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Energy and the universal attractive interaction

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Abstract

The gravitational interaction has two unique features: universal and attractive. These fundamental requirements for any gravity theory are intimately associated with energy and its positivity. From the gravitational response, because it is universal, one can uniquely detect the presence of any physical energy-momentum, even if it is associated with some kind of otherwise non-interacting source, i.e., “dark.” Note that energy and momentum can be exchanged between the gravitational field and its sources, and this happens locally, yet, curiously, the energy of gravitating systems—and hence the energy of all physical systems—is essentially elusive; for fundamental reasons it cannot be localized. It is simply not possible to find a proper energy-momentum density; instead there are only various quasi-local (i.e., associated with a closed 2-surface) expressions, which, moreover, are inherently reference frame dependent. We have found that the Hamiltonian approach tames both of these perplexing classical ambiguities. We identified one quasi-local Hamiltonian boundary expression for Einstein’s gravity theory, general relativity, which is physically distinguished; furthermore we have a procedure which identifies the “best” quasi-local reference frame.

Outline

- symmetry & Noether
- interactions
- special nature of gravity: **universal & attractive**
- **attractive**
  - attraction $\sim$ positive energy
  - positive energy test & positive energy proofs
  - application: the galaxy rotation problem.
    Alternative gravity vs dark matter
  - energy conditions
  - some energy values
  - cosmology
  - dark energy

- some arguments and observations re energy
- universal
  - the energy of gravitating systems, localization
  - quasi-local quantities
  - Hamiltonian approach
  - Hamiltonian boundary term
  - the role of boundary conditions
  - thermodynamics example
  - quasi-local expressions
  - application: electrodynamics
  - application: gravity. pseudotensors rehabilitated
  - achievements, current work & outstanding issues
- summary
- References
- a final word



- 20th century physics theory is mainly about symmetry.
- Noether's first theorem associates conserved quantities with global symmetries
- Noether's 2nd theorem concerns local symmetries: it is the foundation of the modern gauge theories.

PHYSICAL INTERACTIONS

◇ Presently there are 4 known physical interactions:

1. strong,
2. weak,
3. electromagnetic, &
4. gravity.

○ Gravity is very special:
it is the only interaction that is

(i) universal, and
(ii) purely attractive.

▷ Contrast with a familiar example: the **electric and magnetic** force shows
⋆ attraction,
⋆ repulsion, and
⋆ neutrality.

These two unique properties of gravity have deep significance in the scheme of things; they are not at all accidental but rather are essential consequences of fundamental principles, they naturally play key roles in many situations.

Attraction

The purely attractive property of gravity is required by the *fundamental physical principles* of thermodynamics & stability:
⋆ no perpetual motion, no infinite source of energy.

Consider a gravitating system
⋆ repulsion would be caused by a **negative** mass.

Can a negative mass exist?

⋆ Imagine a positive and negative mass pair. They would attract and repel each other, self accelerating to a high speed, gaining large kinetic energy that could be extracted. In this way one could have an unlimited source of energy.

As my thermodynamics Prof explained,
"If this were possible I could get rich, but that will never happen"
▷ This is not allowed.

∘ Purely attractive $\simeq$ positive mass.

Attraction $\approx$ POSITIVE ENERGY

In the far (weak) field regime the energy of a gravitating system is determined by its effective Newtonian mass.
This can be found from Kepler's 1-2-3 law: $(GM)^1 = \omega^2 a^3$.

Einstein's most famous relation,

$$E = MC^2,$$

then tells us that
$\triangleright$ the energy of gravitating systems is positive.

- Moreover physical systems naturally radiate energy until they reach their lowest energy state. If an isolated system could have negative energy, we could combine such systems to make one with a negative energy of arbitrarily large magnitude. Allowing such negative energy states would permit systems to radiate an infinite amount of energy. Physical systems are unstable unless there is a **non-negative** lower bound to energy. Hence energy must be positive, i.e, gravity is purely attractive.

○ This should be satisfied by any acceptable gravity theory.
▷ So it can be used to test proposed gravity theories.

Test and proof

○ Positive energy is a strong test:
it is very hard to create a **relativistic** theory which satisfies this requirement under all conditions.

• For an acceptable gravity theory we want a proof that every physical solution has positive energy.

⋆ That is not at all easy.

• People worked seriously on trying to prove positive energy for Einstein's general relativity theory for at least 20 years until the first proof (by an indirect geometric argument) was obtained by Schoen and S.T. Yau in 1979.

⋆ Witten presented his much more direct spinor proof in 1981, see e.g., Nester ['81,'84]. Later I found several other proofs [IJMPA '89, PRD '94, GRG '99].

Thus Einstein's GR passes this important test

An application at the galactic scale

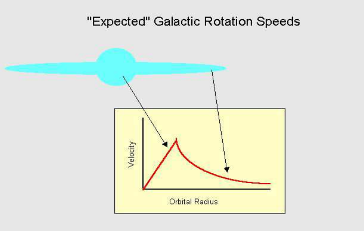
◦ We have very high precision accurate gravity experiments on the lab and solar system scales.

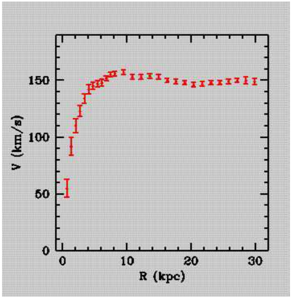
▷ However on much larger scales, like that of our galaxy, the agreement between theory and observations is not clear.

