GRAVITATIONAL ENERGY

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Gravity is the unique interaction which is universal and attractive, distinctive properties directly connected with energy. Identifying a good expression which describes the (quasi-)local energy-momentum of gravitating systems is still an outstanding fundamental puzzle. The traditional pseudotensor approach is considered here along with the more modern quasi-local idea. Using a covariant Hamiltonian boundary-term approach clarifies the geometric and physical ambiguities. Certain criteria can be used as theoretical tests of any proposed quasi-local energy-momentum expression, including positivity, the spatial and null asymptotic limit, and the small region limit. The argument for the positive energy requirement is recalled and some positive energy proofs are noted. Positivity in general is a very strong criterion, but it is not so easy to prove or disprove. Positivity in the small vacuum region limit is simpler, and is also a quite strong test. In particular none of the traditional pseudotensors passes this test. Two natural quasi-local expressions and some other contrived ones do satisfy this small region requirement. The natural expressions have a positive energy proof for finite regions. Conversely, circumstances in which it is appropriate for the energy to be negative are noted. Our covariant Hamiltonian boundary term quasi-local gravitational energy-momentum expression requires, on the boundary, a choice of a displacement vector and field reference values. We have proposed obtaining the reference values from an energy-optimized isometric embedding of the 2-boundary into Minkowski space. This gives reasonable results at least for spherically symmetric regions.

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1. Two Special properties of gravity

The presently known physical interactions: strong, weak, electromagnetic and gravity can all be formulated as local gauge theories (we cite just a few treatments[1–3]). Nevertheless gravity is quite special: it is the only interaction that is *universal* and *purely attractive*. These two *unique* properties of gravity have deep significance in the scheme of things; they are not at all accidental but rather are essential consequences of fundamental principles. They naturally play key roles in many situations.

1.1. *Attraction*

The *purely attractive* property of gravity is a natural requirement following from the fundamental physical principles of thermodynamics and stability (e.g., no perpetual motion, no infinite source of energy) as we briefly argue.

Consider a gravitating system: *repulsion* would be caused by a *negative* mass. Can a negative mass exist? Suppose there were a positive and negative mass pair. Then they would attract and repel each other, self accelerating to a high speed, gaining unlimited kinetic energy that could be extracted—contrary to a fundamental thermodynamic principle. Thus physically there should only be positive masses which attract.

Attraction is associated with *positive energy*. An isolated gravitating system is expected to settle down to its equilibrium state. The gravity field should then asymptotically be Newtonian and not dynamic. The energy of such a system can be determined from its effective Newtonian mass, found from the Kepler orbit of a distant test particle using the 1-2-3 law: $(GM)^1 = \omega^2 a^3$. A bound orbit means attractive gravity, a positive mass, and, via Einstein's famous relation $E = Mc^2$, a positive energy.

Moreover, suppose an isolated system could have negative energy, we could then combine such systems to make one with a negative energy of arbitrarily large magnitude. However, since physical systems will naturally spontaneously radiate energy until they reach their lowest energy state, if unlimited negative energy states were allowed that would then permit systems to radiate an infinite amount of energy. Physical systems are unstable unless there is a non-negative lower bound to energy. Consequently (under appropriate conditions) energy must be positive and gravity is fundamentally purely attractive.

This property should be satisfied by any acceptable gravity theory. Hence it can be used as a test to evaluate proposed gravity theories.

1.1.1. *Test and proof*

Positive energy is a strong test: it is not easy to create a relativistic theory which satisfies this requirement under all conditions.[4–6] For an acceptable gravity theory it is important to have a proof that the energy is positive for every appropriate solution; for a given theory finding such a proof may not be so simple. For Einstein's General Relativity (GR), people worked seriously on trying to prove positive energy for at least 20 years before Schoen and Yau[7] succeeded (by an indirect geometric argument). Soon thereafter Witten (with the aid of a spinor field) presented his much more direct proof,[8–10] other proofs followed later, including some by the present author.[11–14] Thus GR passes this important test.

1.1.2. *Dark energy*

Now a dark cloud has been cast on our thesis. About 12 years ago it was announced that observations indicate that, contrary to all expectations, the expansion of the universe is not slowing down. The scale of the universe is expanding faster and faster, accelerating. There is some kind of global repulsion—a challenge to our thesis that gravity is purely attractive. The cause of this acceleration has been given a name: *dark energy*. Dark energy could be a (quite small) positive cosmological constant or perhaps some unusual type of matter with an effective negative pressure which causes the repulsion (in the usual cosmological model in GR the acceleration is given by $\ddot{a}/a = -(4\pi G/3)(\rho + 3P) + \Lambda$). However—unless perhaps the dark energy comes from gravity itself (which can happen in some alternatives to GR theory[15,16])—gravity itself still would have positive energy and would still be regarded as fundamentally attractive.

1.2. ***Universal***

Gravity is *universal*. The source of gravity is all matter and all interaction fields—including itself. In Newton's theory mass density produced gravity. From Einstein we know that "energy" is equivalent to mass. Hence energy density should also produce gravity. Moreover, from relativity we learn that energy is a part of a 4-vector of energy-momentum. Thus, relativistically, the source of gravity should be the *total energy-momentum density*.

