INTERSECTION OF ALGEBRAIC VARIETIES

QING LIU

Basically, an algebraic variety \( F_1 = \cdots = F_m = 0 \) is an intersection of hypersurfaces \( F_1 = 0, \ldots, F_m = 0 \). So in some sense, the intersection is a concept which is present everywhere in algebraic geometry.

Everybody knows that a polynomial, say over the complex numbers, of degree \( d \geq 1 \), has exactly \( d \) zeros counted with multiplicities. This is a special case of Bezout’s theorem which states that the number of common zeros of two homogeneous polynomials in three variables without common factor is exactly the product of their degrees. We will explain the ideas behind this classical result in §1.

Everybody also knows that, in general, a polynomial of degree \( d \) has \( d \) distinct zeros. This is a special case of Bertini’s theorem. The latter says that if we start with a system of polynomial equations with some nice property, then the new system obtained by affecting to some variable a sufficiently general value, keeps the same nice property (§2). Finally, we will describe some new results of same flavour: over finite fields in §3, and over rings in §4.

1. Bézout’s Theorem

Let \( f(X) \in \mathbb{C}[X] \) be of degree \( d \geq 1 \). Then the zeros of \( f \) can be viewed as the coordinates of the intersection points

\[
\{(x, y) \mid y - f(x) = 0\} \cap \{(x, y) \mid y = 0\}
\]

of two plane curves (one of which is a straight line). If \( x_0 \) is a zero of \( f \), its multiplicity \( \text{ord}_{x_0}(f) \) then measures the contact order of the curve \( C : y - f(x) = 0 \) (which has “degree” \( d \)) and the straight line \( D : y = 0 \) (of “degree” 1). The well known equality

\[
d = \sum_{f(x) = 0} \text{ord}_x(f)
\]

then can be interpreted as \( C, D \) having intersection number (counted with multiplicities) equal to \( d \). In other words, the intersection number \( C.D = \deg C \times \deg D \).

Now what happens if we consider a more general curve \( C : f(x, y) = 0 \) where \( f(x, y) \) is a complex polynomial of (total) degree \( d \geq 1 \)? Some new phenomenon then appears. For instance, let \( f(x, y) = xy - 1 \). Then \( C \cap D = \emptyset \) ! This is because we are working in the affine plane \( \mathbb{C}^2 \) where two lines may not intersect each other. To remedy this problem and get a
satisfactory formula, we have to work in the projective plane\(^1\) \(\mathbb{P}^2(\mathbb{C})\) which is a “compactification” of \(\mathbb{C}^2\) and in which every pair of curves have non-empty intersection. Concretely, the above curve \(C\) becomes \(xy - z^2 = 0\) (while \(D\) keeps the same equation) and its intersection with \(D\) is at \(z^2 = 0\). Therefore, \(C\) and \(D\) intersect at one point \([1,0,0]\) with contact order 2. Their intersection number \(C.D = 2\) equals to \(\deg C \times \deg D\).

Let \(C\) be an arbitrary plane curve in \(\mathbb{P}^2(\mathbb{C})\), defined by a homogeneous polynomial \(F(x, y, z)\) of degree \(d\), not divisible by \(y\). Then \(C \cap D\) consists in \(F(x, 0, z) = 0\). The zeros of \(F(x, 0, z)\) have to be counted (with multiplicities) separately when \(z = 1\) and \(z = 0\). It is easy to check that the total sum of multiplicities is equal to \(d = \deg F(x, 0, z)\). Therefore \(C.D = \deg C \deg D\). This is a special case of:

**Theorem 1.1.** (Bézout’s Theorem) Let \(C, D\) be planes curves in \(\mathbb{P}^2(\mathbb{C})\) defined respectively by homogeneous polynomials \(F(x, y, z), G(x, y, z)\) of degrees \(d, e\). Suppose \(\gcd(F, G) = 1\). Then the intersection number of \(C\) and \(D\) (counted with multiplicities) is \(C.D = d \times e\).

The classical proof uses the resultant. Let us sketch the proof using more geometric tools. We see that the desired formula depends only on the degree of \(C\) and \(D\). A special case is when \(D\) is union of \(e\) distinct lines \(L_1, \ldots, L_e\) (not contained in \(C\)). Then it is clear that \(C.D = \sum_{1 \leq i \leq e} C.L_i\). By the former studies, we have \(C.L_i = d\). Therefore \(C.D = \sum_{1 \leq i \leq e} d = de\). So if we can prove that \(C.D\) depends only on \(\deg D\), then we are done.