▷ Obviously something holds galaxies together, and gravity is the only realistic explanation.

The galactic rotation curve problem

⋆ However when we examine the galaxies in detail something doesn't fit.
[Problems were first noticed by Zwicky in the 1930s]





⋆ In the 1970s Vera Rubin noted that the outer parts of spiral galaxies are rotating several times faster than one would expect based on the observed mass distribution.

$$\text{expected} \quad v^2 \sim \frac{GM}{r}, \qquad \text{observed} \quad v \sim \text{const}$$

How can this be understood?

○ We can learn from **history**.

Discrepancies in the orbit of an outer planet led to the discovery of previously undetected matter: **a new planet**

⋆ This happened twice:

(i) Uranus led to Neptune,

(ii) then Neptun led to Pluto.

○ On the other hand, attempts to account for Mercury's orbit discrepancy by finding a new inner planet failed;
eventually the problem was resolved
by replacing Newton's theory with a new theory: **Einstein's GR**.

○ Similarly, for the galaxy we could account for the rotation problem by assuming that there is some additional undetected material, given the name

▷ **dark matter** (because you cannot see it).

Surprisingly, the necessary amount of dark matter

▷ would have to be

(i) about 10 times as much as the observed matter;

(ii) it must have a very different distribution, &

(iii) must be made of an unfamiliar substance.

naturally many have doubts that

▷ we are totally ignorant of **90%** of the material in the universe.

(We can only conclude that it exists—because gravity is universal).

$\circ$ The alternative is to modify the gravity theory.

(We tried this: [N & Zhytnikov, PRL 1994].)

$\diamond$ It is not so difficult to create a theory which could explain both the solar system and galactic observations.

Unfortunately,

$\triangleright$ all such theories **must violate some principle** that we trust, e.g.,

- $\star$ the positive energy principle,
- $\star$ 1st or 2nd order field eqns,
- $\star$ **linear for weak fields**.

So **our best present idea**: there is a lot of dark matter,

$\star$ and so there is a lot of work yet for scientists to figure out what the universe is made of.

Refinement: material source energy conditions

For all timelike observers u^μ:

weak: energy density

$$T_{\mu\nu}u^\mu u^\nu \geq 0, \qquad \rho \geq 0, \rho + p \geq 0$$

dominant: future timelike e-m flow vector $\Rightarrow$ positive energy

$$p^\mu := -T^\mu{}_\nu u^\nu, \qquad \rho \geq |p|$$

strong: gravity is attractive

$$(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)u^\mu u^\nu \geq 0, \qquad \rho + p \geq 0, \rho + 3p \geq 0$$

The above includes the specific form for a perfect fluid

$$T^{\mu\nu} = (\rho + p/c^2)u^\mu u^\nu + pg^{\mu\nu}.$$

FLRW cosmology

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\rho$$

$$\frac{2\ddot{a}}{c^2 a} + \frac{1}{c}^2 \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \Lambda = \frac{8\pi G}{c^4} p$$

or

$$\begin{aligned} \dot{\rho} &= -3\frac{\dot{a}}{a}(\rho + p/c^2) \\ 3\frac{\ddot{a}}{a} &= -4\pi G(\rho + 3p/c^2) + \Lambda c^2 \end{aligned}$$

DARK ENERGY

◦ After many decades of improving measurements, observations apparently indicate that, contrary to expectations,

▷ the universe is not slowing down!

It is expanding faster and faster, accelerating!

$$\frac{\ddot{a}}{a} > 0 \quad !$$

a kind of global repulsion—this may be seen as a challenge to our gravity is purely attractive principle.

The cause of this has been given a name: **dark energy**.

Dark energy could be a (quite small) positive cosmological constant or by some unusual type of matter with sufficiently large negative pressure which causes the repulsion.

Gravity itself may still have positive energy.

Is positive energy always required?

When should the energy of a physical system be positive?

1. For asymptotically flat regions that are approaching an equilibrium state.
2. For a very small spacetime region.

Note:
A: This is only possible if the non-gravitational sources satisfy some suitable energy condition.
B: Not all useful energy measures will satisfy these conditions.

Some quasi-local energy values

[N, So & Vargas, PRD 2008]
[Wu, Chen, Liu & N, PRD 2011; GRG 2012]

Reissner-Nordström $$E_{\mathrm{RN}} = \frac{2m - q^2/r}{1 + \sqrt{1 - 2m/r + q^2/r^2}}$$

Note 1: not defined between the 2 roots r_+, r_- of the radical,
Note 2: the value is negative for sufficiently small r.

For the Friedmann-Lemaître-Robertson-Walker cosmology

$$E_{\mathrm{FLRW}} = \frac{akr^3}{1 + \sqrt{1 - kr^2}}, \qquad k = -1, 0, +1$$

Note: $E_{\mathrm{FLRW}} \propto k$, the sign of the spatial curvature.

For Bianchi cosmological models, for a homogeneous reference, for all regions, the energy of class A models vanishes and it is negative for all class B models.

Universal

• Gravity is universal. The source of gravity is all matter and all interaction fields including itself.