Energy-momentum should be conserved. When sources interact with the gravity they can exchange (and this happens *locally*) energy-momentum with the gravitational field (e.g., recall how the space probes Voyager and Pioneer acquired the energy to travel so far from the sun). Note also that the binary pulsar provides indirect evidence for both gravitational waves and gravitational energy. Moreover Bondi[17,18] has presented a compelling theoretical argument that (even in Newtonian theory) energy can be transferred through empty space via the gravitational field. A dramatic illustration of this is Io's volcanoes being powered by Jupiter's tidal heating.[19–21]

2. Gravitational local energy-momentum density

Thus gravity itself should have some kind of local energy-momentum density—which should also produce gravity. Hence gravity should be inherently non-linear, and truly universal: it affects and is affected by everything, including itself. (Note that both of the special properties of the gravitational interaction: universal and attractive, are associated with energy.)

Einstein included energy conservation in his search for his gravitational equations. In fact he had his expression for gravitational energy even before he found the correct field equations for his GR theory.[22] However this expression for the energy-momentum density—as well as many others proposed later—has a problematical feature: it is not a true tensor, it is inherently reference frame dependent. Energy-momentum *pseudotensors* can be obtained via the Lagrangian using Noether symmetry (and are thus subject to the usual Noether current ambiguity) or via rearranging field equations (which naturally includes a similar ambiguity).[23]

2.1. *Canonical energy momentum tensor*

Let us briefly review a simple construction of the canonical energy momentum tensor. From the Lagrangian density $\mathcal{L} = \mathcal{L}(\varphi^A, \partial_\mu \varphi^A)$ one obtains the key variational formula

$$\delta\mathcal{L} = \partial_\mu(\delta\varphi^A P^\mu{}_A) + \delta\varphi^A \frac{\delta\mathcal{L}}{\delta\varphi^A}, \tag{1}$$

which implicitly determines the Euler-Lagrange expression and the canonical momentum:

$$\frac{\delta\mathcal{L}}{\delta\varphi^A} := \frac{\partial\mathcal{L}}{\partial\varphi^A} - \partial_\mu P^\mu{}_A, \qquad P^\mu{}_A := \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi^A}. \tag{2}$$

When (1) is integrated over a region with fixed variations on the boundary, from Hamilton's principle for extreme action we get our field equations: the vanishing of the Euler-Lagrange expressions. For invariance under translation along the coordinate directions (assuming that $\mathcal{L}$ depends on position only through the fields) the relation (1) should identically hold with $\delta \to \partial_\nu$:

$$\partial_\nu\mathcal{L} \equiv \partial_\mu(\partial_\nu\varphi^A P^\mu{}_A) + \partial_\nu\varphi^A \frac{\delta\mathcal{L}}{\delta\varphi^A}. \tag{3}$$

This, in turn, shows that the *canonical energy-momentum tensor*,

$$T^\mu{}_\nu := P^\mu{}_A \partial_\nu\varphi^A - \delta^\mu_\nu\mathcal{L}, \tag{4}$$

satisfies

$$\partial_\mu T^\mu{}_\nu \equiv -\frac{\delta\mathcal{L}}{\delta\varphi^A}\partial_\nu\varphi^A. \tag{5}$$

Consequently, "on shell" (i.e., when the field equations are satisfied) the energy-momentum "current" density $T^\mu{}_\nu$ has vanishing divergence, and hence the energy-momentum within a region,

$$p_\nu(V) := \int_V T^\mu{}_\nu d\Sigma_\mu, \tag{6}$$

is *conserved* (i.e., its change is determined by a flux through the boundary).

It is important to note, however, that this energy-momentum density is not the unique quantity of this kind, since

$$T'^\mu{}_\nu := T^\mu{}_\nu + \partial_\lambda V_\nu{}^{[\mu\lambda]} \tag{7}$$

is also likewise likewise divergence free—and thus defines a (generally different) conserved energy-momentum value—for any choice of $V_\nu{}^{[\mu\lambda]}$. As we shall see later (for electodynamics as a specific example), one cannot just always settle for the canonical energy-momentum tensor that follows directly from the Lagrangian; one may want to exploit this freedom to obtain within this class of objects a more suitable energy-momentum density.

2.2. *Einstein and Møller pseudotensors*

Turning to gravity, the Hilbert action $\mathcal{L}_\mathrm{H} := R\sqrt{-g}$ is not suitable for immediate application of the above procedure, as it depends on second derivatives of the dynamical field. Using the metric as the dynamical variable, from $\mathcal{L}_\mathrm{H}(g, \partial g, \partial\partial g)$, one can remove a total divergence to obtain the Einstein Lagrangian:

$$\mathcal{L}_\mathrm{E}(g, \partial g) := \mathcal{L}_\mathrm{H} - \text{total divergence}, \tag{8}$$

which gives the same field equations. The associated canonical energy-momentum tensor is the *Einstein pseudotensor*.[24]

Instead one may use as the dynamic variable an orthonormal frame $e^\alpha{}_j$, where $g_{ij} = \eta_{\alpha\beta}e^\alpha{}_i e^\beta{}_j$ and $\eta_{\alpha\beta} = \mathrm{diag}(-1, +1, +1, +1)$ is the Minkowski metric. Removing a certain total divergence gives the Møller Lagrangian:

$$\mathcal{L}_\mathrm{M}(e, \partial e) := \mathcal{L}_\mathrm{H} - \text{total divergence}. \tag{9}$$

