Equilibrium measures on saddle sets of holomorphic maps on $\mathbb{P}^2$

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Abstract

We consider the case of hyperbolic basic sets $\Lambda$ of saddle type for holomorphic maps $f : \mathbb{P}^2 \mathbb{C} \to \mathbb{P}^2 \mathbb{C}$. We study equilibrium measures $\mu_\phi$ associated to a class of Hölder potentials $\phi$ on $\Lambda$, and find the measures $\mu_\phi$ of iterates of arbitrary Bowen balls. Estimates for the pointwise dimension $\delta_{\mu_\phi}$ of $\mu_\phi$ that involve Lyapunov exponents and a correction term are found, and also a formula for the Hausdorff dimension of $\mu_\phi$ in the case when the preimage counting function is constant on $\Lambda$. For terminal/minimal saddle sets we prove that an invariant measure $\nu$ obtained as a wedge product of two positive closed currents, is in fact the measure of maximal entropy for the restriction $f|_\Lambda$. This allows then to obtain formulas for the measure $\nu$ of arbitrary balls, and to give a formula for the pointwise dimension and the Hausdorff dimension of $\nu$.

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1 Introduction.

The dynamics of holomorphic endomorphisms in higher dimensions presents many interesting geometric and ergodic aspects based on the interplay of complex dynamics, hyperbolic smooth dynamics and ergodic theory (see [9], [6], [8], etc.) In this paper we study the problem of holomorphic endomorphisms of $\mathbb{P}^2 \mathbb{C}$ which are hyperbolic on basic sets of saddle type $\Lambda$ (see [8] and [20] for hyperbolicity in the non-invertible case). An arbitrary holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ is given by three homogeneous polynomials $[P_0 : P_1 : P_2]$ each of them having the same degree $d$. We will say that $d$ is the degree (or the algebraic degree) of $f$. For such maps $f : \mathbb{P}^2 \to \mathbb{P}^2$ Fornaess and Sibony have defined a positive closed current $T = \lim_n \frac{(f^n)^* \omega}{d^n}$ which can be written locally in $\mathbb{C}^3 \setminus \{0\}$ as $dd^cG$ where $G$ is the Green function associated to $f$. This allows them to define a probability measure $\mu = T \wedge T$ which is $f$-invariant and mixing (see [9]). In the case when $f$ is hyperbolic on a basic set $\Lambda$ one may study also the measure of maximal entropy of the restriction of $f$ on $\Lambda$. This measure has different properties than $\mu$; for instance if $\Lambda$ is a saddle set then it has both negative and positive Lyapunov exponents.

Given a compact $f$-invariant set $\Lambda$ one forms the natural extension (or inverse limit) $\hat{\Lambda} := \{(x, x_{-1}, x_{-2}, \ldots)\}$, $f(x_{-i}) = x_{-i+1}, x_{-i} \in \Lambda, i \geq 1$. The natural extension is a compact metric space with the canonical metric (see [20], [8], etc.) On the natural extension $\hat{\Lambda}$ there exists a shift
homeomorphism \( \hat{f} : \hat{\Lambda} \rightarrow \hat{\Lambda} \) defined by \( \hat{f}(\hat{x}) = (f(x), x, x_{-1}, \ldots) \), \( \hat{x} \in \hat{\Lambda} \). We denote the canonical projection by \( \pi : \hat{\Lambda} \rightarrow \Lambda \), \( \pi(\hat{x}) = x \), \( \hat{x} \in \hat{\Lambda} \).

Hyperbolicity for endomorphisms is defined as a continuous splitting of the tangent bundle over \( \hat{\Lambda} \) into stable and unstable directions (see [8], [20]); the stable directions depend only on base points, but unstable directions depend nevertheless on whole prehistories \( \hat{x} \in \hat{\Lambda} \) (i.e past trajectories) and not only on \( x \). Hyperbolic maps on basic sets were also studied for instance in [4], [15], [17], etc.

If \( f \) is hyperbolic on \( \Lambda \) then we have local stable manifolds \( W^s_r(x) \) and local unstable manifolds \( W^u_r(\hat{x}) \) where \( \hat{x} \in \hat{\Lambda} \). Also notice that \( \Lambda \) is not necessarily totally invariant. Thus for \( x \in \Lambda \) we may have some \( f \)-preimages of \( x \) in \( \Lambda \) and others outside \( \Lambda \). Moreover the number of \( f \)-preimages of \( x \) that remain in \( \Lambda \) may vary with \( x \). So the endomorphism case is subtle and very different from the case of diffeomorphisms.

Now given an arbitrary probability measure \( \mu \) on a compact metric space \( X \) one can define the lower pointwise dimension and the upper pointwise dimension at \( x \in X \) respectively by:

\[
\delta_\mu(x) := \liminf_{\rho \rightarrow 0} \frac{\log \mu(B(x, \rho))}{\log \rho}, \quad \text{and} \quad \bar{\delta}_\mu(x) := \limsup_{\rho \rightarrow 0} \frac{\log \mu(B(x, \rho))}{\log \rho}
\]

In case they coincide, we call the common value \( \delta_\mu(x) \) the pointwise dimension of \( \mu \) at \( x \in X \) (see [19]). Also one can define the Hausdorff dimension of \( \mu \) by:

\[
HD(\mu) := \inf \{ HD(Z), \ Z \text{ borelian set with } \mu(X \setminus Z) = 0 \}
\]

In [26] Young proved that for a hyperbolic measure \( \mu \) (i.e without zero Lyapunov exponents) invariated by a smooth diffeomorphism \( f \) of a surface, we have \( \mu \)-a.e the formula

\[
\delta_\mu = \frac{h_\mu}{\chi_u(\mu)} = \frac{1}{\chi_u(\mu)} - \frac{1}{\chi_s(\mu)},
\]

where \( \chi_s(\mu), \chi_u(\mu) \) are the negative, respectively positive Lyapunov exponents of \( \mu \).

For analytic endomorphisms \( f \) on the Riemann sphere \( \mathbb{P}^1 \mathbb{C} \), Manning proved in [12] that if \( f \) is hyperbolic on its Julia set \( J(f) \) and has no critical points in \( J(f) \), then for any ergodic \( f \)-invariant probability measure \( \mu \) on \( J(f) \) the Hausdorff dimension of \( \mu \) is given by:

\[
HD(\mu) = \frac{h_\mu}{\chi(\mu)},
\]

where \( \chi(\mu) \) is the (only) Lyapunov exponent of \( \mu \). This formula was later extended by Mane (see [11]) to the case of all rational maps (i.e not only hyperbolic) and invariant ergodic probabilities with positive Lyapunov exponent. However the situation for higher dimensional endomorphisms and their invariant measures is different (see also [7]). In the case of polynomial endomorphisms Binder and DeMarco gave in [1] estimates for the Hausdorff dimension of the measure of maximal entropy \( \mu = T \wedge T \) which involve Lyapunov exponents of \( \mu \). In [5] Dinh and Dupont extended those estimates to the case of meromorphic endomorphisms of \( \mathbb{P}^k \).

Our case here is different in that we study the measure of maximal entropy of the restriction of \( f \) to a saddle basic set \( \Lambda \), and not the measure of maximal entropy on the whole of \( \mathbb{P}^2 \). In fact we
will consider more generally, equilibrium (Gibbs) measures for a certain class of H"older potentials $\phi$ on $\Lambda$, such that $\phi$ satisfies an inequality relating the number of $f$-preimages remaining in $\Lambda$ and the topological pressure of $\phi$.

In the case when $\Lambda$ is a terminal saddle basic set, i.e when the iterates of $f$ form a normal family on $W^u(\hat{\Lambda}) \setminus \Lambda$, Diller and Jonsson have introduced a measure $\nu_i = \sigma^u \wedge T$ which is $f$-invariant and supported on $\Lambda$. For the case of a minimal saddle basic set $\Lambda$ for an $s$-hyperbolic map on $\mathbb{P}^2$, Fornaess and Sibony introduced in [8] a probability measure $\nu = T \wedge \sigma$ as a wedge product of positive closed currents; this measure is also $f$-invariant and mixing. Examples of terminal sets can be obtained by perturbations of already known examples (see [4], [8]).

Our main results are:

First we study equilibrium measures of H"older potentials on a basic set $\Lambda$. If the smooth endomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ is hyperbolic on $\Lambda$ and if $\phi$ is a H"older potential on $\Lambda$, there exists a unique equilibrium measure $\mu_\phi$ for $\phi$, i.e a measure which maximizes in the Variational Principle (see [25], [10], etc.) $P(\phi) = \sup \{ h_\mu + \int \phi d\mu, \mu \text{ $f$-invariant} \}$. Equilibrium measures for stable potentials were also used for instance in [15] to show that the stable dimension cannot be 2 on a basic saddle set $\Lambda$ in $\mathbb{P}^2$.