○ In Newton's theory mass density produced gravity.

* From Einstein's relation we know that "energy" is equivalent to mass. Hence energy (density) should also produce gravity.

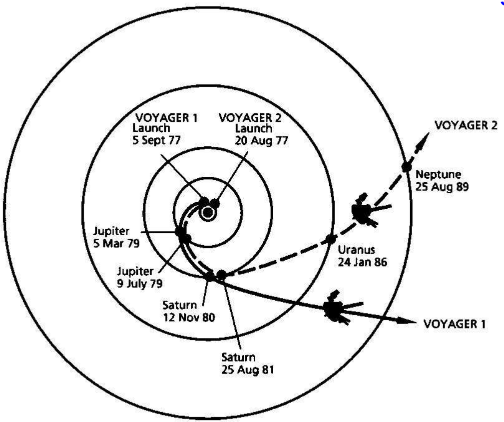
* Moreover, from special relativity covariance we learn that energy is a part of a 4-vector of energy-momentum.

* Hence, relativistically, we expect the total energy-momentum density to produce gravity.

○ Energy-momentum should be conserved. Sources interact with the gravitational field, and exchange energy-momentum with gravity.

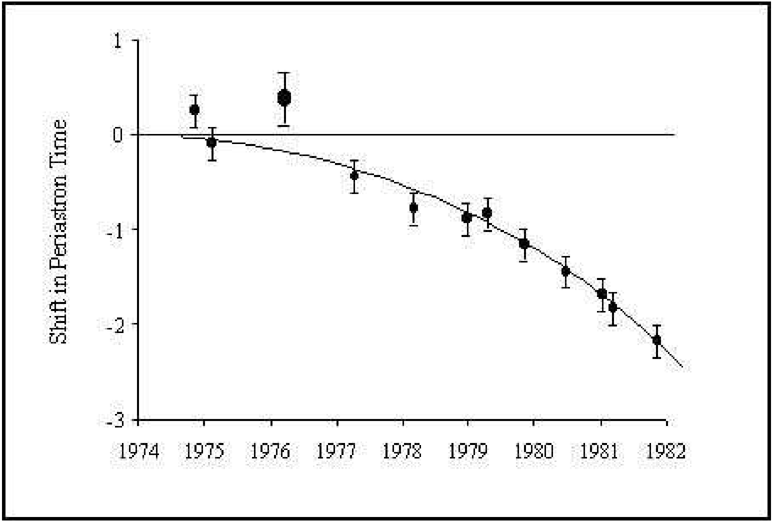
Voyager

energy and momentum can



be exchanged through the gravitational field.
This happens locally.

The binary pulsar



Indirect evidence for
gravitational waves and gravitational energy

The Tweedle twins

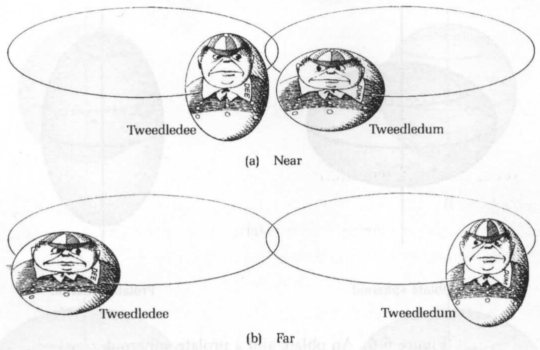
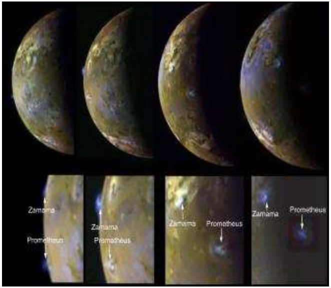
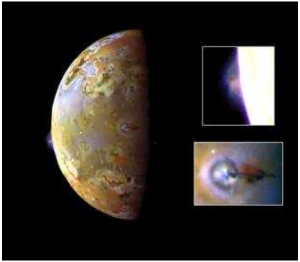


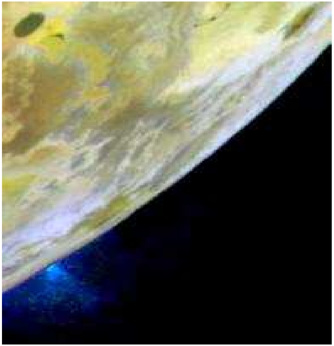
Figure 6-7. Tweedledum gains energy at the expense of Tweedledee by changing shape in accordance with tidal forces.

Bondi's gedanken experiment

energy transferred through empty space by gravity







Jupiter’s tidal heating powers Io’s volcanoes
[Purdue (1999), Booth & Creighton (2000), Favata (2001)]

doubts

Some question the whole idea, e.g.,

- F. I. Cooperstock, Ann Phys (NY) **282** (2000) 115 arXiv:gr-qc/9904046.

$$\text{gravitational e-m} + \text{material e-m} = -\frac{c^4}{8\pi G} G_{\mu\nu} + T_{\mu\nu} = 0$$

- the total energy-momentum density vanishes everywhere
- The Einstein tensor is the energy-momentum density of gravity
- This was first suggested by Levi-Civita.
- According to my thinking it is almost correct! This is the correct volume density, but—as we shall see—one also needs to include a surface density term.