Now the canonical energy-momentum tensor construction using $e^\alpha{}_i$ as the dynamical variable gives the Møller 1961 tetrad-teleparallel pseudotensor.[25]

2.3. *Other famous pseudotensors*

All the classical pseudotensors can be obtained by suitably rearranging Einstein's equation:[23,24,26–30]

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{10}$$

(where $\kappa := 8\pi G/c^4$). One may choose a *superpotential* $U^{\mu\lambda}{}_{\nu} \equiv U^{[\mu\lambda]}{}_{\nu}$ and *define* the related gravitational energy-momentum density pseudotensor by

$$2\kappa t^{\mu}{}_{\nu} := \partial_\lambda U^{\mu\lambda}{}_{\nu} - 2|g|^{\frac{1}{2}} G^{\mu}{}_{\nu}. \tag{11}$$

Einstein's equation now takes the form

$$2\kappa \mathcal{T}^{\mu}{}_{\nu} := 2\kappa(|g|^{\frac{1}{2}} T^{\mu}{}_{\nu} + t^{\mu}{}_{\nu}) = \partial_\lambda U^{\mu\lambda}{}_{\nu}. \tag{12}$$

Because of the antisymmetry of the superpotential the total energy-momentum density complex is automatically conserved: $\partial_\mu \mathcal{T}^{\mu}{}_{\nu} \equiv 0$; this is a true conservation law, in contrast to

$$\nabla_\mu \sqrt{-g} T^{\mu}{}_{\nu} \equiv \partial_\mu(\sqrt{-g} T^{\mu}{}_{\nu}) - \Gamma^{\lambda}{}_{\nu\mu} T^{\mu}{}_{\lambda} = 0, \tag{13}$$

which shows the change of material energy-momentum due to the gravitational intereaction.

There are some variations on the above formulation. The classical pseudotensorial total energy-momentum density complexes all follow from superpotentials according to one of the patterns

$$2\kappa \mathcal{T}^{\mu}{}_{\nu} = \partial_\lambda U^{\mu\lambda}{}_{\nu}, \quad 2\kappa \mathcal{T}^{\mu\nu} = \partial_\lambda U^{\mu\lambda\nu}, \quad 2\kappa \mathcal{T}^{\mu\nu} = \partial_{\alpha\beta} H^{\alpha\mu\beta\nu}, \tag{14}$$

where the superpotentials have certain symmetries which automatically guarantee conservation: specifically $U^{\mu\lambda}{}_{\nu} \equiv U^{[\mu\lambda]}{}_{\nu}$, $U^{\mu\lambda\nu} \equiv U^{[\mu\lambda]\nu}$, while $H^{\alpha\mu\beta\nu}$ has the symmetries of the Riemann tensor (this latter form guarantees also a good formula for angular momentum). In particular the Einstein total energy-momentum density follows from the Freud superpotential[31]

$$U_{\mathrm{F}}^{\mu\lambda}{}_{\nu} := -|g|^{\frac{1}{2}} g^{\beta\sigma} \Gamma^{\alpha}{}_{\beta\gamma} \delta^{\mu\lambda\gamma}_{\alpha\sigma\nu}; \tag{15}$$

while the Bergmann-Thompson,[32] Landau-Lifshitz,[33] Papapetrou,[34] Weinberg[35,37] and Møller[38] expressions can be obtained respectively from

$$U_{\mathrm{BT}}^{\mu\lambda\nu} := g^{\nu\delta} U_{\mathrm{F}}^{\mu\lambda}{}_{\delta}, \tag{16}$$

$$U_{\mathrm{LL}}^{\mu\lambda\nu} := |g|^{\frac{1}{2}} U_{\mathrm{BT}}^{\mu\lambda\nu}, \quad \text{equivalently} \quad H_{\mathrm{LL}}^{\alpha\mu\beta\nu} := |g| \delta^{\mu\alpha}_{ma} g^{a\beta} g^{m\nu}, \tag{17}$$

$$H_{\mathrm{P}}^{\alpha\mu\beta\nu} := \delta^{\mu\alpha}_{ma} \delta^{\nu\beta}_{nb} \bar{g}^{ab} (|g|^{\frac{1}{2}} g^{mn}), \tag{18}$$

$$H_{\mathrm{W}}^{\alpha\mu\beta\nu} := \delta^{\mu\alpha}_{ma} \delta^{\nu\beta}_{nb} |\bar{g}|^{\frac{1}{2}} \bar{g}^{ab} (-\bar{g}^{mc} \bar{g}^{nd} + \frac{1}{2} \bar{g}^{mn} \bar{g}^{cd}) g_{cd}, \tag{19}$$

$$U_{\mathrm{M58}\nu}^{\mu\lambda} := -|g|^{\frac{1}{2}} g^{\beta\sigma} \Gamma^{\alpha}{}_{\beta\nu} \delta^{\mu\lambda}_{\alpha\sigma} \equiv |g|^{\frac{1}{2}} g^{\beta\mu} g^{\lambda\delta} (\partial_\beta g_{\delta\nu} - \partial_\delta g_{\beta\nu}). \tag{20}$$

(all indicies in these expressions refer to spacetime and range from 0 to 3, otherwise our conventions follow MTW.)