In the sequel we shall use H"older continuous potentials $\phi$ which satisfy the following inequality

$$\phi + \log d' < P(\phi) \text{ on } \Lambda,$$

where $d'$ is an upper bound on the number of $f$-preimages in $\Lambda$ of an arbitrary point. Notice that $d'$ may even be 1, i.e the restriction to $\Lambda$ is a homeomorphism (as in the examples from [17]). In Theorem 1 we will give precise estimates of the measure $\mu_\phi$ of an arbitrary iterate of a Bowen ball. This will help us obtain estimates and in some cases even exact formulas for the pointwise dimension and the Hausdorff dimension of $\mu_\phi$. In particular we prove that the measure $\mu_\phi$ is exact dimensional in those cases. In Corollary 1 we give such estimates for $\delta_{\mu_\phi}$ in the case when the number of preimages remaining in $\Lambda$ is not constant.

Also we obtain in Corollary 2 the Jacobian (in the sense of Parry [18]) of $\mu_\phi$ with respect to an iterate $f^m$ in the case when the preimage counting function is constant on $\Lambda$.

We prove in Theorem 2 that for a terminal set $\Lambda$ the measure $\nu_i$ from above, is in fact the measure of maximal entropy $\mu_0$ of the restriction $f|_\Lambda$; and if $\Lambda$ is minimal and $c$-hyperbolic for the Axiom A holomorphic map $f$, then the measure $\nu$ from above is equal to $\mu_0$ as well.

In Corollaries 2, 3 we estimate, and in certain cases give formulas for the pointwise dimension of the measures $\nu_i$, $\nu$ on terminal, respectively minimal saddle sets; in Corollary 4 we give complete formulas for the pointwise dimension of $\nu$, for all the possible minimal $c$-hyperbolic saddle sets of a map of degree 2.

In the end we will give also examples of holomorphic maps and equilibrium measures of H"older potentials on terminal saddle sets, for which the (upper/lower) pointwise dimension can be estimated/computed.
2 Equilibrium measures of Hölder potentials on saddle basic sets.

In the sequel we consider a holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$, of degree $d$; this means that $f$ is given as $[f_0 : f_1 : f_2]$, where $f_0, f_1, f_2$ are homogeneous polynomials in coordinates $(z_0, z_1, z_2)$, of common degree $d$. We will work on a basic set $\Lambda$, i.e., an $f$-invariant compact set $\Lambda$ for which there exists a neighbourhood $U$ such that $\cap_{n \in \mathbb{Z}} f^n(U) = \Lambda$ and $f|_\Lambda$ is topologically transitive. If the non-invertible map $f$ is hyperbolic on the basic set $\Lambda$, then from the Spectral Decomposition Theorem (see [20], [10]) $\Lambda$ can be written as the union of finitely many mutually disjoint subsets $\Lambda_i$ s.t there exists a positive integer $m$ with $f^m(\Lambda_i) = \Lambda_i$, and $f^m|_{\Lambda_i}$ topologically mixing.

Notice that we only have forward invariance of $\Lambda$, but not total invariance; this means that $f(\Lambda) = \Lambda$ but an arbitrary point $x \in \Lambda$ may have in general also $f$-preimages outside $\Lambda$.

On a basic set $\Lambda$ for $f$, let us now consider a continuous potential $\phi : \Lambda \to \mathbb{R}$. Then from the Variational Principle, we know that the topological pressure satisfies $P(\phi) = \sup \{h_\mu + \int \phi \, d\mu; \mu \text{-- invariant probability measure on } \Lambda \}$. If $\phi$ is Hölder continuous on $\Lambda$ and $f$ is hyperbolic on $\Lambda$, then there exists a unique measure $\mu_\phi$ which attains the supremum in the Variational Principle, and it is called the equilibrium measure of $\phi$ (see [10] for the diffeomorphism case and [16] for the endomorphism case). It follows from above that $h_{\mu_\phi} + \int \phi \, d\mu_\phi = P(\phi)$.

Denote by $B_n(x, \varepsilon) := \{y \in \Lambda, d(f^i y, f^i x) < \varepsilon, 0 \leq i \leq n - 1\}$ a Bowen ball, i.e., the set of points which $\varepsilon$-follow the orbit of order $n$ of $x$.

Recall now that there exists a unique correspondence between $f$-invariant measures $\mu$ on $\Lambda$ and $\hat{f}$-invariant measures $\hat{\mu}$ on $\hat{\Lambda}$ and that $\pi_* \hat{\mu} = \mu$. Then by going to $\hat{\Lambda}$ and then projecting, we showed in [16] that the equilibrium measure $\mu_\phi$ satisfies the following estimates on Bowen balls:

\[
\frac{1}{C} e^{S_n \phi(x) - nP(\phi)} \leq \mu_\phi(B_n(x, \varepsilon)) \leq Ce^{S_n \phi(x) - nP(\phi)},
\]

for every $n > 0$, where $S_n \phi(x) := \phi(x) + \ldots + \phi(f^{n-1}(x))$ is the consecutive sum and $C$ is a positive constant independent of $x, n$.

**Definition 1.** Given a basic set $\Lambda$ for the map $f$, denote by $d(x) := Card\{f^{-1}(x) \cap \Lambda\}, x \in \Lambda$ and call it the preimage counting function on $\Lambda$.

If $\Lambda$ is a connected basic set such that the critical set does not intersect $\Lambda$, i.e. $C_f \cap \Lambda = \emptyset$ and if there exists a neighbourhood $U$ of $\Lambda$ with $f^{-1}(\Lambda) \cap U = \Lambda$, then the preimage counting function $d(\cdot)$ is constant on $\Lambda$ (see [15]). Notice also that the preimage counting function is not necessarily preserved when taking perturbations; see Example 1) at the end of paper, where by perturbing a 2-to-1 basic set we may obtain a basic set on which the restriction is 1-to-1.

Now let us recall the following Lemma proved in [14] which relates the measures of various subsets of Bowen balls, which by iteration go to the same image:

**Lemma 1.** Let $f$ be an endomorphism, hyperbolic on a basic set $\Lambda$; consider also a Holder continuous potential $\phi$ on $\Lambda$ and $\mu_\phi$ be the unique equilibrium measure of $\phi$. Let a small $\varepsilon > 0$, two
disjoint Bowen balls $B_k(y_1, \varepsilon), B_m(y_2, \varepsilon)$ and a borelian set $A \subset f^k(B_k(y_1, \varepsilon)) \cap f^m(B_m(y_2, \varepsilon))$, s.t. $\mu_\phi(A) > 0$; denote by $A_1 := f^{-k}A \cap B_k(y_1, \varepsilon), A_2 := f^{-m}A \cap B_m(y_2, \varepsilon)$ and assume that $\mu_\phi(\partial A_1) = \mu_\phi(\partial A_2) = 0$. Then there exists a positive constant $C_\varepsilon$ independent of $k, m, y_1, y_2$ such that

$$\frac{1}{C_\varepsilon} \mu_\phi(A_2) \cdot \frac{e^{S_k\phi(y_1)}_m}{e^{S_m\phi(y_2)}_m} \cdot e^{(m-k)P(\phi)} \leq \mu_\phi(A_1) \leq C_\varepsilon \mu_\phi(A_2) \cdot \frac{e^{S_k\phi(y_1)}_m}{e^{S_m\phi(y_2)}_m} \cdot e^{(m-k)P(\phi)}$$

We shall say that two quantities $Q_1(n, x)$ and $Q_2(n, x)$ are comparable if there exists a positive constant $C$ such that $\frac{1}{C} Q_1(n, x) \leq Q_2(n, x) \leq C Q_1(n, x)$ for all $n > 0$ and $x$; this will be denoted by $Q_1 \approx Q_2$. The constant $C$ is sometimes called a comparability constant.

The next Definition is similar to that of s-hyperbolicity (see [8]), but it refers only to a fixed basic set, not to the whole nonwandering set.

**Definition 2.** Let $\Lambda$ be a basic set for a map $f$ on a manifold $M$. We say that $f$ is c-hyperbolic on $\Lambda$ if $f$ is hyperbolic on $\Lambda$, there exists a neighbourhood $U$ of $\Lambda$ with $f^{-1}(\Lambda) \cap U = \Lambda$ and if the critical set $C_f$ of $f$ does not intersect $\Lambda$.

Define now the set

$$B(n, k, z, \varepsilon) := f^n(B_{n+k}(z, \varepsilon)), z \in \Lambda, n > 0, k > 0$$

This set is an iterate of a Bowen ball and when $n$ and $k$ vary, we can adjust the sides of $B(n, k, z, \varepsilon)$ arbitrarily; in particular we can make it have (almost) equal sides in the stable and unstable directions.

**Theorem 1.** Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map of degree $d$ and $\Lambda$ a basic set such that $f$ is c-hyperbolic on $\Lambda$ and the preimage counting function is constant and equal to $d'$ on $\Lambda$. Consider also a Hölder continuous potential $\phi$ on $\Lambda$ which satisfies $\phi(x) + \log d' < P(\phi), \forall x \in \Lambda$. Then

$$\mu_\phi(B(n, k, z, \varepsilon)) \approx \frac{e^{S_{n+k}\phi(z)}}{(d')^k}$$

Moreover the pointwise dimension of $\mu_\phi$ exists $\mu_\phi$-a.e, is denoted by $\delta_{\mu_\phi}$ and we have $\mu_\phi$-a.e:

$$\delta_{\mu_\phi} = HD(\mu_\phi) = h_{\mu_\phi} \left( \frac{1}{\chi_u(\mu_\phi)} - \frac{1}{\chi_s(\mu_\phi)} \right) + \log d' \cdot \frac{1}{\chi_s(\mu_\phi)}$$

**Proof.** Recall that $S_n \phi(y)$ is defined as the consecutive sum $\phi(y) + \phi(f(y)) + \ldots + \phi(f^{n-1}(y))$, $y \in \Lambda$. Let us take a point $z \in \Lambda$, a positive integer $n$ and $x := f^n(z)$. By definition $B(n, k, z, \varepsilon) = f^n(B_{n+k}(z, \varepsilon))$. Now we assumed that the preimage counting function $d(\cdot)$ is constant on $\Lambda$ and equal to $d'$. Thus every point $y$ from $\Lambda$ has exactly $d'$ $f$-preimages remaining in $\Lambda$.