Feynman & Christodoulou

In 1957 Feynman gave a simple argument that gravity waves carried energy and could be detected.
This involved 2 beads sliding on a rod and heating the rod.

In 1968 Wheeler asked Christodoulou to show “the formation of black holes in pure general relativity, by the focusing of incoming gravitational waves”.

After 40 years he finally succeeded, see
Christodoulou *The formation of black holes in General Relativity* (European Math Society, 2009)

Christdoulou’s result should dispel all doubts regarding the reality of gravitational energy in the vacuum and its essentially attractive/positive-energy nature.

gravitational local energy-momentum density

Thus gravity itself should have some kind of local energy-momentum density—which should also produce gravity:

Hence gravity should be inherently non-linear, truly universal: it affects and is affected by everything, including itself.

Note that both special properties: universal and attractive are associated with energy.

Einstein’s energy expressions of 1913

4.3 The Derivation of the *Entwurf* Field Equations

The awkward episode of his 1912 field equation in his earlier theory of static fields seems to have convinced Einstein of the necessity of ensuring from the very beginning that any new field equation satisfy the conservation laws. This means that one should be able to construct a gravitational field stress tensor, or a stress-energy tensor in the four-dimensional case, using the field equations. This recognition provides the key to the understanding of some of the pages in the Zurich notebook and leads us directly to the derivation of the *Entwurf* field equations.

The law of conservation of energy-momentum, written as the vanishing of the covariant divergence of the stress-energy tensor $\Theta_{\mu\nu}$, takes the form

$$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu}(\sqrt{-g}g_{\sigma\mu}\Theta_{\mu\nu}) - \frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\Theta_{\mu\nu} = 0 \qquad (12)$$

in equation (10) of Einstein’s part of the *Entwurf* paper. The second term of

(15), which corresponds to the satisfaction of the conservation laws, would have confirmed for Einstein the correctness of this expectation.

We can now step directly to the method of deriving the field equations used in the *Entwurf* paper, which amount to a simple inversion of the method used to construct equation (13). As Einstein showed in section 5 of his part of the paper, once an identity has been decided upon to stand for equation (13), then one reads the gravitation tensor and gravitational field stress-energy tensor directly from it. The derivation of this identity,

$$\sum_{\alpha\beta\tau\rho}\frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\gamma_{\alpha\beta}\frac{\partial\gamma_{\tau\rho}}{\partial x_\beta}\frac{\partial g_{\tau\rho}}{\partial x_\sigma}\right)-\frac{1}{2}\sum_{\alpha\beta\tau\rho}\frac{\partial}{\partial x_\sigma}\left(\sqrt{-g}\gamma_{\alpha\beta}\frac{\partial\gamma_{\tau\rho}}{\partial x_\alpha}\frac{\partial g_{\tau\rho}}{\partial x_\beta}\right)$$
$$=\sum_{\mu\nu}\sqrt{-g}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\left\{\sum_{\alpha\beta}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\sqrt{-g}\frac{\partial\gamma_{\mu\nu}}{\partial x_\beta}\right)\right.$$
$$-\sum_{\alpha\beta\tau\rho}\gamma_{\alpha\beta}g_{\tau\rho}\frac{\partial\gamma_{\mu\tau}}{\partial x_\alpha}\frac{\partial\gamma_{\nu\rho}}{\partial x_\beta}+\frac{1}{2}\sum_{\alpha\beta\tau\rho}\gamma_{\alpha\mu}\gamma_{\beta\nu}\frac{\partial g_{\tau\rho}}{\partial x_\alpha}\frac{\partial\gamma_{\tau\rho}}{\partial x_\beta}$$
$$\left.-\frac{1}{4}\sum_{\alpha\beta\tau\rho}\gamma_{\mu\nu}\gamma_{\alpha\beta}\frac{\partial g_{\tau\rho}}{\partial x_\alpha}\frac{\partial\gamma_{\tau\rho}}{\partial x_\beta}\right\}, \tag{16}$$

is given by Grossmann in his section of the paper. It amounts to a generalization of equation (14) from the weak-field case to the general case. Equation (14) was constructed originally by expanding the terms on its right-hand side and retaining only quantities of second order to yield the left-hand side. The bulk of Grossmann's derivation of identity (16) is devoted to making this process exact. He took the terms of third order, which were dropped in constructing equation (14), and reworked and redistributed them until the identity had the form required by equation (13). This yielded identity (16) directly.

- such energy-momentum expressions are not tensors
  - they are inherently reference frame dependent
  - such energy-momentum pseudotensors can be obtained
  - via Lagrangian & Noether symmetry
    (Note: Noether current ambiguity)
  - via rearranging field equations
    (Note: equally ambiguous)
- Amazingly, in the paper with her 2 famous theorems that are so important for fundamental physics theory Emmy Noether proved also that the energy-momentum density—not only for Einstein's GR but also for any geometric gravity theory—cannot be a proper quantity.
- pseudotensor ambiguities: which pseudotensor, which reference frame?

Famous pseudotensors: Einstein, Landau & Lifshitz, Papapetrou ('48), Bergmann ('53), Goldberg ('58), Møller ('58), Møller ('61), Weinberg ('72).

