beta g \|_\nu^2,
\]
\[
\left( w^{2l} \partial_\beta \Gamma(g_1, g_2), \partial_\beta g_3 \right) \leq C \{ \| w^l \partial_\beta^2 g_1 \| \| w^l \partial_\beta^2 g_2 \| \| w^l \partial_\beta^2 g_1 \| \| w^l \partial_\beta^2 g_2 \| \| w^l \partial_\beta^2 g_1 \| \| w^l \partial_\beta^2 g_2 \| \}
\]
\times \| w^l \partial_\beta^2 g_3 \|_\nu.
\]

For the soft potential case $-D < \gamma < 0$, for $l \geq 0$, 
\[
\left( w^{2l(\gamma|\beta|)} \partial_\beta \mathcal{L} g, \partial_\beta g \right) \geq \frac{1}{2} \| w^{l(|\beta|)} \partial_\beta g \|_\sigma^2 - C_{|\beta|} \| \partial_\beta g \|_\sigma^2,
\]
\[
\left( w^{2l(\gamma|\beta|)} \partial_\beta \Gamma(g_1, g_2), \partial_\beta g_3 \right) \leq C \{ \| w^{l(|\beta|)} \partial_\beta g_1 \| \| w^{l(|\beta|)} \partial_\beta g_2 \| \| \partial_\beta g_3 \| \sigma \}
\]
\times \| w^{l(|\beta|)} \partial_\beta g_3 \|_\sigma.
\]

where all above summations are over $|\alpha| + |\beta| \leq N$ and $|\alpha_1| + |\beta_1| \leq |\alpha| + |\beta| \leq \frac{N}{2} + 4$ and $\alpha_2 \leq \alpha$. And we remark that $C_{\beta} = 0$ if $\beta = 0$.

Lemma A.2. In the averaging, the oscillatory integral
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-s\sqrt{-1}A(k)} \phi(s) ds,
\]
for any integrable function $\phi(s)$ vanishes when $A(k) \neq 0$. The only non-zero contributions that survive the averaging process are the resonance $A(k) = 0$. Here $A(k)$ is any polynomial of $k$ such that the above integral is integrable. For the proof, which can be found in [8].

Lemma A.3. For $s \geq 0$, we have the identities and estimate
\[
(Q_{2r}(\Pi \Pi^\perp V), \Pi \Pi^\perp V)_{H^s} = 0 \text{ for } s \geq 0;
\]
\[
(Q_{3r}(\Pi \Pi^\perp V), \Pi \Pi^\perp V)_{L^2} = 0;
\]
\[
\| Q_{2r}(\Pi \Pi^\perp W) \|_{H^s} \leq C \| W \|_{B^{1/2}} \| W \|_{H^{s+1}} + \| W \|_{B^{1/2}} \| V \|_{H^{s+1}}, \text{ for } s > 0,
\]

where $B^{1/2}$ is the standard Besov space.

Proof. We only prove the first properties. The others can be proven in a similar way by Dauchin [6]. We employ the symmetry of $Q_{2r}$. Noticing that $\Phi_k^\alpha = \Phi_{-k}^\alpha$ and that $U_k^\alpha = U_{-k}^\alpha$. Then
\[
(Q_{2r}(\Pi \Pi^\perp V), \Pi \Pi^\perp V)_{H^s} = \sqrt{-1} \sum_{\gamma, m} \sum_{k+l=m \atop \alpha_{sg}(k) = \gamma_{sg}(m)} U_k^\gamma \bar{U}_{-m}^\gamma \| m \|^2 \left[ \frac{k \cdot \hat{m}}{|k||m|} + \frac{1}{m} \right] \hat{w}_l \cdot m + \frac{1}{2} \sqrt{\frac{m+2}{d-4 \alpha_{sg}(k)}} \cdot (\hat{v}_l m).
\]
In the above summation, exchange $\alpha$ and $\gamma$, and $k$ to $-m$, $m$ to $-k$. Notice that under the change of index, the relation $l = m - k$ is invariant, so

\[
(Q_{2r}(\Pi V, \Pi V), \Pi V)_{H^s} = \sqrt{-1} \sum_{\gamma, m} \sum_{k+l=m, |k|=|m|} \hat{U}_k \hat{U}_\gamma \hat{\vartheta}_m \|m\|^{2s} \left[ - \left( \frac{k}{|k|} m + \frac{1}{2m} \right) \hat{\vartheta}_m \cdot m - \frac{1}{2} \sqrt{\frac{D+2}{D}} \alpha s g(k) \frac{k}{|k|} \cdot (\hat{\vartheta}_m m) \right] = -(Q_{2r}(\Pi V, \Pi V), \Pi V)_{H^s},
\]

which finish our proof. We remark that here we have used the fact $\hat{\vartheta}_m \cdot m = \hat{\vartheta}_m \cdot k$ since $w$ is divergence-free. \hfill $\square$

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