Next we assumed $\phi + \log d' < P(\phi)$; so if we define the real-valued function $\tilde{\phi} := \phi - P(\phi) + \log d'$, then $P(\tilde{\phi}) = P(\phi) - P(\phi) + \log d' = \log d'$ and also $\tilde{\phi} < 0$ on $\Lambda$. Then since $\phi$ and $\tilde{\phi}$ are cohomologous, they have the same equilibrium measure $\mu_\phi$. Therefore we will assume in the sequel that $\phi < 0$ on $\Lambda$ and $P(\phi) = \log d'$. 

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Consider now prehistories in \( \hat{\Lambda} \) of an arbitrary point \( y \in B(n, k, z, \varepsilon) \subset \Lambda \); for such a prehistory \( \hat{y} = (y, y_{-1}, y_{-2}, \ldots) \) let us denote by \( n(\hat{y}) \) the smallest positive integer \( m \) satisfying \( S_m \phi(y_m) \leq S_n \phi(z) \); since \( \phi < 0 \) on \( \Lambda \), it is clear that such an \( m \) must exist for any prehistory \( \hat{y} \in \hat{\Lambda} \). So \( S_{n(\hat{y})-1} \phi(y_{n(\hat{y})+1}) > S_n \phi(z) \) while \( S_{n(\hat{y})} \phi(y_{-n(\hat{y})}) \leq S_n \phi(z) \). Call such a finite prehistory \( (y, y_{-1}, \ldots, y_{-n(\hat{y})}) \) a "maximal prehistory".

We intend to get an estimate of the measure \( \mu_{\phi}(B(n, k, z, \varepsilon)) \) from the \( f \)-invariance of \( \mu_{\phi} \) and the comparison estimates between the different pieces of its preimage set using Lemma 1. We will write \( B(n, k, z, \varepsilon) \) as a union of subsets \( E \) which are contained in forward iterates of Bowen balls; the question is how these iterates intersect and what is the relation between various components of preimage sets of different orders.

In fact since we know that \( B(n, k, z, \varepsilon) = f^n(B_{n+k}(z, \varepsilon)) \), it means that every point in \( B(n, k, z, \varepsilon) \) has an \( f^n \)-preimage in \( B_{n+k}(z, \varepsilon) \). But in estimating \( \mu_{\phi}(B(n, k, z, \varepsilon)) \) we have to consider all \( f^n \)-preimages in \( \Lambda \) of points from \( B(n, k, z, \varepsilon) \), in order to use the \( f \)-invariance of \( \mu_{\phi} \); so we will compare various \( f^m \)-preimages of subsets of \( B(n, k, z, \varepsilon) \) with the corresponding \( f^{n}_m \)-preimages from \( B_{n+k}(z, \varepsilon) \). Our standard for comparison of all these preimages of various orders of points in \( B(n, k, z, \varepsilon) \), will be those preimages belonging to \( B_{n+k}(z, \varepsilon) \).

Take an arbitrary point \( y \in B(n, k, z, \varepsilon) \) and a prehistory \( \hat{y} \in \hat{\Lambda} \) of \( y \); then \( y \in f^n(B_{n+k}(z, \varepsilon)) \cap f^n(\hat{y})(B_{n(\hat{y})}(y_{-n(\hat{y})}, \varepsilon)) \). Let us take all the prehistories \( \hat{y} \) of \( y \) in \( \hat{\Lambda} \); along such each such prehistory we go until reaching the preimage of order \( n(\hat{y}) \). It is clear that there exists only a finite collection \( \mathcal{P}(y) \) of such maximal prehistories of \( y \), since \( \phi < 0 \) on the compact set \( \Lambda \) and since we cannot continue to add indefinitely values of \( \phi \) on consecutive preimages until reaching the value \( S_n \phi(z) \). Denote by \( E(y) \) the intersection of the iterates \( f^n(\hat{y})(B_{n(\hat{y})}(y_{-n(\hat{y})}, \varepsilon)) \) over all the prehistories \( \hat{y} \) of \( y \) in \( \hat{\Lambda} \) (i.e in fact over the finite prehistories of \( \mathcal{P}(y) \)).

We shall cover the set \( B(n, k, z, \varepsilon) \) with mutually disjoint subsets of various sets of type \( E(y) \), \( y \in B(n, k, z, \varepsilon) \). Actually we can cover \( B(n, k, z, \varepsilon) \) with a collection \( \mathcal{F} \) of sets \( F \), each such \( F \) belonging to \( E(y) \) for some \( y \in B(n, k, z, \varepsilon) \). Now if \( F \subset E(y) \), denote by

\[
F(\hat{y}) := B(y_{-n(\hat{y})}, \varepsilon) \cap f^{-n(\hat{y})}(F)
\]

From the definition of \( E(y) \) we know that \( F = f^n(\hat{y})(F(\hat{y})) \), where we recall that \( n(\hat{y}) \) was defined above with respect to \( S_n \phi(z) \). Recall also that for \( y \in \Lambda \), the set of "maximal" prehistories of type \( (y, y_{-1}, \ldots, y_{-n(\hat{y})}) \) for \( \hat{y} \in \hat{\Lambda} \) prehistory of \( y \), is denoted by \( \mathcal{P}(y) \).

Take now two prehistories \( \hat{y}, \hat{y}' \) of \( y \) belonging to \( \hat{\Lambda} \) and go along these prehistories until we reach \( n(\hat{y}) \) and \( n(\hat{y}') \) respectively. We want to compare the measure \( \mu_{\phi} \) on the preimages of \( F \) along these maximal prehistories by using Lemma 1.

\[
\frac{1}{C} \mu_{\phi}(F(\hat{y})) e^{S_{n(\hat{y})} \phi(y_{-n(\hat{y})})} e^{(n(\hat{y}) - n(\hat{y}')) P(\phi)} \leq \mu_{\phi}(F(\hat{y}')) \leq C \mu_{\phi}(F(\hat{y})) e^{S_{n(\hat{y})} \phi(y'_{-n(\hat{y}'})} e^{(n(\hat{y}) - n(\hat{y}')) P(\phi)},
\]

where \( C \) is a positive constant independent of \( y, \hat{y}, \hat{y}', x \). On the other hand from the definition of \( n(\hat{y}), n(\hat{y}') \) we know that \( |S_{n(\hat{y})} \phi(y_{-n(\hat{y})}) - S_n \phi(z)| \leq M \) and \( |S_{n(\hat{y})} \phi(y'_{-n(\hat{y}'}) - S_n \phi(z)| \leq M \), for
some positive constant $M$. Therefore since $P(\phi) = \log d'$ we obtain (by taking perhaps a larger $C$) that:

$$\frac{1}{C} \mu(\phi)(F(\hat{y})) e^{(n(\hat{y})-n(\hat{y}')) \log d'} \leq \mu(\phi)(F(\hat{y}')) \leq C \mu(\phi)(F(\hat{y})) e^{(n(\hat{y})-n(\hat{y}')) \log d'}$$

(4)

We add now the measures of various preimages $F(\hat{y})$ of $F$, over all finite "maximal" prehistories from $\mathcal{P}(y)$, in order to obtain the measure $\mu(\phi)$, where recall that $F \subset E(y)$. We take into consideration the fact that any point in $\Lambda$ has $(d')^m f^m$-preimages in $\Lambda$, $m > 0$. Thus if $n(\hat{y})$ is say the largest maximal order associated to any prehistory from $\mathcal{P}(y)$ and if $\hat{y}'$ is another prehistory with $n(\hat{y}') = n(\hat{y}) - 1$, then from (4) it follows that

$$\mu(\phi)(F(\hat{y})) \approx \mu(\phi)(F(\hat{y}')) \cdot \frac{1}{d'},$$

where the comparability constant is $C$ above (i.e a positive universal constant). Now if we add the measures $\mu(\phi)(F(\hat{y}))$ over all prehistories which coincide with $\hat{y}'$ up to order $n(\hat{y}) - 1$, we obtain $\mu(\phi)(F(\hat{y}'))$. Similarly we order the integers $n(\hat{y}), \hat{y} \in \mathcal{P}(y)$ in decreasing order and then add successively the measures of preimages $F(\hat{y})$ using (4) and the fact that each point has exactly $d'$ $f$-preimages in $\Lambda$. Thus if we compare the measures of $F(\hat{y})$ with the measure of the $f^n$-preimage $F(f^n z, \ldots, z)$ of $F$ in $B_{n+k}(z, \varepsilon)$, we obtain that

$$\sum_{\hat{y} \in \mathcal{P}(y)} \mu(\phi)(F(\hat{y})) \approx \mu(\phi)(F(f^n z, f^{n-1} z, \ldots, z)) \cdot (d')^n$$