- Misner, Thorne & Wheeler
  “Anyone who looks for a magic formula for “local gravitational energy-momentum” is looking for the right answer to the wrong question. Unhappily, enormous time and effort were devoted in the past to trying to “answer this question” before investigators realized the futility of the enterprize.” MTW **Gravitation** p 467.

Geometry and gravity: Quasi-local energy-momentum

Gravity is necessarily connected with geometry.

• The source of gravity is energy-momentum.

◦ By Noether's first theorem associating physically conserved quantities with symmetry, energy-momentum is related to the translation symmetry of space-time geometry.

According to the equivalence principle "gravity cannot be detected at a point."

• A consequence is that gravitational energy-momentum—and hence the energy-momentum of gravitating systems—and hence the energy-momentum of all physical systems—is fundamentally non-local.

◦ All physical energy-momentum is quasi-local!
(associated with a closed 2-surface).

Outline for technical part

- Energy-momentum and its localization
- Quasi-local quantities:
  - Hamiltonian approach
  - role of the boundary terms
  - covariant Hamiltonian formalism
  - boundary conditions from boundary variation principle
  - quasi-local expressions

  - Application: electromagnetism
  - Application: Einstein gravity
  - How to choose the reference
- Summary
- References

pseudotensor Hamiltonian boundary term association

Choose any $H^{\nu\lambda}{}_{\mu} \equiv H^{[\nu\lambda]}{}_{\mu}$, and use it to split the Einstein tensor. Define the gravitational energy-momentum pseudotensor by

$$\kappa\sqrt{-g}t^{\mu}{}_{\nu} := -\sqrt{-g}G^{\mu}{}_{\nu} + \frac{1}{2}\partial_{\lambda}H^{\mu\lambda}{}_{\nu}.$$

Then Einstein's equation, $G^{\mu}{}_{\nu} = \kappa T^{\mu}{}_{\nu}$, takes a form with the total effective energy-momentum pseudotensor as its source: [Chang, N, Chen, PRL '99]

$$\partial_{\lambda}H^{\mu\lambda}{}_{\nu} = 2\kappa\sqrt{-g}\mathcal{T}^{\mu}{}_{\nu} := 2\kappa\sqrt{-g}(t^{\mu}{}_{\nu} + T^{\mu}{}_{\nu}).$$

$$\begin{aligned} -P(N) &:= -\int_V N^{\mu}\mathcal{T}^{\nu}{}_{\mu}\sqrt{-g}(d^3x)_{\nu} \\ &\equiv \int_V \Big[N^{\mu}\sqrt{-g}(\frac{1}{\kappa}G^{\nu}{}_{\mu} - T^{\nu}{}_{\mu}) - \frac{1}{2\kappa}\partial_{\lambda}(N^{\mu}H^{\nu\lambda}{}_{\mu})\Big](d^3x)_{\nu} \\ &\equiv \int_V N^{\mu}\mathcal{H}_{\mu} + \oint_{S=\partial V}\mathcal{B}(N) \equiv H(N). \end{aligned}$$

Quasi-local quantities: Hamiltonian approach

- Energy can be identified as the value of the Hamiltonian associated with a timelike displacement vector field N.
- The Hamiltonian $H(N)$ is given by an integral of a suitable Hamiltonian 3-form (density) $\mathcal{H}(N)$ over a 3-dimensional (spacelike) region Σ along N.
- This 3-form is always linear in the displacement N and its derivatives which can always be written in the form

$$\mathcal{H}(N) = N^\mu \mathcal{H}_\mu + d\mathcal{B}(N).$$

- Thus the Hamiltonian generally includes an integral over the boundary of the region.

$$H(N) = \int_\Sigma \mathcal{H} = \int_\Sigma N^\mu \mathcal{H}_\mu + \oint_{\partial\Sigma} \mathcal{B}(N).$$

- The two parts of the Hamiltonian have distinct roles.

Quasi-local quantities: role of the boundary term

- The 3-form part $N^\mu \mathcal{H}_\mu$ generates the equations of motion.
  $\triangleright$ As we shall see, for diffeomorphic invariant theories it has vanishing value.
- The Hamiltonian generally includes a boundary term $\mathcal{B}(N)$.
- It plays two key roles:
  (i) It determines the values of the quasi-local quantities,
  (ii) It determines the boundary conditions.

Technical details: Noether translational current

- The first order Lagrangian for an f-form field φ and its conjugate momentum is p is given by

$$\mathcal{L} = d\varphi \wedge p - \Lambda(\varphi, p).$$

- The variation

$$\delta\mathcal{L} = d(\delta\varphi \wedge p) + \delta\varphi \wedge \frac{\delta\mathcal{L}}{\delta\varphi} + \frac{\delta\mathcal{L}}{\delta p} \wedge \delta p,$$

gives the EOM

$$\frac{\delta\mathcal{L}}{\delta p} := d\varphi - \partial_p\Lambda = 0, \qquad \frac{\delta\mathcal{L}}{\delta\varphi} := -\varsigma dp - \partial_\varphi\Lambda = 0,$$

and $\varsigma := (-1)^f$.