What criteria can one use to decide which, if any, of these expressions are satisfactory descriptions of energy-momentum? Perhaps the most basic property is that, for asymptotically flat space, the expression should give the desired ADM[36] and linearized theory values[37] for the energy and momentum at spatial infinity. It turns out that all of these famous pseudotensors—*except* for the Møller 1958 expression—are satisfactory at spatial infinity. (There is also a requirement concerning angular momentum; angular momentum will not be considered in detail in this report.)

2.4. *The small sphere limit*

Another requirement concerns the limiting value assigned to a small region. Taylor expand the pseudotensor obtained from the superpotential. According to the *equivalence principle* one should get— to zeroth order—the matter energy-momentum. This requirement turns out to be even weaker than getting good asymptotic values; once again all the famous pseudotensors except the Møller 1958 expression satisfy this property.

Continuing the expansion to higher order, *in vacuum* the first nonvanishing contribution should appear at second order. It has been argued[39] that to this order the expression should be proportional to the Bel-Robinson tensor:[37,39,40]

$$\mathcal{T}^{\mu\nu} \propto B^{\mu\nu}{}_{\alpha\beta} x^\alpha x^\beta. \tag{21}$$

Why? Because the Bel-Robinson tensor assures *positive energy* (and in fact the *dominant energy condition* for the small region energy-momentum vector). This requirement is a strong constraint.

For holonomic frames, using Riemann normal coordinates, *none* of the traditional pseudotensors satisfies this requirement[29,30] (it is satisfied by certain contrived linear combinations of the classical pseudotensors[29,30] as well as by many new pseudotensors which we constructed.[29,41]). For *orthonormal frames*, using Riemann normal coordinates and *normal frames*:[42]

$$\vartheta^\alpha_i = \delta^\alpha_i, \qquad \partial_j \vartheta^\alpha{}_k = 0, \tag{22}$$

$$\Gamma^\alpha{}_{\beta k} = 0, \qquad \partial_j \Gamma^\alpha{}_{\beta k} = \frac{1}{2} R^\alpha{}_{\beta jk}, \tag{23}$$

we found that the Møller 1961 tetrad/teleparallel energy-momentum pseudotensor[25] *does not* satisfy this requirement. However, both our favorite *quasi-local* expression (to be discussed below) as well as the *translational* *gauge current* associated with the *teleparallel* formulation of GR[43] do satisfy this important small vacuum region "positivity" requirement.[44]

2.5. *Doubts*

Some have questioned the whole idea of trying to find a local energy density for gravitating systems. We note just two examples: (1) Cooperstock,[45] has argued that the gravitational energy-momentum density vanishes outside of matter. (2) A very influential textbook states: *Anyone who looks for a magic formula for "local gravitational energy-momentum" is looking for the right answer to the wrong question. Unhappily, enormous time and effort were devoted it the past to trying to "answer this question" before investigators realized the futility of the enterprise."*[37]

Certainly, the serious ambiguities of the pseudotensor approach must be recognized. First, given all the above classical pseudotensors along with the infinite number of possible ones constructed from all the possible choices of superpotentials (which essentially corresponds to the freedom noted in Eq. (7) above), which, if any, gives the true energy-momentum localization? Second, since pseudotensors are inherently reference frame dependent objects, which reference frame gives the physically correct localization?

Below it will be argued that these pseudotensor ambiguities are much mitigated by the Hamiltonian approach.

3. Geometry and gravity: quasi-local energy-momentum

Gravity is necessarily connected with geometry. The source of gravity is energy-momentum. By Noether's first theorem associating physically conserved quantities with symmetry, energy-momentum is related to the translation symmetry of space-time geometry.

According to the *equivalence principle* gravity cannot be detected at a point. There is thus, when it comes to energy, a fundamental incompatibility between the requirements of the equivalence principle and the *principle of general covariance*. There is simply no suitable generally covariant expression which can completely describe the local energy-momentum density for gravity. A consequence is that *gravitational energy-momentum*—and hence the energy-momentum of gravitating systems—and hence the energy-momentum of *all physical systems*—is *fundamentally non-local*. The modern idea is that all physical energy-momentum is *quasi-local*, i.e., it is associated with a closed 2-surface (and not with a density at a point).