But now the sets $F \in \mathcal{F}$ were chosen mutually disjoint modulo $\mu(\phi)$, hence their preimages will be mutually disjoint too (recall that $C_f \cap \Lambda = \emptyset$); thus by adding over $F \in \mathcal{F}$

$$\sum_{F \in \mathcal{F}} \sum_{\hat{y} \in \mathcal{P}(y), F \subset E(y)} \mu(\phi)(F(\hat{y})) \approx \mu(\phi)(B_{n+k}(z, \varepsilon)) \cdot (d')^n$$

Thus from the $f$-invariance of $\mu(\phi)$ and by adding as above all the measures of preimages along maximal prehistories, we obtain the following formula for the measure $\mu(\phi)$ of an arbitrary "rectangle" with sides in the stable and in the unstable directions centered on $\Lambda$:

$$\mu(\phi)(B(n, k, z, \varepsilon)) \approx \mu(\phi)(B_{n+k}(z, \varepsilon)) \cdot (d')^n \approx \frac{\varepsilon^{S_{n+k} \phi(z)}}{(d')^k},$$

(5)

where the comparability constants are independent of $n, k, z, x$. Then the above formula helps us obtain the measure $\mu(\phi)$ of an arbitrary ball centered on $\Lambda$ (not only balls with radii which are indefinitely and non-uniformly small).

As an application we obtain the pointwise dimension of such an equilibrium measure $\mu(\phi)$. We know from definition that $B(n, k, z, \varepsilon) = f^n(B_{n+k}(z, \varepsilon))$. Since $f$ is holomorphic on $\mathbb{P}^2$, it follows from conformality on the local stable/unstable manifolds that the set $B(n, k, z, \varepsilon)$ has a side comparable to $\varepsilon |Df^n_s(z)|$ in the stable direction, and a side comparable to $\varepsilon |Df^n_u(z)||Df^{n+k}_u(z)|^{-1} = \varepsilon |Df^n_u(x)|^{-1}$ in the unstable direction. Now for some $n, k$ the set $B(n, k, z, \varepsilon)$ becomes "round", i.e the stable side and the unstable side become comparable with a fixed comparability constant. So we want $\rho := \varepsilon |Df^n_s(z)| \approx \varepsilon |Df^n_u(x)|^{-1}$, where recall that $x = f^n(z)$. 

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In general for a continuous function \( \psi : \Lambda \to \mathbb{R} \), an \( f \)-invariant ergodic probability measure \( \mu \) on \( \Lambda \) and \( \tau > 0 \), let us define the following set of well-behaved points with respect to \( \mu \):

\[
G_n(\psi, \mu, \tau) := \{ y \in \Lambda, |\frac{1}{n}S_n \psi(y) - \int \psi d\mu| < \tau \}, \quad n > 0
\]

Then from Birkhoff Ergodic Theorem we have \( \mu(G_n(\psi, \mu, \tau)) \to 1 \) when \( n \to \infty \); so for every \( \tau' > 0 \) there is \( n(\tau') > 0 \) such that \( \mu(G_n(\psi, \mu, \tau)) > 1 - \tau' \) for \( n > n(\tau') \).

We apply this to our case for the ergodic measure \( \mu_\phi \) and the functions \( \log |Df_s| \) and \( \log |Df_u| \) which are continuous and bounded on \( \Lambda \) (as \( f \) has no critical points in \( \Lambda \)). If \( z \in G_n(\log |Df_s|, \mu_\phi, \tau) \) and \( x = f^n(z) \in G_k(\log |Df_u|, \mu_\phi, \tau) \), then

\[
|\frac{1}{n}S_n \log |Df_s|(z) - \int \log |Df_s|d\mu_\phi| < \tau \quad \text{and} \quad |\frac{1}{k}S_k \log |Df_u|(x) - \int \log |Df_u|d\mu_\phi| < \tau
\]

The question is how large is the set of such \( z' \)'s. From above it follows that \( \mu_\phi(f^{-n}(G_k(\log |Df_u|, \mu_\phi, \tau)) = \mu_\phi(G_k(\log |Df_u|, \mu_\phi, \tau)) > 1 - \tau' \) and \( \mu_\phi(G_n(\log |Df_s|, \mu_\phi, \tau)) > 1 - \tau' \), for \( n > n(\tau') \). Thus for \( n \) large enough:

\[
\mu_\phi(G_n(\log |Df_s|, \mu_\phi, \tau) \cap f^{-n}G_k(\log |Df_u|, \mu_\phi, \tau)) > 1 - 2\tau'
\]

We now come back to the problem of the pointwise dimension of \( \mu_\phi \). It is clear from above that if \( B(n, k, z, \varepsilon) \) has comparable sides (i.e it is ”round”), then \( k \) must depend on \( n \), so denote it by \( k(z, n) \). Let us consider also

\[
\chi_s(\mu_\phi) := \int \log |Df_s|d\mu_\phi, \quad \chi_u(\mu_\phi) := \int \log |Df_u|d\mu_\phi,
\]

the Lyapunov exponents of the ergodic measure \( \mu_\phi \) in the stable, respectively unstable directions; they will be denoted also by \( \chi_s, \chi_u \) for simplicity. As \( \mu_\phi \) is ergodic (see [25]) we have that \( \frac{1}{n}S_n \log |Df_s| \to \chi_s \) and \( \frac{1}{k}S_k \log |Df_u| \to \chi_u \). Thus if \( |Df_s^k(z)| \approx |Df_u^k(z)|^{-1} \), it follows that

\[
\frac{-\log C}{n} \cdot S_{k(z,n)} \log |Df_s^k(z)| > \frac{k(z,n)}{n} \leq \frac{S_n \log |Df_s^k(z)|}{n} \leq \frac{-\log C}{n} \cdot S_{k(z,n)} \log |Df_u^k(z)| > \frac{k(z,n)}{n}
\]

Thus since \( n \to \infty \), we have \( k(z,n) \to \frac{\chi_s}{\chi_u} \). But then from (5) and (8) one sees that if \( B(n, k, z, \varepsilon) \) is a ”round” ball, i.e with sides of comparable size \( \rho = \varepsilon |Df_s^k(z)| \) then for \( \mu_\phi \cdot a.e \ z \in \Lambda \)

\[
\frac{\log \mu_\phi(B(n,k(z,n),z,\varepsilon))}{\log \rho} \to \int \phi d\mu_\phi - \frac{\chi_s}{\chi_u} \cdot \int \phi d\mu_\phi + \frac{\chi_s}{\chi_u} \cdot \log d'
\]

Therefore the pointwise dimension of \( \mu_\phi \) is well-defined and for \( \mu_\phi \cdot a.e \ z \in \Lambda \), it is given by:

\[
\delta_{\mu_\phi}(z) = \int \phi d\mu_\phi \cdot \left( \frac{1}{\chi_s} - \frac{1}{\chi_u} \right) + \frac{\log d'}{\chi_u}
\]

But \( \mu_\phi \) is the equilibrium measure for \( \phi \) and we assumed \( P(\phi) = \log d' \), so \( P(\phi) = \log d' = h_{\mu_\phi} + \int \phi d\mu_\phi \). Hence from above for \( \mu_\phi \cdot a.e \ z \in \Lambda \) we obtain

\[
\delta_{\mu_\phi}(z) = h_{\mu_\phi}(\frac{1}{\chi_u} - \frac{1}{\chi_s}) + \log d' \cdot \frac{1}{\chi_s}
\]
In conclusion the measure \( \mu_\phi \) is exact dimensional on \( \Lambda \) and it satisfies the above formula.

The fact that the Hausdorff dimension of \( \mu_\phi \) takes the same value as \( \delta_{\mu_\phi} \) follows from a criterion of Young (see [26]), since the pointwise dimension is constant \( \mu_\phi \)-a.e.

Even if the preimage counting function \( d(\cdot) \) is not constant on \( \Lambda \), still we obtain bounds for the measure of iterates of Bowen balls, and estimates for the lower pointwise dimension:

**Corollary 1.** In the setting of Theorem 1 assume the preimage counting function satisfies \( d(x) \leq d' \) for \( \mu_\phi \)-a.e \( x \in \Lambda \) and that \( \phi(x) + \log d' < P(\phi) \) for all \( x \in \Lambda \); then for \( \mu_\phi \)-a.e \( x \in \Lambda \)

\[
\delta_{\mu_\phi}(x) \geq h_{\mu_\phi}(\frac{1}{\chi_u(\mu_\phi)} - \frac{1}{\chi_s(\mu_\phi)}) + \log d' \cdot \frac{1}{\chi_s(\mu_\phi)}
\]

*Proof.* In (5) we have to consider only that now \( d(x) \leq d' \), therefore when adding the measures of different pieces of local backwards iterates we will obtain \( \mu_\phi(B(n,k,z,\varepsilon)) \leq C \mu_\phi(B(n+k(z,\varepsilon) \cdot (d')^n \leq C' \cdot \frac{\delta_{\mu_\phi}(\varepsilon)}{(d')^k} \), for some constants \( C, C' \).