- Diffeomorphism invariance (in terms of the Lie derivative) requires

$$d i_N \mathcal{L} \equiv \pounds_N \mathcal{L} \equiv d(\pounds_N \varphi \wedge p) + \pounds_N \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \pounds_N p.$$

- Hence the "translational current" (3-form)

$$\mathcal{H}(N) = \pounds_N \varphi \wedge p - i_N \mathcal{L}$$

satisfies the conservation law

$$-d\mathcal{H}(N) \equiv \pounds_N \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \pounds_N p. \qquad (*)$$

- Like other Noether conserved currents, $\mathcal{H}(N)$ is not unique: it can be modified by adding the differential of any 2-form.

Technical details: local diffeomorphism invariance

- With geometric gravity included, we have also local diffeomorphism invariance, which gives rise to a differential identity.
- Explicit calculation shows that $\mathcal{H}(N) = \pounds_N\varphi \wedge p - i_N\mathcal{L}$ always has the form

$$\mathcal{H}(N) = N^\mu \mathcal{H}_\mu + d\mathcal{B}(N).$$

- By substituting into Eq.(*) we find that

$$d(N^\mu \mathcal{H}_\mu + d\mathcal{B}(N)) \equiv dN^\mu \wedge \mathcal{H}_\mu + N^\mu d\mathcal{H}_\mu$$

  is proportional to the field equations, therefore $\mathcal{H}_\mu$ vanishes "on shell".
- Hence for gravitating systems the Noether translational "charge" — energy-momentum — is quasi-local, it is given by the integral of the boundary term, $\mathcal{B}(N)$.
- But this boundary term can be completely modified to any value.

Quasi-local quantities: covariant Hamiltonian formalism

- The Hamiltonian approach tames the ambiguity.
- Generalizing $L = \dot{q}p - H$, from the first order Lagrangian one can construct the Hamiltonian 3-form by projecting along a "timelike" displacement vector field:

$$i_N \mathcal{L} = \pounds_N \varphi \wedge p - \mathcal{H}(N).$$

- As noted it can be written in the form

$$\mathcal{H}(N) = N^\mu \mathcal{H}_\mu + d\mathcal{B}(N),$$

consequently the quasi-local energy is then determined only by the surface integral

$$E(N) = \int_\Sigma \mathcal{H}(N) = \int_\Sigma N^\mu \mathcal{H}_\mu + d\mathcal{B}(N) = \oint_{\partial\Sigma} \mathcal{B}(N).$$

Quasi-local quantities: boundary conditions

- The Hamiltonian boundary terms determines the values of the quasi-local quantities.
  - **Energy** is given by a suitable timelike displacement
  - Linear momentum is obtained from a spatial translation
  - Angular momentum from a suitable rotational displacement
  - A spacetime displacement which is asymptotically a boost will give the center-of-mass moment.
- But our Noether analysis has revealed that $\mathcal{B}(N)$ can be adjusted, changing to a new conserved value.
- However the **variational principle** contains an additional (largely overlooked) feature which distinguishes all of these choices.

- The boundary variation principle, i.e. the boundary term in the variation, tells us what to hold fixed on the boundary — it determines the boundary conditions.
- The different Hamiltonian boundary terms are each associated with distinct boundary conditions.
- As in thermodynamics or electrostatics there are various "energies" which correspond to how the system interacts with the outside through its boundary.
- An example: Thermodynamics

(volume V, pressure P) (temperature T, entropy S)

$$\begin{aligned} dU &= TdS - PdV, && \text{internal energy} \\ dF &= - SdT - PdV, && \text{Helmholtz free energy} \\ dH &= TdS + VdP, && \text{enthalpy} \\ dG &= - SdT + VdP, && \text{Gibbs free energy} \end{aligned}$$

Quasi-local quantities: quasi-local expressions

- In general (in particular for gravity) it is necessary (in order to guarantee functional differentiability of the Hamiltonian on the phase space with the desired boundary conditions) to adjust the boundary term $\mathcal{B}(N) = i_N\varphi \wedge p$ which is naturally inherited from the Lagrangian.
- A reference configuration, $\bar{\varphi}$ and $\bar{p}$, (which determine the ground state) is essential for gravity to give the correct value for $\delta\mathcal{L} = 0$.
- With $\Delta\varphi := \varphi - \bar{\varphi}$, $\Delta p := p - \bar{p}$, we found two covariant boundary choices (essentially Dirichlet and Neumann)

$$\begin{aligned}
\mathcal{B}_\varphi &= i_N\varphi \wedge \Delta p - \varsigma\Delta\varphi \wedge i_N\bar{p}, && i_N(\delta\varphi \wedge \Delta p) \\
\mathcal{B}_p &= i_N\bar{\varphi} \wedge \Delta p - \varsigma\Delta\varphi \wedge i_N p, && -i_N(\Delta\varphi \wedge \delta p)
\end{aligned}$$

and two other physical interesting choices

$$\begin{aligned}
\mathcal{B}_{\rm dyn} &= i_N\bar{\varphi} \wedge \Delta p - \varsigma\Delta\varphi \wedge i_N\bar{p}, && \varsigma\delta\varphi \wedge i_N\Delta p - i_N\Delta\varphi \wedge \delta p \\
\mathcal{B}_{\rm con} &= i_N\varphi \wedge \Delta p - \varsigma\Delta\varphi \wedge i_N p, && i_N\delta\varphi \wedge \Delta p - \varsigma\Delta\varphi \wedge i_N\delta p
\end{aligned}$$

Application: electromagnetism

- Hamiltonian

$$H = \int \left[\frac{1}{2}(E^2 + B^2) + \phi \vec{\nabla} \cdot \vec{E} \right] d^3x$$

$$\delta H \sim \oint \phi \, \delta(\vec{E} \cdot \vec{n}) dS$$

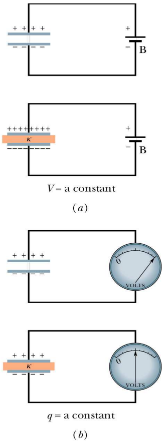
essentially fix surface charge,

$$H = \int \left[\frac{1}{2}(E^2 + B^2) - \vec{E} \cdot \vec{\nabla} \phi \right] d^3x$$

$$\delta H \sim -\oint \delta\phi \, (\vec{E} \cdot \vec{n}) dS$$

fix potential.