Intuitively, the intersection number \(C.D\) varies continuously with \(D\), and \(D\) runs in a connected topological space (parametrized by its coefficients in \(\mathbb{C}\)). As this function takes discrete values, it must be constant, hence depends only on \(\deg D\). To be more rigorous, one can show that \(C.D\) can be computed in terms of the Euler-Poincaré characteristic of some line bundles on \(\mathbb{P}^2(\mathbb{C})\). It is known that the Euler-Poincaré characteristic various continuously with the line bundles.

A slightly different idea is to compare \(C.D\) with \(C.D_0\) when \(\deg D = \deg D_0\). If \(D_0\) is defined by \(G_0(x, y, z)\), then \(G(x, y, z)/G_0(x, y, z)\) becomes a rational function on \(\mathbb{P}^2(\mathbb{C})\) and its restriction \(h\) to \(C\) is a rational function on \(C\). The difference \(C.D - C.D_0\) is equal to the degree of \(h\) (sum of the order of the zeros minus the sum of the orders of poles of \(h\), all counted with multiplicities). Now it is known that the degree of a rational function on a projective curve is always zero. So \(C.D - C.D_0 = 0\) and \(C.D_0\) depends only on \(\deg D\).

More information can be found in Fulton’s introductory book [2].

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\(^1\)The projective space \(\mathbb{P}^n(\mathbb{C})\) is the set of vector lines in \(\mathbb{C}^{n+1}\) up to scalar multiplication by invertible elements of \(\mathbb{C}\). Every point has homogeneous coordinates \([a_0, \ldots, a_n]\) (a direction vector of the corresponding line), and \([a_0, \ldots, a_n] = [b_0, \ldots, b_n]\) in \(\mathbb{P}^n(\mathbb{C})\) if and only if there exists \(\lambda \in \mathbb{C}\) such that \(b_i = \lambda a_i\) for all \(i = 0, \ldots, n\).
2. Bertini’s theorem

An algebraic variety is locally defined by polynomial equations

\[ F_1(x_1, \ldots, x_n) = 0, \ldots, F_m(x_1, \ldots, x_n) = 0. \]

We will say it is smooth or non-singular if the Jacobian matrix of \( F_1, \ldots, F_m \) has the maximal rank (that is, the codimension of the variety in the ambient affine space) everywhere. Over the complex numbers (resp. real numbers), the same equations then define a complex manifold (resp. smooth manifold).

The variety defined by a single polynomial in one variable is smooth if and only if the polynomial has no multiple zeros.

A polynomial \( f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{C}[x] \) has in general \( d \) distinct roots (equivalently, the discriminant \( \Delta(a_0, \ldots, a_{d-1}) \), which is a polynomial function in the \( a_i \)'s, doesn’t vanish). The algebraic variety defined by the equation \( x^d + a_{d-1}x^{d-1} + \cdots + a_0 = 0 \) (the \( a_i \)'s are viewed as variables) is smooth. The previous statement says that if we intersect this smooth variety with a sufficiently general line (by fixing the values of the \( a_i \)'s), then we still have a smooth algebraic variety. This is a special form of the following

**Theorem 2.1. (Bertini’s theorem)** Let \( X \) be a smooth subvariety of a projective space \( \mathbb{P}^n(\mathbb{C}) \). Then there exists a hyperplane \( H \) in \( \mathbb{P}^n(\mathbb{C}) \) such that \( X \cap H \) is smooth.

The proof is fairly easy and involves just some dimension countings. First, a hyperplane \( H \) is defined by an equation

\[ a_0x_0 + \cdots + a_nx_n = 0 \]

for some non-zero vector \((a_0, \ldots, a_n)\). Moreover, two non-zero vectors define the same hyperplane if and only if they are colinear. So the set of hyperplanes is parametrized by the points of \( \mathbb{P}^n(\mathbb{C}) \). This set is denoted by \( \mathbb{P}^n(\mathbb{C})^\vee \) and called the dual space of \( \mathbb{P}^n(\mathbb{C}) \). We just saw that it can be endowed with the structure of a projective space of dimension \( n \).