Hence by dividing \( \log \mu_\phi(B(n,k(z,n),z,\varepsilon)) \) with the negative quantity \( \log \rho \), we obtain the conclusion as in the proof of Theorem 1.

In addition the above formula in (5) gives the Jacobian (in the sense of Parry [18]) of the equilibrium measure \( \mu_\phi \), with respect to an arbitrary iterate \( f^m \), by taking the iterate of a round ball \( B(n,k,z,\varepsilon) \):

**Corollary 2.** In the same setting as in Theorem 1, if the preimage counting function is constant and equal to \( d' \) on \( \Lambda \), then the Jacobian \( J_{\mu_\phi}(f^m) \) of an arbitrary iterate satisfies \( J_{\mu_\phi}(f^m) \approx (d')^m \) on \( \Lambda \), where the comparability constant does not depend on \( m \).

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In [8] Fornaess and Sibony studied \( s \)-hyperbolic holomorphic maps on \( \mathbb{P}^2 \) and minimal saddle basic sets, in the sense of the ordering between saddle basic sets \( \Lambda_i \succ \Lambda_j \) if \( \mathcal{W}^u(\hat{\Lambda}_i) \cap \mathcal{W}^s(\Lambda_j) \neq \emptyset \). A related notion introduced in [4] is that of a terminal set in the case of a holomorphic map \( f \) on \( \mathbb{P}^2 \). Here \( f \) is not assumed to have Axiom A and the condition refers only to \( \Lambda \) itself. A saddle set \( \Lambda \) is called terminal if for any \( \hat{x} \in \hat{\Lambda} \), the iterates of \( f \) restricted to \( \mathcal{W}^{u}_{loc}(\hat{x}) \setminus \Lambda \) form a normal family. Notice that if \( f \) is Axiom A and if \( \Lambda \) is minimal, then for any \( \hat{x} \in \hat{\Lambda} \) the global unstable set \( \mathcal{W}^u(\hat{x}) \) does not intersect any global stable set of any other basic set, thus \( \mathcal{W}^u(\hat{\Lambda}) \setminus \Lambda \) is contained in the union of basins of attraction of attracting cycles. Therefore in this case minimal sets are also terminal. Examples of minimal sets for holomorphic maps on \( \mathbb{P}^2 \) are given in [8], and examples of terminal sets are given in [4].

In [8], Fornaess and Sibony constructed positive closed currents \( \sigma \) on minimal sets for \( s \)-hyperbolic maps, by using forward iterates of unstable disks (or enough disks which are transverse to local stable directions); if \( D \) is an unstable disk then

\[
\frac{f^\ast_n([D])}{d^n} \rightarrow \sigma \cdot \int D \wedge T
\]
Then using the positive closed \((1,1)\) current \(\sigma\), they constructed an invariant measure \(\nu\) on \(\Lambda\) as

\[
\nu = \sigma \wedge T
\]

In [4] Diller and Jonsson introduced a positive current \(\sigma^u\) by using transversal measures (see also the diffeomorphism case in [22], [23]); namely in a neighbourhood of \(x \in \Lambda\),

\[
<\sigma^u, \chi> = \int_{W^s_{loc}(\hat{x})} \int_{W^u_{loc}(\hat{y})} \chi d\hat{\mu}^s_x(\hat{y}),
\]

where \(\hat{\mu}^s_x\) are transversal measures on \(\hat{W}^s_{loc}(x) := \pi^{-1}(W^s_{loc}(x)).\) Here we use a different notation for these measures, in order to emphasize that they are supported on lifts of local stable manifolds. If \(\Lambda\) is terminal, then they defined an invariant probability measure on \(\Lambda\),

\[
\nu_i = \sigma^u \wedge T
\]

We use the notation \(\nu_i\) in order to emphasize the way the current \(\sigma^u\) was constructed with the help of the inverse limit. In general from the Spectral Decomposition Theorem a basic set can be written as a union \(\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_m\) of mutually disjoint compact subsets, and there exist positive integers \(n_1, \ldots, n_m\) s.t \(f^{n_i}\) invariates \(\Lambda_j\) and \(f^{n_i}\) is topologically mixing on \(\Lambda_j\) (see [10]). In the next Theorem we want to prove that the measures \(\nu, \nu_i\) are both equal to the measure of maximal entropy of \(f|\Lambda\) if \(\Lambda\) is (topologically) mixing.

**Theorem 2.** (a) Let \(f : \mathbb{P}^2 \to \mathbb{P}^2\) be a holomorphic map of degree \(d\) and \(\Lambda\) be a terminal mixing saddle set. Then \(\nu_i\) is equal to the measure of maximal entropy \(\mu_0\) on \(\Lambda\).

(b) Let \(f : \mathbb{P}^2 \to \mathbb{P}^2\) be an Axiom A holomorphic map of degree \(d\) and assume \(f\) is c-hyperbolic on the minimal and mixing saddle basic set \(\Lambda\). Then \(\nu_i = \nu = \mu_0\), where \(\mu_0\) is the measure of maximal entropy on \(\Lambda\).

**Proof.** (a) First let us remind the properties of the transversal measures \(\hat{\mu}^{s}_x\); they are built in the same fashion as in [22] (see also [23]), but on the natural extension \(\hat{\Lambda}\). The key to that proof is the existence for diffeomorphisms of a Markov partition; in our endomorphism case, we have instead a Markov partition on the inverse limit \(\hat{\Lambda}\) (see [21]). Moreover the inverse limit \(\hat{\Lambda}\) has local product structure, in fact it is a Smale space (see [21], [10]). One obtains then a system of transversal measures \(\hat{\mu}^{s}_x\) on \(\hat{W}^s_{loc}(x)\), where we denote by \(\hat{W}^s_{loc}(x)\) and \(\hat{W}^u_{loc}(\hat{x})\) the lifts to \(\hat{\Lambda}\) of the local stable intersection \(W^s_{loc}(x) \cap \Lambda\), respectively of the local unstable intersection \(W^u_{loc}(\hat{x}) \cap \Lambda\). More precisely \(\hat{W}^s_{loc}(x) := \pi^{-1}(W^s_{loc}(x) \cap \Lambda)\) and \(\hat{W}^u_{loc}(\hat{x}) := \pi^{-1}(W^u_{loc}(\hat{x}) \cap \Lambda)\), \(\hat{x} \in \hat{\Lambda}\). Let us assume without loss of generality that all the stable and unstable local manifolds we work with, are of size \(r\) for some \(r > 0\) small enough. The measures \(\hat{\mu}^{s}_x\) satisfy the following properties:

i) if \(\chi^s_{x,y} : \hat{W}^s_r(x) \to \hat{W}^s_r(y)\) is the holonomy map given by \(\chi^s_{x,y}(\hat{\xi}) = \hat{W}^u_r(\hat{\xi}) \cap \hat{W}^s_r(y)\), then \(\hat{\mu}^s_x(A) = \hat{\mu}^s_y(\chi^s_{x,y}(A))\) for any borelian set \(A\).

ii) \(\hat{f}_* \hat{\mu}^s_x = e^{h_{top}(f|\Lambda)} \hat{\mu}^s_x|f(\hat{W}^s_r(x))\)

iii) \(\text{supp } \hat{\mu}^s_x = \hat{W}^s_r(x)\)
In fact from [22] and [23] applied to our case on \( \hat{\Lambda} \), it follows that there exist also unstable transversal measures, denoted by \( \hat{\mu}_x^u \) on \( \hat{W}_r^u(\hat{x}) \), \( \hat{x} \in \hat{\Lambda} \) with similar properties. And moreover the measure of maximal entropy on \( \hat{\Lambda} \) denoted by \( \hat{\mu}_0 \), can be written as the product of transversal stable measures \( \hat{\mu}_y^s \) with transversal unstable measures \( \hat{\mu}_x^u \) i.e

\[
\hat{\mu}_0(\phi) = \int_{\hat{W}_r^u(\hat{x})} \left( \int_{\hat{W}_r^u(\hat{y})} \phi \, d\hat{\mu}_y^s \right) \, d\hat{\mu}_x^u(\hat{y}),
\]

(7)

for any function \( \phi \) defined on a neighbourhood of \( \hat{x} \in \hat{\Lambda} \).

Now in [4] the measure \( \nu_i \) is defined as the wedge product \( \sigma^u \wedge T \), where the positive closed current \( \sigma^u \) is constructed with the help of the stable transversal measures \( \hat{\mu}_x^s \), \( x \in \Lambda \). Recall also that \( \pi|_{\hat{W}_r^u(\hat{x})} : \hat{W}_r^u(\hat{x}) \to W_r^u(\hat{x}) \) is a bijection (see [16]), so any function \( \phi \) on \( W_r^u(\hat{x}) \) determines uniquely a function denoted again with \( \phi \) on \( W_r^u(\hat{x}) \). Then from the measure \( \nu_i \) we can form a system of measures on the lifts of local unstable manifolds \( \hat{W}_r^u(\hat{x}) \), \( \hat{x} \in \hat{\Lambda} \) in the following way:

\[
\hat{\nu}_x^u(\phi) = \int_{W_r^u(\hat{x})} \phi T|_{W_r^u(\hat{x})}
\]

We assumed that \( f \) is mixing on \( \Lambda \); in fact (topological) mixing of \( f \) on \( \Lambda \) is equivalent to mixing of \( \hat{f} \) on \( \hat{\Lambda} \). Define stable holonomy maps between lifts to \( \hat{\Lambda} \) of local unstable manifolds, namely \( \chi_x^u : \hat{W}_r^u(\hat{x}) \to W_r^u(\hat{y}) \), \( \chi_x^u(\hat{x}) = \hat{W}_r^u(\hat{y}) \cap \hat{W}_r^u(\hat{x}) \), \( \hat{\xi} \in \hat{W}_r^u(\hat{x}) \).