How much work to insert/remove a dielectric?



(i) Connect the battery. The potential on the capacitor plates is fixed. Insert/remove the dielectric.
(ii) Connect the battery, charge up the capacitor then disconnect the battery. Now the charge on the plates is fixed. Insert/remove the dielectric.

Application: Einstein gravity

- Einstein's (vacuum) gravity theory: first order Lagrangian

$$\mathcal{L}_{\mathrm{GR}} = R^{\alpha}{}_{\beta} \wedge \eta_{\alpha}{}^{\beta},$$

where $\Gamma^{\alpha}{}_{\beta}$ is the connection 1-form;
$R^{\alpha}{}_{\beta} := d\Gamma^{\alpha}{}_{\beta} + \Gamma^{\alpha}{}_{\gamma} \wedge \Gamma^{\gamma}{}_{\beta},$ is the curvature 2-form and
$\eta^{\alpha\beta} := *(\vartheta^{\alpha} \wedge \vartheta^{\beta})$.

Preferred Boundary Term for GR

Chen, N, Tung (1995) [also found by Katz, Bičák & Lynden-Bel]

$$\mathcal{B}(N) = \frac{1}{2\kappa}(\Delta\Gamma^{\alpha}{}_{\beta} \wedge i_N \eta_{\alpha}{}^{\beta} + \bar{D}_{\beta} N^{\alpha} \Delta\eta_{\alpha}{}^{\beta}) \qquad \eta^{\alpha\beta\cdots} := *(\vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge \cdots)$$

fix the orthonormal coframe ϑ^{μ} ($\sim$ metric) on the boundary:

$$\delta\mathcal{H}(N) \sim d i_N (\Delta\Gamma^{\alpha}{}_{\beta} \wedge \delta\eta_{\alpha}{}^{\beta})$$

Like other choices, at spatial infinity it gives the ADM, MTW (1973), Regge-Teitelboim (1974), Beig-Ó Murchadha (1987), Szabados (2003) energy, momentum, angular-momentum, center-of-mass

some special virtues:
(i) at null infinity: the Bondi-Trautman energy & the Bondi energy flux
(ii) it is “covariant”
(iii) it has positive energy [on maximal slices in the SOF gauge]
(iv) for small spheres, a positive multiple of the Bel-Robinson tensor
(v) first law of thermodynamics for black holes
(vi) for spherical solutions it has the hoop property
(vii) for a certain reference the Minkowski quasi-local quantities vanish

A “good” resolution for one ambiguity

The reference and the quasi-local quantities

- Note: For all other fields it is appropriate to choose vanishing reference values as the reference ground state—the vacuum.
- But for geometric gravity the standard ground state is the non-vanishing Minkowski metric. A non-trivial reference is essential.
- With standard Minkowski coordinates y^i, a Killing field of the reference has the form $N^k = N_0^k + \lambda_0^k{}_l y^l$, with N_0^k and $\lambda_0^{kl} = \lambda_0^{[kl]}$ being constants. The 2-surface integral of the Hamiltonian boundary term then gives the value

$$\oint_S \mathcal{B}(N) = -N_0^k p_k(S) + \frac{1}{2}\lambda_0^{kl} J_{kl}(S),$$

  i.e., not only a quasi-local energy-momentum but also a quasi-local angular momentum/center-of-mass. The integrals $p_k(S),\ J_{kl}(S)$ in the spatial asymptotic limit agree with accepted expressions for these quantities.

the reference

- For energy-momentum take N^μ to be a translational Killing field of the Minkowski reference. Then the second quasi-local term vanishes.
- Remark: Holonomically (with vanishing reference) the first term is Freud's 1939 superpotential.
  Thus we are in effect making a proposal for best choice of coordinates for the Einstein pseudotensor.

To construct a reference choose, in a neighborhood of the desired spacelike boundary 2-surface S, 4 smooth functions $y^i,\ i = 0, 1, 2, 3$ with $dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 \neq 0$ and then define a Minkowski reference by $\bar{g} = -(dy^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2$.
equivalent to finding a diffeomorphism for a neighborhood of the 2-surface into Minkowski space. The reference connection is obtained from the pullback of the flat Minkowski connection. Then with constant N^k our quasi-local expression takes the form

$$\mathcal{B}(N) = N^k x^\mu{}_k (\Gamma^\alpha{}_\beta - x^\alpha{}_j\, dy^j{}_\beta) \wedge \eta_{\mu\alpha}{}^\beta .$$

Isometric matching of the 2-surface

The reference metric on the dynamical space has the components

$$\bar{g}_{\mu\nu} = \bar{g}_{ij} y^i{}_\mu y^j{}_\nu. \tag{1}$$

Consider the usual embedding restriction: isometric matching of the 2-surface S. This can be expressed quite simply in terms of quasi-spherical foliation adapted coordinates t, r, θ, ϕ as

$$g_{AB} = \bar{g}_{AB} = \bar{g}_{ij} y^i_A y^j_B = -y^0_A y^0_B + \delta_{ij} y^i_A y^j_B$$

on S, where A, B range over $2, 3 = \theta, \phi$.