4. Pseudotensors and the Hamiltonian boundary term

Choose any superpotential $U^{\nu\lambda}{}_{\mu} \equiv U^{[\nu\lambda]}{}_{\mu}$, and use it to split the Einstein tensor, thereby defining the associated gravitational energy-momentum pseudotensor via Eq. (11). Einstein's equation, $G^{\mu}{}_{\nu} = \kappa T^{\mu}{}_{\nu}$, is thus transformed into (12), a form with the total effective energy-momentum pseudotensor as source.[23,26,46,47]

Fix a coordinate system and (for later convenience) multiply (12) by a vector field N^{μ} having constant components in this system. Now integrate over a finite spacetime 3-volume Σ to get the total energy-momentum associated with the chosen vector field:

$$\begin{aligned} -N^{\mu}p_{\mu} &:= -\int_{\Sigma} N^{\mu}\mathcal{T}^{\nu}{}_{\mu}\sqrt{-g}(d^3x)_{\nu} \\ &\equiv \int_{\Sigma}\big[N^{\mu}\sqrt{-g}(\frac{1}{\kappa}G^{\nu}{}_{\mu} - T^{\nu}{}_{\mu}) - \frac{1}{2\kappa}\partial_{\lambda}(N^{\mu}U^{\nu\lambda}{}_{\mu})\big](d^3x)_{\nu}. \end{aligned} \tag{24}$$

This expression has the form

$$H(N,\Sigma) = \int_{\Sigma} N^{\mu}\mathcal{H}_{\mu} + \oint_{S=\partial\Sigma}\mathcal{B}(N) = \oint_{S=\partial\Sigma}\mathcal{B}(N), \tag{25}$$

since $\mathcal{H}_{\mu}$ vanishes by the field equation; $\mathcal{B}(N)$ is a certain 2-boundary integrand which is here linear in the superpotential. In a later section it will be shown that the GR Hamiltonian always has this same form.

5. A first order geometric formulation

For a more efficient formulation, we will now use the notation of differential forms. Paralleling at first the treatment in Section 2.1, consider a *first order* Lagrangian 4-form for a form field φ (which might have suppressed indicies):

$$\mathcal{L} = d\varphi \wedge p - \Lambda(\varphi, p). \tag{26}$$

Its variation leads to the key relation

$$\delta\mathcal{L} = d(\delta\varphi \wedge p) + \delta\varphi \wedge \frac{\delta\mathcal{L}}{\delta\varphi} + \frac{\delta\mathcal{L}}{\delta p} \wedge \delta p. \tag{27}$$

On the one hand, this relation implicitly defines the two variational derivatives. Integrating (27) and invoking Hamilton's principle (with vanishing $\delta\varphi$ on the boundary) one finds that the vanishing of these two expressions constitute a *pair of first order equations*. (The explicit form, which contains the first differentials of the dynamic quantities φ, p will not be needed here.)

On the other hand, for geometric theories $\mathcal{L}$ should be diffeomorphic invariant. The changes induced by an infinitesimal diffeomorphism are given by the Lie derivative with respect to some vector field. On form fields $\pounds_N \equiv di_N + i_N d$ (here i_N is the interior product, aka contraction). Consequently, for diffeomorphism invariant systems, one can let $\delta \to \pounds_N$, then the above key relation (27) should be identically satisfied:

$$di_N\mathcal{L} \equiv \pounds_N\mathcal{L} \equiv d(\pounds_N\varphi \wedge p) + \pounds_N\varphi \wedge \frac{\delta\mathcal{L}}{\delta\varphi} + \frac{\delta\mathcal{L}}{\delta p} \wedge \pounds_N p. \tag{28}$$

This invites the definition of the *translational current* 3-form:

$$\mathcal{H}(N) := \pounds_N\varphi \wedge p - i_N\mathcal{L} =: N^\mu\mathcal{H}_\mu + d\mathcal{B}(N), \tag{29}$$

which is (on shell) a conserved "current" (Noether's first theorem):

$$-d\mathcal{H}(N) \equiv \pounds_N\varphi \wedge \frac{\delta\mathcal{L}}{\delta\varphi} + \frac{\delta\mathcal{L}}{\delta p} \wedge \pounds_N p. \tag{30}$$

Substituting the rhs of (29) into the lhs of (30) gives

$$-dN^\mu \wedge \mathcal{H}_\mu - N^\mu d\mathcal{H}_\mu. \tag{31}$$

Since N is locally a free choice, one can conclude (this is Noether's second theorem), by expanding out the Lie derivatives on the rhs of (30), that both terms of (31) must be separately proportional to certain field equation expressions. Hence, in particular, $\mathcal{H}_\mu$ must vanish "on shell".

The conserved quantity obtained from the translational current 3-form is the energy-momentum. The energy-momentum associated with a given spacetime 3-volume and a vector field is, since $\mathcal{H}_\mu$ vanishes (on shell), given by an integral of the boundary term:

$$E(N,\Sigma) := \int_\Sigma \mathcal{H}(N) = \int_\Sigma N^\mu\mathcal{H}_\mu + d\mathcal{B}(N) = \oint_{\partial\Sigma} \mathcal{B}(N). \tag{32}$$

However, it is clear that one can still freely modify the boundary term $\mathcal{B}(N)$ (and thereby the value of the conserved energy-momentum) without changing the conservation property. In the next section we will see that the Hamiltonian approach tames this ambiguity.