Let \( x \in X \) and let \( H \in \mathbb{P}^n(\mathbb{C})^\vee \) be a hyperplane passing through \( x \). Using the Jacobian criterion of smoothness that we saw before, it is easy to see that \( x \) is a smooth point of \( X \cap H \) if and only if the tangent space \( T_x(X) \) to \( X \) at \( x \) is not contained in \( H \) (there is a slight abuse of the notion of tangent spaces here, but it is essentially correct). Now consider the subvariety

\[ Z := \{(x, H) \in X \times \mathbb{P}^n(\mathbb{C})^\vee \mid T_x(X) \subseteq H\} \]

of \( X \times \mathbb{P}^n(\mathbb{C})^\vee \), endowed with the canonical projections \( p, q \) to \( X, \mathbb{P}^n(\mathbb{C})^\vee \) respectively. The above observations imply that Bertini’s theorem is equivalent to \( q : Z \to \mathbb{P}^n(\mathbb{C})^\vee \) non-surjective (an \( H \notin q(Z) \) will intersect \( X \) along a smooth subvariety).

Let us compute the dimension of \( Z \). Let \( d \) be the dimension of \( X \). Consider the surjective map \( p : Z \to X \). For every \( x \in X \), \( p^{-1}(x) \) is the set of hyperplanes \( H \) containing the projective linear subspace \( T_x(X) \) of dimension
It is easy to see that $p^{-1}(x)$ is a projective linear subspace of $\mathbb{P}^n(\mathbb{C})^\vee$ of dimension $n - (d + 1)$. So $\dim p^{-1}(x) = n - (d + 1)$ for all $x \in X$ and we get

$$\dim Z \leq \dim X + n - (d + 1) = n - 1.$$ 

As $\dim \mathbb{P}^n(\mathbb{C})^\vee = n > \dim Z$, the map $q : Z \rightarrow \mathbb{P}^n(\mathbb{C})^\vee$ can’t be surjective, and the theorem is proved.

**Remark 2.2** Contrarily to Bézout’s theorem, Bertin’s theorem exists in the affine (and not projective) form.

**Remark 2.3** Analogous statement for smooth submanifolds of $\mathbb{R}^n$ can be proved using Sard’s theorem.

**Remark 2.4** We worked over the complex numbers, but the proof is exactly the same over any algebraically closed field $K$. Moreover, without any hypothesis on the base field $K$, with the notation in the proof, we can show that $q(Z)$ is contained is some hypersurface $\{F = 0\}$ of $\mathbb{P}^n(\mathbb{C})^\vee$. So if $K$ is infinite, there exists $(a_0, \ldots, a_n) \in K^{n+1}$ non-zero such that $F(a_0, \ldots, a_n) \neq 0$. The corresponding hyperplane $H \notin q(Z)$ and we have $X \cap H$ smooth (just regular if the base field is not necessarily perfect). Therefore, Bertini’s theorem holds over any perfect infinite field. For the finite fields case, see the next section.

**Remark 2.5** (*Embedding problem*) Whitney proved that a smooth manifold of dimension $d$ (Hausdorff and with a countable basis) can be embedded into $\mathbb{R}^{2d}$. It is well known that any projective smooth variety (over an infinite field) of dimension $d$ can be embedded in $\mathbb{P}^{2d+1}$. If $X$ is a possibly singular projective algebraic variety, define for all $e \geq 0$, $X_e$ be the set of points of $X$ with $\dim T_x(X) = e$ and let $r = \max_e \{e + \dim X_e\}$. Using Bertini’s type techniques, it can be shown [5] that $X$ can be embedded into a smooth projective variety of dimension $r$.

### 3. The case of finite fields

Let’s start with an example of N. Katz. Let $p$ be a prime number and consider the hypersurface $X$ in $\mathbb{P}^{2n+1}$ over $\mathbb{F}_p$ defined by the equation

$$(x_0^p y_0 - x_0 y_0^p) + \cdots + (x_n^p y_n - x_n y_n^p) = 0.$$ 

The tangent space to $X$ at a point $(a_0, \ldots, a_n, b_0, \ldots, b_n)$ is

$$-b_0 x_0 - \cdots - b_n x_n + a_0 y_0 + \cdots + a_n y_n = 0.$$ 

As every point of $\mathbb{P}^{2n+1}$ with coordinates in $\mathbb{F}_p$ (the so called rational points) belongs to $X$, we see that every hyperplane is tangent to $X$ at some point. Therefore Bertini’s theorem fails in this case. This is because there are only finitely many hyperplanes in a given projective space over a finite field. To remedy this problem, we can consider hypersurfaces of various degrees.