From a classic closed 2-surface into $\mathbb{R}^3$ embedding theorem, we expect that that—as long as one restricts S and $y^0(x^\mu)$ so that on S

$$g'_{AB} := g_{AB} + y^0_A y^0_B$$

is convex—one has a unique embedding.

Wang & Yau used this type of embedding in their recent quasi-local work.

Note that the choice of y^0 on S determines y^1, y^2, y^3 on S.

Complete 4D isometric matching

• Our "new" proposal complete isometric matching on S:
[already suggested by Szabados in 2000]

$$\text{10 constraints : } \quad g_{\mu\nu}|_S = \bar{g}_{\mu\nu}|_S = \bar{g}_{ij}{y^i}_\mu {y^j}_\nu|_S.$$

on 12 embedding functions on the 2-surface of constant t, r:

$$y^i (\Longrightarrow y^i_\theta, y^i_\phi), \quad y^i_t, \quad y^i_r$$

In terms of the orthonormal coframe ϑ^α with 6 local Lorentz gauge d.o.f.
Lorentz transform to match the reference coframe dx^α on the 2-surface.
Integrability condition: the 2-forms $d\vartheta^\alpha$ should vanish when restricted to the 2-surface:

$$d\vartheta^\alpha|_S = 0, \quad \text{4 restrictions}$$

The “best matched” reference geometry

- 12 embedding variables subject to 10 isometric conditions
- equivalently, 6 local Lorentz gauge subject to 4 embedding conditions
- To fix the remaining 2, regard the quasi-local value as a measure of the difference between the dynamical and the reference boundary values. The critical points are distinguished

For a given 2 surface S 2 different quantities can be considered:
$m^2 = -\bar{g}^{ij} p_i p_j$ and $E(N, S)$,
for the latter there are 2 different ways to fix N.
I: Find the critical points of m^2. This should determine the reference up to Poincaré transformations.

II: Much more simply, find the critical points of $E(\partial_T, S)$. (As long as $m^2 > 0$ this is equivalent to **I**.) Both lead to quasi-local quantities associated with S

An alternative gives a quasi-local energy associated with an observer.
[Afterward one could extremize over the choice of N.]

Based on some physical and practical computational arguments it is reasonable to expect a unique solution.

Recently we have applied this approach to general axisymmetric systems and have, in particular, found some nice results for the Kerr solution.

Bonus

The above procedure for selecting our Hamiltonian boundary term reference frame has much wider application.

First, note that our reference selection strategy simultaneously provides an answer for selections needed for other related energy expressions, in particular:

- How to choose the coordinate system for the Freud superpotential and Einstein pseudotensor;
- How to select the tetrad for the teleparallel gauge current;
- It gives a spinor field selection for the Witten spinor Hamiltonian boundary term;
- It gives a frame and spinor selection for Tung's Quadratic Spinor Lagrangian formulation.

Furthermore

- 4D isometric matching with critical energy value can be applied to select the reference frame for all of the pseudotensors.
- 4D isometric matching with critical energy value can be applied to select the reference for the other GR boundary terms corresponding to GR Hamiltonians with other boundary conditions.
- Indeed 4D isometric matching with critical energy values can be applied to all of the different boundary terms that have been proposed for the most general metric-affine gravity theory and all its special subcases, including the Poincaré gauge theory and teleparallel theory.
- We explicitly considered Minkowski as the fixed reference, but actually almost any geometry could serve as a reference. Cases of special interest might be (Anti-)de Sitter, FLRW and Schwarzschild.

The main result

With a suitable choice of boundary condition and a technique for finding the Minkowski space “best matched” to a given boundary 2-surface, we have a satisfactory way of fixing the Hamiltonian boundary term and thereby resolving the ambiguities in determining the quasi-local energy-momentum and angular momentum/center-of-mass of classical physical systems.

Achievements & current work

- **a good Hamiltonian boundary term**
  - good quasi-local expressions
  - at spatial infinity & future null infinity
  - energy-momentum & angular momentum/center-of-mass: compatible with ADM ('61), Regge & Teitelboim ('74), Beig & O Murchadha ('85), Szabados ('03,'04)
  - Bondi energy flux: distinguishes a unique expression
  - small sphere distinguishes the same unique expression
- current work: reference choice [arXiv:1307.1510]
- via “best matching”, 4D isometric and extremize energy
- a resolution of the second traditional ambiguity
- application axisymmetric systems [arXiv:1307.1039]

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A final thought

Gravity is the universal attractive interaction,

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Gravity is the universal attractive interaction,
moreover it connects all of existence together
and is the prime cause of the order in the cosmos

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Is there a deeper meaning to this similarity?
I think so.

With appreciation for your attention.