Discontinuous Galerkin Finite Element Methods for High Order Nonlinear Partial Differential Equations

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Summary

Discontinuous Galerkin (DG) finite element methods were first designed to solve hyperbolic conservation laws utilizing successful methodologies from high resolution finite difference and finite volume schemes such as approximate Riemann solvers and nonlinear limiters. More recently the DG methods have been generalized to solve convection dominated convection-diffusion equations (e.g. high Reynolds number Navier-Stokes equations), convection-dispersion (e.g. KdV equations) and other high order nonlinear wave equations or diffusion equations.

In this talk we first give an introduction to the DG method for hyperbolic conservation laws. We start with the weak formulation of the partial differential equation (PDE), and then explain how to convert it to a DG scheme which can be implemented on the computer. We emphasize the importance of the design of the numerical fluxes, indicating that the success of the DG method, including its stability and accuracy, depends crucially on a good design of the numerical fluxes. We point out that the choice of the numerical fluxes could borrow from the vast research in finite volume and finite difference schemes, namely we can choose the monotone fluxes for the scalar case and exact or approximate Riemann solvers for the system case. The time discretization is by the standard, nonlinearly stable TVD (or SSP) Runge-Kutta methods. We then point out the major advantages of DG schemes for solving hyperbolic conservation laws, including easy handling of complicated geometry and boundary conditions and the allowance of hanging nodes in the mesh, compactness due to the minimal communication with immediate neighbors regardless of the order of the scheme, the explicit nature of the scheme without any need to convert large linear or nonlinear systems, and excellent parallel efficiency. Theoretically, the method can be proved to satisfy a cell entropy inequality and L^2 stability, for arbitrary scalar equations and symmetric systems in any spatial dimension and any triangulation, for any order of accuracy, without limiters. Error estimates can be obtained for smooth solutions, and the method allows for easy h-p adaptivity.

We then move on to discuss the generalization of DG methods to PDEs containing higher order derivatives. We first indicate that a naive generalization of the DG method to a PDE containing higher order spatial derivatives could have disastrous results. An example of a naive generalization of the DG method to the heat equation is to replace $f(u)$ by $-u_x$ everywhere and to use a simple average for the numerical flux. Numerical results indicate that this method seems to converge to a wrong solution. A detailed analysis

indicates that the method is actually consistent but weakly unstable. We then introduce the idea of local DG (LDG) methods, which involves the introduction of auxiliary variables approximating derivatives of the solution and conversion of the original higher order PDE to a first order system. A DG method is then formally constructed to this first order system. The crucial ingredient then is the correct design of the numerical fluxes, which may not be able to follow just upwind principle as in the hyperbolic case. The fluxes also need to be designed such that the auxiliary variables can be locally eliminated (hence the reason that the methods are called *local* DG methods). We spend some time to discuss patterns for the choice of numerical fluxes leading to stable and accurate LDG schemes for various types of nonlinear, high order PDEs, including the convection-diffusion equations, the KdV type equations, the bi-harmonic type equations, the Kuramoto-Sivashinsky type equations, PDEs in semi-conductor device simulations, the Cahn-Hilliard equations, the surface diffusion equations, the $K(m, n)$ equation, the Kawahara equation, the fifth-order fully nonlinear $K(n, n, n)$ equations, the generalized nonlinear Schrödinger (NLS) equation, the Kadomtsev-Petviashvili (KP) equations, the Zakharov-Kuznetsov (ZK) equations, the Camassa-Holm (CH) equation, the Hunter-Saxton (HS) equation, the generalized Zakharov system, and the Degasperis-Procesi (DP) equation.

Finally, we point out some of the current and future work in the study of DG schemes for higher order PDEs, including the design of stable DG methods for more nonlinear dispersive wave equations and diffusion equations in applications, the study of efficient time discretization (preconditioning, multigrid, exponential type time discretization, deferred correction, ...), and error estimates that can help the design of adaptivity.

A general review of DG methods can be found in [1]. A survey on LDG methods for higher order PDEs can be found in [3]. A simple lecture notes on DG method can be found in [2].

References

[1] B. Cockburn and C.-W. Shu, *Runge-Kutta discontinuous Galerkin methods for convection-dominated problems*, Journal of Scientific Computing, v16 (2001), pp.173-261.

[2] C.-W. Shu, *Discontinuous Galerkin methods: general approach and stability*, Numerical Solutions of Partial Differential Equations, S. Bertoluzza, S. Falletta, G. Russo and C.-W. Shu, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, Basel, 2009, pp.149-201.

[3] Y. Xu and C.-W. Shu, *Local discontinuous Galerkin methods for high-order time-dependent partial differential equations*, Communications in Computational Physics, v7 (2010), pp.1-46.