6. The Hamiltonian approach

The Hamiltonian 3-form associated with a given first order Lagrangian 4-form (26) can be obtained simply by contracting with the timelike spacetime displacement vector field that specifies the time evolution:

$$i_N\mathcal{L} = \pounds_N\varphi \wedge p - \mathcal{H}(N), \tag{33}$$

(analogous to $L = \dot{q}p - H$). This specifies the same 3-form earlier referred to as the translational gauge current (29). However its integral,

$$H(N, \Sigma) = \int_\Sigma \mathcal{H}(N) = \int_\Sigma N^\mu \mathcal{H}_\mu + \oint_{\partial\Sigma} \mathcal{B}(N), \tag{34}$$

is now seen as the generator of "time" evolution. The boundary term now has a two-fold role. One the one-hand, as we have already seen, it determines the quasi-local energy-momentum. This boundary term can (and, as we shall see below, often should) be modified. A modification to the boundary term does not change the evolution, yet it does change the value of the the conserved quantities. Here in this Hamiltonian formulation, however, this freedom is not arbitrary; it has a clear physical significance. Which brings us to the other important (often overlooked) role of the Hamiltonian boundary term. Namely, it *controls what should be held fixed on the boundary*, i.e., *the boundary conditions*.[48,49] Here we do not have the space to give a detailed discussion (which can be readily found in certain of our published works[23,46,47,50–53]). Let us illustrate this idea by two familiar examples.

6.1. *Example: thermodynamics*

With volume V, pressure P, temperature T and entropy S, a thermodynamic system can be described by various energies. The most common are

$$\begin{aligned} dU &= \quad TdS - PdV, && \text{internal energy} \\ dF &= -\, SdT - PdV, && \text{Helmholtz free energy} \\ dH &= \quad TdS + VdP, && \text{enthalpy} \\ dG &= -\, SdT + VdP, && \text{Gibbs free energy.} \end{aligned}$$

Note that there is not one unique physical energy. Rather there are several physically meaningful energies: $U(S,V)$, $F(T,V)$, $H(S,P)$ and $G(T,P)$, each associated with a certain distinct specific pair of independent variables that can be externally controlled.

6.2. *Application: electromagnetism*

Consider the following Hamiltonian for vacuum electrodynamics (using for convenience here ordinary vector notation):

$$H(\sigma) = \int \left[\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \phi \nabla \cdot \mathbf{E} \right] d^3x, \tag{35}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$. Note the familiar energy density and the *gauge* term (which both generates the gauge transformation and gives the Gauss constraint). The variation of this Hamiltonian includes the boundary term

$$\delta H(\sigma) \sim \oint \phi\, \delta(\mathbf{E} \cdot \mathbf{n})\, dS. \tag{36}$$

This will vanish—if we fix on the boundary the normal component of the electric field. Physically this means fixing σ, the surface charge density.

On the other hand one could use the alternative Hamiltonian

$$H(\phi) = \int \left[\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - \mathbf{E} \cdot \nabla\phi\right] d^3x = H(\sigma) - \oint \phi \mathbf{E} \cdot \mathbf{n}\, dS; \tag{37}$$

this differs simply by a boundary term, which does not change the evolution equations. However the variation of the Hamiltonian now includes a different boundary term, namely,

$$\delta H \sim -\oint \delta\phi\, \mathbf{E} \cdot \mathbf{n}\, dS. \tag{38}$$

This will vanish—if we fix the scalar potential on the boundary.

Each of these Hamiltonians accurately describes a certain class of physically realizable systems. In particular, for (36) one could connect a battery to a parallel plate capacitor until it was charged up, and then disconnect the battery and measure the work required to insert/remove a dielectric. Conversely, if we leave the battery connected (this will allow current to flow) one can measure the work with the fixed potential boundary condition. This is an example of a physical system described by (37). As is well known the amount of work required is not the same in these two cases.

This is a good example of the point we wish to emphasize: typically we have a choice between a Dirichlet or Neumann type boundary condition. In fact the Hamiltonian (36) corresponds to the Hilbert (symmetric) energy-momentum tensor, while the Hamiltonian (37) is essentially the canonical energy-momentum tensor of electromagnetism. Although both of these have physical meaning, they are nevertheless not of equal value. One should naturally prefer to use the first Hamiltonian (36), because (since the Gauss constraint vanishes on shell) it is *gauge invariant.* Along with this comes a bonus: with vanishing Gauss constraint the energy is nonnegative and, moreover, vanishes only for vanishing field. On the other hand, the Hamiltonian (37) is not gauge invariant (of course, for the fixed potential boundary condition is gauge dependent) and, moreover, its value can have any sign or magnitude, it can even vanish for non-vanishing fields. For similar reasons, we generally favor the Hilbert energy-momentum tensor over the canonical—they are related by a transformation of the form (7)—for most physical applications. This resolves the classical ambiguity for the choice of energy-momentum for all material sources as well as for all the gauge interaction fields—except for gravity.

7. The quasi-local Hamiltonian boundary term for GR

We have developed a covariant Hamiltonian formalism for geometric gravity theories. In this formulation the ambiguity regarding the choice of energy expression is given a clear physical and geometric meaning. Briefly, there are an infinite number of possible energy-momentum expressions simply because there are an infinite number of possible types of boundary conditions. Each of the possible energy expressions (which includes all the classical pseudotensors) corresponds to a Hamiltonian with evolution satisfying some specific boundary condition. Naturally not all boundary conditions are equally nice or equally physically reasonable or meaningful.