We wish to prove that the measures \( \hat{\nu}_x^u \) are transversal and invariant with respect to stable holonomy maps in the Smale space structure of \( \hat{\Lambda} \), in the sense of Bowen and Marcus ([3]). From the way the local unstable manifolds were constructed as determined by prehistories, it follows that there is a bijection between \( W_r^u(\hat{x}) \cap \Lambda \) and its lift \( \hat{W}_r^u(\hat{x}) \) (see also [16]). Given a borelian set \( \hat{A} \subseteq \hat{W}_r^u(\hat{x}) \), there exists a unique borelian set \( A \subseteq W_r^u(\hat{x}) \cap \Lambda \) such that \( \pi \) is a bijection between \( \hat{A} \) and \( A \). From the definition of \( \hat{\nu}_x^u \), we know that \( \hat{\nu}_x^u(\hat{A}) = \int_{A \cap W_r^u(\hat{x})} ddG|_{W_r^u(\hat{x})} \).

Let us remind some geometric properties of the positive closed current \( T \) used in the definition of \( \hat{\nu}_x^u \) (see [9], for more details). First there exists a continuous plurisubharmonic function \( G \) on \( \mathbb{C}^3 \setminus \{0\} \) called the Green function of \( f \), satisfying \( G(F(z)) = d \cdot G(z) \) where \( F : \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}^3 \setminus \{0\} \) is the lift of \( f \) relative to the canonical projection \( \pi_2 : \mathbb{C}^3 \to \mathbb{P}^2 \). We have \( G \in \mathcal{P}_1 \), where \( \mathcal{P}_1 \) is the cone of plurisubharmonic functions \( u \) on \( \mathbb{C}^3 \setminus \{0\} \) satisfying the homogeneity condition \( u(\lambda z) = \log |\lambda| + u(z) \), \( \lambda \in \mathbb{C} \) and \( z \in \mathbb{C}^3 \setminus \{0\} \). Recall that \( \pi_2^* T = dd^c G \).

Denote now the unstable intersection \( W_r^u(\hat{x}) \cap \Lambda \) by \( Z(\hat{x}) \) for \( \hat{x} \in \hat{\Lambda} \). Consider points \( x, y \) in a subset of \( \Lambda \) belonging to an open set \( V \subseteq \mathbb{P}^2 \) so there exists a holomorphic inverse \( s : V \to \mathbb{C}^3 \setminus \{0\} \) of \( \pi_2 \). Then for \( r \) small we can identify \( Z(\hat{x}), Z(\hat{y}) \) with their respective lifts to \( \mathbb{C}^3 \setminus \{0\} \) for any prehistories \( \hat{x}, \hat{y} \in \hat{\Lambda} \). Since there are no critical points of \( f \) in \( \Lambda \) and since we work on \( \Lambda \), it follows that \( Z(\hat{x}) \) can be split into mutually disjoint subsets on which \( f^n \) is injective, i.e \( Z(\hat{x}) = \cup_i Z_{i,n}(\hat{x}) \), \( f^n|_{Z_{i,n}(\hat{x})} : Z_{i,n}(\hat{x}) \to Z_{i,n}^2(\hat{x}) \) is bijective, and moreover \( Z_{i,n}^2(\hat{x}), i \) are mutually disjoint. It follows that \( f^n(Z(\hat{x})) = \cup_i Z_{i,n}^2(\hat{x}) \). Now if \( Z(\hat{x}) \) is contained in \( V \), then \( f^n(Z(\hat{x})) \) may not be contained in \( V \); but, if \( f^n(Z(\hat{x})) \) is contained say in \( V_1 \cup V_2 \) where \( V_1, V_2 \) are open sets in \( \mathbb{P}^2 \) as above, with respective local inverses \( s_1, s_2 \) of \( \pi_2 \), and if \( V_1 \cap V_2 \neq \emptyset \), then there exists a holomorphic function \( \rho \) on \( V_1 \cap V_2 \)
so that \( s_1 = \rho s_2 \) on \( V_1 \cap V_2 \). So \( dd^c(G \circ s_1) = dd^c(G(\rho s_2)) = dd^c(\log |\rho|) + dd^c(G \circ s_2) = dd^c(G \circ s_2) \); this implies that working with \( dd^cG \) on \( C^3 \setminus \{0\} \) is the same as working on \( \mathbb{P}^2 \). Now \( G \circ F = d \cdot G \) and \( f^n : Z_{i,n}(x) \to Z^n_{i}(x) \) is bijective hence similarly as in [8], \( \int_{Z^n_{i}(x)} dd^cG = d^n \int_{Z_{i,n}(x)} dd^cG \). Thus by adding over all the indices \( i \) we obtain:

\[
\int_{f^n(Z(x))} dd^cG = d^n \int_{Z(x)} dd^cG
\]  

(8)

Now let \( x, y \in \Lambda \) closer than \( r/2 \) and iterate \( Z(x) \) and \( Z(y) \) for some prehistories \( \hat{x}, \hat{y} \in \hat{\Lambda} \). We also take as above, the subsets \( Z_{i,n}(\hat{y}) \) such that \( f^n : Z_{i,n}(\hat{y}) \to Z^n_{i}(\hat{y}) \) is a bijection, \( Z^n_{i}(\hat{y}), i \) are mutually disjoint and \( f^n(Z(\hat{y})) = \cup Z^n_{i}(\hat{y}) \). If \( Z_{i,n}(\hat{x}), Z_{i,n}(\hat{y}) \) has diameter small enough, then it follows that \( Z^n_{i}(\hat{x}), Z^n_{i}(\hat{y}) \) both have diameter bounded above by \( r \) and they are very close to each other, in fact \( d(Z^n_{i}(\hat{x}), Z^n_{i}(\hat{y})) \to 0 \) for each \( i \), when \( n \to \infty \). This follows as in the Laminated Distortion Lemma (see [15]) since the distances between iterates of points on stable manifolds decrease exponentially, and the unstable derivative \( |Df_u| \) is Hölder continuous. Now if \( \psi \) is a smooth test function equal to 1 on a fixed neighbourhood of \( Z^n_{i} \) we have \( \int_{Z^n_{i}(\hat{x})} dd^cG = \int_{Z^n_{i}(\hat{y})} G dd^c\psi \) hence since \( dd^c\psi \) is continuous and \( Z^n_{i}(\hat{x}) \) and \( Z^n_{i}(\hat{y}) \) are close, we obtain \( \int_{f^n(A) \cap Z^n_{i}(\hat{x})} dd^cG - \int_{f^n(A) \cap Z^n_{i}(\hat{y})} dd^cG \leq \varepsilon m_2(Z^n_{i}(\hat{x})) \) for \( n \) large enough, where \( m_2 \) is the Lebesgue measure on \( \mathbb{P}^2 \). Now we add these inequalities over \( i \) and use the fact proved in Proposition 5.3 of [8] that \( m_2(f^n(Z(\hat{x}))) \leq C d^n, n > 0 \). Hence by dividing with \( d^n \), using (8), and letting \( n \to \infty \) we obtain \( \int_{A \cap Z(\hat{x})} dd^cG = \int_{\chi_{\hat{x}, \hat{y}}(A) \cap Z(\hat{y})} dd^cG \). We lift then to the natural extension, keeping in mind that there exists a homeomorphism between \( Z(\hat{x}) \) and \( \hat{W}^u_{r}(\hat{x}) \). Hence on \( \hat{\Lambda} \) we have:

\[
\hat{\nu}^u_{\hat{x}}(\hat{\Lambda}) = \hat{\nu}_y(\chi_{\hat{x}, \hat{y}}(\hat{\Lambda})), \ \hat{\Lambda} \text{ borelian set in } \hat{W}^u_{r}(\hat{x})
\]

The above equality can be extended next to general borelian sets contained in global unstable sets \( \hat{W}^u_{r}(\hat{x}) = \bigcup_{n \geq 0} f^n(\hat{W}^u_{r}(\hat{x})), \hat{x} \in \hat{\Lambda} \). Thus by a theorem of Bowen and Marcus ([3]) extended to the mixing homeomorphism \( \hat{f} \) on \( \hat{\Lambda} \), it follows that there exists a positive constant \( \gamma \) such that \( \hat{\nu}^u_{\hat{x}} = \gamma \hat{\mu}^u_{\hat{0}, \hat{x}} \), for any \( \hat{x} \in \hat{\Lambda} \), where \( \hat{\mu}^u_{\hat{0}, \hat{x}} \) are the transversal measures given by the measure of maximal entropy \( \hat{\mu}_0 \) on \( \hat{\Lambda} \) (as in [22], [23]); see also (7). In fact if \( \mu_0 \) is the unique measure of maximal entropy on \( \Lambda \) and if \( \hat{\mu}_0 \) is the unique measure of maximal entropy on \( \hat{\Lambda} \) then

\[
\mu_0 = \pi_* \hat{\mu}_0 \text{ and } h_{\mu_0} = h_{top}(f|\Lambda) = h_{top}(\hat{f}|\Lambda) = h_{\hat{\mu}_0}
\]