**Theorem 3.1.** (Gabber [3], Poonen [7]) Let $X$ be a smooth subvariety of $\mathbb{P}^n$ over any field. Then there exists a hypersurface $T$ of $\mathbb{P}^n$ such that $X \cap T$ is smooth.
If $X$ is geometrically connected (i.e. connected over the algebraic closure of the base field), then $T$ can be choosen to ensure that $X \cap T$ is smooth and geometrically connected.

In fact, there are infinitely many such $T$. Suppose we are working over a finite field. For every degree $r \geq 1$, we can count the ratio $R_r$ of the “good hypersurfaces” ($X \cap T$ smooth and geometrically connected) of degree $r$ with respect to all hypersurfaces of degree $r$. Then we have the following beautiful result:

**Theorem 3.2.** (Poonen [7]) Suppose $X$ is a smooth subvariety of dimension $d$ in some projective space $\mathbb{P}^n$ over a finite field of cardinality $q$. Then

$$\lim_{r \to +\infty} R_r = \zeta_X(d+1)^{-1} > 0,$$

where $\zeta_X$ is the zeta function defined by

$$\zeta_X(s) = \prod_{x \in X} \left(1 - q^{-(\deg x)s}\right)^{-1}.$$

There are various (and sometimes surprising) applications of this result. See [7] for more information.

4. The relative situation

What happens if instead of a single smooth subvariety, we have a family of smooth subvarieties? We now use scheme language. Namely, if $A$ is a noetherian ring (say, $\mathbb{Z}$) and if $X$ is a smooth subscheme of some projective space $\mathbb{P}^n$ over $A$ (so $X$ is a family of smooth projective varieties, parametrized by the points of $\text{Spec } A$), does a “general” hypersurface $S$ in $\mathbb{P}^n$ cut out $X$ along a smooth subscheme? An example of Fakhruddin (with $\mathbb{P}^2_\mathbb{Z}$ embedded in $\mathbb{P}^9_\mathbb{Z}$, [7], 5.15) shows that the answer is no. There are however positive answers if we allow hypersurfaces with coefficients in a finite extension of $A$.

**Theorem 4.1.** (Autissier [1], Tamagawa [8]) Let $A$ be the ring of integers (or $S$-integers) of a numbers fields or the ring of regular functions of an affine regular curve over a finite field (so in all cases $\text{Frac } A$ is a global field). Let $X$ be a smooth closed subscheme of some projective space $\mathbb{P}^n_A$ (or more generally a projective bundle over $A$). Then there exists a finite extension $B/A$ and a hyperplane $H$ in $\mathbb{P}^n_B$ such that $H \cap X_B$ is smooth over $B$.

Moreover, in the function field case, $B$ can be taken to be finite étale over $A$.

The proof is partly based on a theorem of Rumely and Moret-Bailly (see [6]) which states that an irreducible quasi-projective scheme over $A$ with geometrically irreducible generic fiber admits a section after a finite flat extension of $A$. 
When the base ring $A$ is more general, the above theorem is unlikely true. Instead, we can look for Bertini’s type theorem for weaker properties and a Rumely-Moret-Bailly’s type theorem.

**Theorem 4.2.** ([4]) Let $A$ be a noetherian ring. Let $X$ be a reduced subscheme of a projective space $\mathbb{P}^n_A$, flat over $A$. Then there exists a hypersurface $T$ in $\mathbb{P}^n_A$ such that $X \cap T$ is flat over $A$. If for all $s \in \text{Spec } A$, the fiber $X_s$ is Cohen-Macaulay (resp. locally complete intersection), then we can further require the same property for $X \cap T$.

Note that smooth $\implies$ locally complete intersection $\implies$ Cohen-Macaulay.

**Corollary 4.3.** Let $A$ be a noetherian ring.

1. Let $X$ be a projective flat scheme over $A$. Then $X$ has a finite quasi-section (i.e. a closed subscheme of $X$ finite surjective over $\text{Spec } A$).

2. Let $M$ be a finitely generated projective module over $A$. Then there exists a finite faithfully flat extension $B/A$ such that $M \otimes_A B$ is a direct sum of invertible modules over $B$.

Part (1) is a generalization of Rumely-Moret-Bailly, though we have to suppose $X$ projective (instead of quasi-projective). But Rumely-Moret-Bailly is known to be false if $A$ doesn’t satisfy the hypothesis of Theorem 4.1.

**References**