For GR, using the orthonormal coframe ϑ^α and the connection one-form $\Gamma^\alpha{}_\beta$ as the basic variables, we identified several nice boundary terms which correspond to certain Dirichlet/Neumann physical boundary condition choices for these quantities. One stands out above all the others as having the nicest properties. Our preferred Hamiltonian boundary term for GR is

$$\mathcal{B}(N) = \frac{1}{2\kappa}(\Delta\Gamma^\alpha{}_\beta \wedge i_N\eta_\alpha{}^\beta + \bar{D}_\beta N^\alpha \Delta\eta_\alpha{}^\beta), \tag{39}$$

where $\eta^{\alpha\beta} := *(\vartheta^\alpha \wedge \vartheta^\beta)$ is the dual coframe, and for any quantity we define

$$\Delta\varphi := \varphi - \bar{\varphi}, \tag{40}$$

where $\bar{\varphi}$ is a *reference value*.

Reference values specify the choice of zero point, the "ground state" of the variable. They can be used with other fields (e.g., for electromagnetism to include a background field) but they are not essential, simply because for all other fields it is possible—and usually desirable—to take the ground state as vanishing field value. However, for gravity the ground state is not a vanishing metric but rather the non-vanishing Minkowski metric values. Thus for gravity non-trivial reference values are unavoidable. Physically, the effect of including the reference values in our quasi-local expressions is the following: if the fields take on the reference values on the boundary, then all the geometric quasi-local quantities (energy-momentum etc.) vanish, indicating that the interior of the region is flat empty Minkowski space.

Thus our Hamiltonian boundary term quasi-local energy-momentum expression still has an ambiguity: namely, the explicit choice of reference (it is not enough to say it is the Minkowski metric, one needs to effectively give a reference coordinate system on the boundary in which the metric takes its standard Minkowski value). This reference choice freedom here plays essentially the same role as the pseudotensor coordinate ambiguity. However, we now have a geometric formulation which clarifies the physical meaning of the choice of reference and clearly shows that we only need to make a choice on the boundary of the region. We will consider below the problem of how to choose the reference.

8. Some properties of our quasi-local expression

From the integral of our preferred choice of Hamiltonian boundary term (39) over the 2-boundary of any region one can obtain values for the quasi-local quantities. With a suitable choice of reference, one can get quasi-local energy, momentum, angular momentum and center-of-mass by choosing, respectively, the space time displacement vector field to be an appropriate infinitesimal timelike or spacelike translation, or a rotation or boost.

The expression reduces to some other well regarded expressions in appropriate limits. At spatial infinity it has good limits to the asymptotic weak field expressions[37] and to the ADM energy-momentum[36] and the angular momentum/center-of-mass, as given by the expressions of Regge and Teitelboim,[54] or better the refined expressions of Beig and Ó Murchadha[55] and the further refinement of Szabados.[56] At future null infinity the expression gives the Bondi energy and, via a remarkable variational identity, the Bondi energy flux.[52,57] In the small region vacuum limit our quasi-local expression is, as desired, proportional to the Bel-Robinson tensor.[53]

Our expression has two terms: one, linear in the vector field, is essentially the Freud superpotential[31] (which generates the Einstein pseudotenor), the other, linear in the derivative of the displacement, is rather like Komar's expression.[58] The second term is essential for the center-of-mass[59] and also can contribute to some angular momentum calculations.[60]

It should be noted that essentially the same boundary term expression was (using a quite different approach with holonomic methods) independently found by Katz, Bičák and Lynden-Bel,[61] who have worked out a number of nice applications. Under suitable conditions the first term in our expression reduces exactly to the famous Brown-York energy, momentum and angular momentum quasi-local expressions.[62]

There is a proof of positive total energy for asymptotically flat gravitating systems that applies to our quasi-local expression.[11] It can be adapted to give a positive quasi-local energy proof.

9. Reference choices

To give specific values for the quasi-local energy-momentum our Hamiltonian boundary term expression must be supplied with a choice of spacetime displacement vector field and reference values for the dynamical variables. The usual choice of reference geometry is Minkowski space. A reasonable choice of evolution vector could be a constant timelike vector in this Minkowski reference. What is needed is, effectively, an embedding of the 2-boundary surface from the dynamic geometry into Minkowski space. Locally this can be described by four functions of two variables. Finding a good embedding, satisfying appropriate conditions, is presently an active pursuit. It is usually presumed that one would like to embed the spatial 2-boundary isometrically. That imposes three conditions. The standard quasi-local criteria[40,63] are that one should have positive energy, with vanishing energy only for Minkowski space. These criteria have been used to select both the embedding and the energy expression.[63–66] A reasonable proposal is to regard the energy as a function of the embedding variable(s) and examine its extreme. For our Hamiltonian boundary term quasi-local expression we will next give a brief report of the results we have obtained for the special case of spherically symmetric spacetimes.

9.1. *The optimal reference choice for spherical systems*

For the Schwarzschild spacetime with the evolution vector as the timelike Killing field of the reference corresponding to a static observer, we find our optimal quasi-local energy to have the standard Brown-York value:[62]

$$E = r\left(1 - \sqrt{1 - \frac{2m}{r}}\right) = \frac{2m}{1 + \sqrt{1 - \frac{2m}{r}}}. \tag{41}$$

Note: at spatial infinity the value is m, at the horizon it is $2m$, and it is not defined inside the horizon (there is no static observer inside).