The measure \( \nu_i \) is constructed with the transversal stable measures \( \hat{\mu}^s_{\hat{x}} \) (which we denote also by \( \hat{\mu}^s_{\hat{0}, \hat{x}} \)). Now from [21] we know that any \( f \)-invariant measure \( \mu \) on \( \Lambda \) can be lifted uniquely to an \( \hat{f} \)-invariant measure \( \hat{\mu} \) on \( \hat{\Lambda} \) such that \( \pi_* \hat{\mu} = \mu \). In our case we denote by \( \hat{\nu}_i \) this unique lift of \( \nu_i \) to \( \hat{\Lambda} \). Since both \( \hat{\nu}_i \) and \( \hat{\mu}_0 \) are ergodic probabilities on \( \Lambda \), it follows that \( \gamma = 1 \) and that

\[
\hat{\mu}_0 = \hat{\nu}_i, \ \text{hence } \mu_0 = \nu_i
\]

b) Let us assume now that \( f \) has Axiom A, that \( \Lambda \) is a minimal basic set (i.e the unstable set of \( \Lambda \) does not intersect the stable set of any other basic set \( \Lambda' \)) and that \( f \) is c-hyperbolic on
\[\Lambda.\] Then Fornaess and Sibony ([8]) constructed a positive closed \((1, 1)\) current \(\sigma\) supported on the global unstable set \(W^u(\hat{A})\) such that if \(D\) is a local disk transverse to the stable direction, then \(f^\nu([D]) \to (f[D] \cap T)\sigma\). Without loss of generality assume that the disk \(D\) is chosen such that \(f[D] \cap T = 1\); and also that \(T\) has no mass on the boundary \(\partial D\) of \(D\).

We have from [8] that locally on a neighbourhood \(\Delta\) of a point \(x \in \Lambda\) there exists a measure \(\lambda\) defined on the space of holomorphic maps from a local unstable disk \(\Delta_1\) to a local stable disk \(\Delta_2\) such that \(\sigma = f[W^u_r(\hat{y})]d\lambda(\hat{y})\) where \(W^u_r(\hat{y})\) are local unstable manifolds intersecting \(\Delta\) and \(g_{\hat{y}} : \Delta_1 \to \Delta_2\) is a holomorphic map whose graph is \(W^u_r(\hat{y})\).

Then \(\nu = \sigma \wedge T\) is supported only on \(\Lambda\); hence we can define measures \(\hat{\nu}^s_x\) on \(\hat{W}^s(x)\) by \(\hat{\nu}^s_x(\hat{A}) = \lambda(\{y : \hat{y} \in \hat{A}\})\).

Thus from the way the function \(g_{\hat{y}}\) was defined, namely as a function whose graph is \(W^u_r(\hat{y})\), it follows that these measures are invariant to the local holonomy map between \(\hat{W}^s(x)\) and \(\hat{W}^s(y)\) for \(x, y\) close. Also by covering with small flow boxes it follows we can extend this property globally. Therefore from [3] we obtain that \(\hat{\nu}^s_x = \gamma \hat{\mu}^s_{0,x}\), where the constant \(\gamma > 0\) does not depend on \(x \in \Lambda\). Now \(\nu\) was defined as integration of \(T\) on local unstable manifolds followed by integration with respect to transversal measures; by using a) we obtain that \(\hat{\nu} = \hat{\mu}_0\), and thus \(\nu = \mu_0\). Hence on minimal saddle basic sets the measure \(\nu\) is equal to the measure \(\nu_i\) and both are equal to the measure of maximal entropy \(\mu_0\) on \(\Lambda\).

\[\square\]

Now that we know that the measure \(\nu_i\) is equal to the measure of maximal entropy \(\mu_0\), and to \(\nu\) when \(f\) has Axiom A and \(\Lambda\) is minimal, we find its pointwise dimension.

**Corollary 3.** a) Let \(\Lambda\) be a mixing terminal saddle set for a holomorphic map \(f : \mathbb{P}^2 \to \mathbb{P}^2\) of degree \(d\), s.t. \(\Lambda\) does not intersect the critical set \(C_f\) of \(f\). If each point in \(\Lambda\) has at most \(d'\) \(f\)-preimages in \(\Lambda\) and if \(d' < d\), then for \(\mu_0\)-a.e. \(z\),

\[\delta_\nu_i(z) \geq \log d \cdot \left(\frac{1}{\chi_u(\nu_i)} - \frac{1}{\chi_s(\nu_i)}\right) + \log d' \cdot \frac{1}{\chi_s(\nu_i)}\]

b) If \(\Lambda\) is a mixing terminal saddle set for a holomorphic map \(f\) on \(\mathbb{P}^2\) of degree \(d\), if \(C_f \cap \Lambda = \emptyset\) and if the preimage counting function is constant equal to \(d'\) on \(\Lambda\) for \(d' \leq d\), then we have:

\[\delta_\nu_i = HD(\nu_i) = \log d \cdot \left(\frac{1}{\int \log |Df_u|d\nu_i} - \frac{1}{\int \log |Df_s|d\nu_i}\right) + \log d' \cdot \frac{1}{\int \log |Df_s|d\nu_i}\]

c) Let \(f : \mathbb{P}^2 \to \mathbb{P}^2\) be a holomorphic Axiom A map of degree \(d\), which is c-hyperbolic on a connected minimal saddle set \(\Lambda\). Then

\[\delta_\nu = HD(\nu) = \log d \cdot \left(\frac{1}{\chi_u(\nu)} - \frac{1}{\chi_s(\nu)}\right) + \log d' \cdot \frac{1}{\chi_s(\nu)},\]

where \(d'\) denotes the constant number of \(f\)-preimages in \(\Lambda\) of a point.

**Proof.** a) We use the result of [4] that the topological entropy of \(f|_\Lambda\) is equal to \(\log d\) if \(\Lambda\) is a terminal saddle basic set. From Theorem 2 we have that \(\nu_i = \mu_0\), the measure of maximal entropy
on Λ; hence \( h_{\mu_0} = h_{\text{top}}(f|_\Lambda) \). Then the inequality follows from Corollary 1 in case the number of preimages in \( \Lambda \) is bounded above by \( d' \).

b) If \( \Lambda \) is terminal, then from Theorem 2 we know that \( \nu_1 \) is equal to \( \mu_0 \) the measure of maximal entropy of \( f|_\Lambda \); also since the topological entropy of \( f|_\Lambda \) is \( \log d \), it follows that \( h_{\nu_1} = \log d \).

If the preimage counting function is constant and equal to \( d' \), then in case \( d' < d \), we have that condition \( \phi + \log d' < P(\phi) \) is satisfied on \( \Lambda \) for \( \phi \equiv 0 \). So one can apply Theorem 1 in order to obtain the pointwise dimension of \( \nu_i \); and since \( \delta_{\nu_i} \) is constant, then \( HD(\nu_i) = \delta_{\nu_i} \).

There remains only the case \( d' = d \). In this case every point in \( \Lambda \) has \( d \) \( f \)-preimages in \( \Lambda \) and \( h_{\text{top}}(f|_\Lambda) = \log d \) (from [4]). Thus the unique zero of the function \( t \to P(t \log |Df_s| - \log d) \) is equal to 0; so from [13] the function \( f|_\Lambda \) is expanding. In this expanding case we have that \( B_n(z, \varepsilon) \) is itself a round ball of \( \mu_0 \)-measure comparable to \( \frac{1}{d^n} \) (from the estimates of equilibrium measures on Bowen balls in (2)). At the same time the radius of this ball is comparable to \( \varepsilon |Df^n_s(z)|^{-1} \). Hence the lower and upper pointwise dimensions of \( \nu_i \) coincide and the pointwise dimension and Hausdorff dimension of \( \nu_i \) are both equal to

\[
\delta_{\mu_0} = HD(\mu_0) = \frac{\log d}{\chi_u(\mu_0)}
\]

c) In this case if \( \Lambda \) is connected and \( f \) is \( c \)-hyperbolic on \( \Lambda \), it follows from [15] that the preimage counting function is constant on \( \Lambda \). Also \( \Lambda \) cannot be written as a disjoint union of compact sets, so it is mixing for an iterate of \( f \). So we can apply Theorem 1 for the measure \( \nu \) which, according to Theorem 2 is equal to \( \mu_0 \), and thus has entropy \( \log d \).

Remark. If \( f \) is a smooth endomorphism hyperbolic on a basic set \( \Lambda \), then by taking a smooth perturbation \( g \) of \( f \), it follows that \( g \) has also a basic set \( \Lambda_g \) on which it is hyperbolic (see [20]). Also if \( \Lambda \) is connected then \( \Lambda \) is connected, so from the conjugacy of \( \hat{f}|_\Lambda \) to \( \hat{g}|_{\Lambda_g} \), \( \Lambda_g \) is connected and thus \( \Lambda_g \) is connected too. However the dynamics of perturbations may be very different; for instance perturbations of toral endomorphisms are not conjugated necessarily to the original maps, and may even have infinitely many unstable manifolds through a given point.

For minimal \( c \)-hyperbolic sets of maps of degree 2 we can determine the possible values of the pointwise dimension of \( \nu \); recall that the preimage counting function is constant if \( \Lambda \) is connected.

**Corollary 4.** Let \( f \) be an Axiom A holomorphic map on \( \mathbb{P}^2 \) of degree 2, which is \( c \)-hyperbolic on a connected minimal saddle set \( \Lambda \). Then we have exactly one of the following two possibilities:

1) the preimage counting function of \( f \) is equal to 1 on \( \Lambda \); then \( f|_\Lambda \) is a homeomorphism and

\[
\delta_\nu = \log 2 \cdot \left( \frac{1}{\int \log |Df_u| d\nu} - \frac{1}{\int \log |Df_s| d\nu} \right)
\]

2) or, the preimage counting function of \( f \) is equal to 2 on \( \Lambda \); then \( f|_\Lambda \) is expanding and

\[
\delta_\nu = \log 2 \cdot \frac{1}{\int \log |Df_u| d\nu}
\]
Proof. If $f$ has Axiom A and $\Lambda$ is minimal then $\Lambda$ is terminal, thus from [4] it follows that $h_{top}(f|\Lambda) = \log 2$. Now if $\Lambda$ is connected and if there exists a neighbourhood $U$ of $\Lambda$ with $f^{-1}(\Lambda) \cap U = \Lambda$ then the preimage counting function is constant on $\Lambda$. If the preimage counting function is equal to $d'$ on $\Lambda$ it follows that $d' \leq 2$, otherwise from Misiurewicz-Przytycki Theorem (see [10]) we would have $h_{top}(f|\Lambda) \geq \log d' > \log 2$ which is impossible, as we saw above.

So we either have $d' = 1$ or $d' = 2$. In the first case we can apply Theorem 1 for the potential $\phi \equiv 0$ since $\log d' < P(0) = \log d$ on $\Lambda$. In this case $f|\Lambda$ is a homeomorphism (like for instance the family of polynomial perturbations constructed in [17]).

In the second case if $d' = 2$, the stable dimension $\delta_s := HD(W^s_r(z) \cap \Lambda)$ is equal to the unique zero of the pressure function $t \to P(t \log |Df| - \log 2)$ (see [15] and references therein); but since $h_{top}(f|\Lambda) = \log 2$, the zero of this pressure function is indeed equal to 0. Then $\delta_s = 0$ and we can apply the result of [13] saying that in this case, $f$ must be expanding on $\Lambda$.

From Theorem 2 the measure $\nu$ is equal to $\mu_0$, i.e the measure of maximal entropy. Therefore from Theorem 1 the pointwise dimension of $\nu$ is $\delta_{\nu} = \log 2 \cdot \frac{1}{\int \log |Df_v| dv}$.

\square

Remark and examples.

1) First notice that the map $f_0(z, w) = (z^2 + c, w^2)$ is 2-to-1 and expanding on $\Lambda_0 = \{p_0(c)\} \times S^1$, where $p_0(c)$ is the fixed attracting point of $z \to z^2 + c$ for $|c|$ is small enough. We gave in [17] a class of examples of perturbations of $f_0$ by polynomial maps which are homeomorphic on their respective basic sets. This shows in particular that the preimage counting function is not necessarily preserved by perturbations. These are maps of type

$$f_\varepsilon(z, w) = (z^2 + c + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2, w^2),$$

for $b \neq 0$, $|c|$ small and $0 < \varepsilon < \varepsilon(a, b, c, d, e)$. Then $f_\varepsilon$ has a basic set $\Lambda_\varepsilon$ (close to $\Lambda_0$), on which it is hyperbolic and has a homeomorphic restriction. For $0 < \varepsilon < \varepsilon(a, b, c, d, e)$ it also follows that $f_\varepsilon$ is $c$-hyperbolic on $\Lambda_\varepsilon$ since there are no critical points of $f_\varepsilon$ in $\Lambda_\varepsilon$ and since there exists a neighbourhood $U$ of $\Lambda_\varepsilon$ such that $f_\varepsilon^{-1}(\Lambda_\varepsilon) \cap U = \Lambda_\varepsilon$. Indeed for $\varepsilon$ fixed if there were no such neighbourhood, then for any neighbourhood $V$ of $\Lambda_\varepsilon$ there would exist a point $y \in V \setminus \Lambda_\varepsilon$ with $f_\varepsilon(y) \in \Lambda_\varepsilon$. Thus for any $n > 0$ there would exist points $y_n \in B(\Lambda_\varepsilon, \frac{1}{n})$ with $f_\varepsilon(y_n) = x_n \in \Lambda_\varepsilon$, and since $\Lambda$ is compact we can assume $x_n \to x \in \Lambda_\varepsilon$. Since there are no critical points of $f_\varepsilon$ in $\Lambda_\varepsilon$, it follows that there exists a positive distance $\eta_0$ s.t $d(y_n, z_n) > \eta_0$, where $z_n$ is another preimage of $x_n$ belonging to $\Lambda_\varepsilon$ (such $z_n$ must exist since $f_\varepsilon(\Lambda_\varepsilon) = \Lambda_\varepsilon$). Then without loss of generality we can assume that $y_n \to y$ so $y \in \Lambda$ since $d(y_n, \Lambda) < \frac{1}{n}$. But perhaps after passing to a subsequence, $z_n \to z \in \Lambda$; then from above $d(y, z) > \eta_0/2$. But this is a contradiction since $f_\varepsilon$ is homeomorphic on $\Lambda_\varepsilon$. Hence there must exist a neighbourhood $U$ of $\Lambda_\varepsilon$ satisfying

$$f_\varepsilon^{-1}(\Lambda_\varepsilon) \cap U = \Lambda_\varepsilon$$

Also notice that if we fix $a, b, c, d, e, \varepsilon$ as above and perturb now $f_\varepsilon$, we obtain another map $g$ which has a saddle basic set $\Lambda_g$ on which $g$ is hyperbolic and homeomorphic.
Examples of terminal sets can be obtained by perturbations of known examples $f$ and $\Lambda$; if $W^u(\hat{\Lambda}) \setminus \Lambda$ is contained in the union of basins of finitely many attracting cycles of $f$, then any small perturbation $g$ has a saddle basic set $\Lambda_g$ close to $\Lambda$, and $W^u(\hat{\Lambda}_g) \setminus \Lambda_g$ is also contained in the union of basins of attraction of $g$; hence $\Lambda_g$ is terminal too.

Also the topological entropy of restrictions is preserved by perturbations, i.e $h_{top}(f|\Lambda) = h_{top}(g|\Lambda_g)$. Thus by perturbing the known examples (like products of hyperbolic rational maps, or Ueda type examples of [8], see [8]), we obtain more examples of terminal sets. As noticed before if $\Lambda$ is connected, then the basic set $\Lambda_g$ is connected too. And if $f$ is mixing on $\Lambda$ then $\hat{f}$ is mixing on $\hat{\Lambda}$; hence from the conjugacy on inverse limits, we obtain that $g$ is mixing on $\Lambda_g$ as well.

We can take for instance examples constructed by Ueda’s method (see [24]); if $\Phi: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is the Segre map $\Phi([z_0 : z_1], [w_0 : w_1]) = [z_0w_0 : z_1w_1 : z_0w_1 + z_1w_0]$ and $f: \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map then there exists $F: \mathbb{P}^2 \to \mathbb{P}^2$ holomorphic of the same degree as $f$, so that $\Phi(f, f) = F \circ \Phi$. If $f$ is hyperbolic on its Julia set $J(f)$ (i.e expanding), then $F$ is hyperbolic on basic sets of type

$$\Lambda = \Phi(\{\text{periodic sink of } f\} \times J(f))$$

The saddle set $\Lambda$ is terminal and topologically mixing for $F$. Let us consider now also a holomorphic perturbation $G$ of $F$ with a corresponding basic set $\Lambda_G$, which is close to $\Lambda$. From above it follows that $\Lambda_G$ is terminal and mixing saddle set for $G$. Consider also Hölder potentials $\phi$ on $\Lambda_G$ satisfying inequality (1) with respect to $G$; for instance, in the setting of Corollary 3 we can take $\phi$ sufficiently small in $C^0$-norm s.t (1) is still satisfied.

Now for each such $\phi$ we have an equilibrium measure $\mu_\phi$ on $\Lambda_G$. Then it follows that one can apply Theorem 1, Corollary 1 and Corollary 3 in order to obtain the values of $\mu_\phi$ on iterates of Bowen balls in $\Lambda_G$, and also in order to estimate the (upper/lower) pointwise dimensions of $\mu_\phi$.

In particular we obtain information about the (upper/lower) pointwise dimensions for the measure $\mu_{0,G}$ of maximal entropy of the restriction $G|\Lambda_G$.

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References


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