Similarly, for the static observer in Reissner-Nördstrom spacetime

$$E = r\left(1 - \sqrt{1 - \frac{2m}{r} + \frac{Q^2}{r^2}}\right) = \frac{2m - \frac{Q^2}{r}}{1 + \sqrt{1 - \frac{2m}{r} + \frac{Q^2}{r^2}}}. \tag{42}$$

Note that this energy is *negative* at $r < \frac{Q^2}{2m}$, which is is exactly the turnaround radius inside which the gravitational force is repulsive.

However if we consider a radial geodesic observer who falls initially with velocity v_0 from a constant distance $r = a > 2m$. Then our energy for Schwarzschild is

$$E = r\left(1 - \sqrt{\frac{1-2m/a}{1-v_0^2}}\right). \tag{43}$$

When the initial velocity v_0 is less, equal, or greater than the escape velocity $\sqrt{2m/a}$, the energy is positive, zero, or negative, respectively.

9.2. *For the FLRW spacetimes*

Our program also works well for dynamic spherically symmetric spacetimes. For the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime,

$$ds^2 = -dt^2 + \frac{a^2(t)}{1-kr^2}dr^2 + a^2(t)r^2 d\Omega^2, \tag{44}$$

for a freely falling co-moving observers the quasi-local energy is

$$E = \frac{kar^3}{1+\sqrt{1-kr^2}}, \tag{45}$$

which vanishes for $k = 0$, is positive for $k = +1$ (but vanishing for the whole universe) and *negative* for $k = -1$.

9.3. *Application to Bianchi cosmologies*

Our Hamiltonian-boundary-term quasi-local energy-momentum ideas have been applied to homogenous cosmological models.[68]

Cosmological models which are homogeneous, but not (in general) isotropic can be described in terms of a metric of the form

$$ds^2 = -dt^2 + \delta_{ab}\vartheta^a \otimes \vartheta^b, \qquad a, b = 1, 2, 3 \tag{46}$$

where the spatial coframe,

$$\vartheta^a = h^a{}_j(t)\sigma^j(x), \tag{47}$$

is spatially homogeneous, i.e.,

$$d\sigma^i = \frac{1}{2}C^i{}_{jk}\sigma^j \wedge \sigma^k, \qquad i, j, k = 1, 2, 3 \tag{48}$$

where $C^i{}_{jk}$ are certain constants. The distinct possibilities have been systematically classified. Briefly, there are nine Bianchi types of such frames, falling into two classes:

Class A: $A_j := C^k{}_{jk} = 0$ (Types I, II, VI_0, VII_0, VIII, IX), Class B: $A_k \neq 0$ (Types III, IV, V, VI_h, VII_h).

Here we will not need any more detail, except to note that for Type I the spatial curvature vanishes, for Type IX it is positive, and for all other types the spatial curvature is negative.

Within this framework we examined the energy of all such models with completely general sources (e.g., matter, radiation, dark matter, dark energy, cosmological constant etc.) We used, as seems appropriate, the *co-moving time evolution vector* and *homogeneous boundary conditions* and a *homogeneous reference*. With this specialization our favored Hamiltonian boundary term quasi-local energy coincides with some other respectable energy expressions, including the teleparallel gauge current and the Hamiltonian associated with the Witten positive energy proof. The value of this common energy for these models works out to be

$$E(V) = -\frac{1}{\kappa} A_k A_l g^{kl}(t) V(t) \leq 0. \tag{49}$$

There are two noteworthy features: (i) The energy vanishes for all regions for all class A models (this is reasonable as Class A models are compactifiable, and the energy must vanish for a closed universe). (ii) The energy is *negative* for all regions for all class B models.

Thus, according to this *reasonable measure* of quasi-local energy for these models, one can have (i) negative energy, and (ii) vanishing energy for a non-trivial dynamic geometry!

10. Negative energy

We have used *the same energy expressions that give positive energy* for asymptotically flat isolated gravitating system and found, for physically and geometrically reasonable choices of evolution vector and reference, in some cases *negative quasi-local energy*. However, it should be noted that for the cosmological models for which this happens the gravitating systems are not at all like asymptotically flat isolated gravitating systems approaching a static or stationary equilibrium (for which there are compelling arguments in favor of positivity). For these dynamic models the negative spatial curvature geometry acts like a concave lens causing null geodesics to be defocused, as if they were being repelled by a negative mass, so a negative quasi-local energy value may be appropriate.

Under certain appropriate circumstances our quasi-local Hamiltonian boundary term is expected to have positive values. In some other circumstances it seems reasonable to have a negative value. Deeper investigation is required to sharpen the criteria for when the value should be positive, for when it is acceptable or even appropriate to have a negative value, and under what conditions it is acceptable to have a non-trivial geometry with zero energy.

11. Concluding thoughts

To better understand our work it may help to note that our principle aim has not been to find a unique “best quasi-local energy”. Rather it has aimed to find the best general choice for the Hamiltonian boundary term.

Going back to our opening theme, gravity is the universal attractive interaction, moreover it connects all of existence together and is the prime cause of the order in the cosmos. It seems that gravity is like love, something worthy of meditation.

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