On Singularity Formation of a 3D Model for Incompressible Navier-Stokes Equations

Thomas Y. Hou* Zuoqiang Shi† Shu Wang‡

Abstract

We investigate the singularity formation of a 3D model that was recently proposed by Hou and Lei in [2] for axisymmetric 3D incompressible Navier-Stokes equations with swirl. The main difference between the 3D model of Hou and Lei and the reformulated 3D Navier-Stokes equations is that the convection term is neglected in the 3D model. This model shares many properties of the 3D incompressible Navier-Stokes equations. One of the main results of this paper is that we prove rigorously the finite time singularity formation of the 3D inviscid model for a class of initial boundary value problems with smooth initial data of finite energy.

The question of whether a solution of the 3D incompressible Navier-Stokes equations can develop a finite time singularity from smooth initial data with finite energy is one of the most outstanding mathematical open problems. Most regularity analysis for the 3D Navier-Stokes equations relies on energy estimates. Due to the incompressibility condition, the convection term does not contribute to the energy norm of the velocity field or any L^p $(1 < p \leq \infty)$ norm of the vorticity field. As a result, the main effort has been to use the diffusion term to control the nonlinear vortex stretching term without making use of the convection term explicitly.

In a recent paper by Hou and Lei, the authors investigated the effect of convection by constructing a new 3D model for axisymmetric 3D incompressible Navier-Stokes equations with swirl. Specifically, their 3D model is given below:

$$\partial_t u_1 = \nu(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)u_1 + 2\partial_z\psi_1 u_1, \tag{1}$$

$$\partial_t \omega_1 = \nu(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)\omega_1 + \partial_z((u_1)^2), \tag{2}$$

$$-(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)\psi_1 = \omega_1. \tag{3}$$

Note that (1)-(3) is already a closed system. The only difference between this 3D model and the reformulated Navier-Stokes equations is that the convection term is neglected in the

*Applied and Comput. Math, Caltech, Pasadena, CA 91125. *Email: hou@acm.caltech.edu.*

†Mathematical Sciences Center, Tsinghua University, Beijing China. *Email: zqshi@math.tsinghua.edu.cn.*

‡College of Applied Sciences, Beijing University of Technology, Beijing 100124, China. *Email: wangshu@bjut.edu.cn*

model. If one adds the convection term back to the left hand side of (1) and (2), one would recover the full Navier-Stokes equations. This model preserves almost all the properties of the full 3D Navier-Stokes equations, including the energy identity for smooth solutions of the 3D model and the divergence free property of the reconstructed 3D velocity field given by $u^\theta = ru_1$, $u^r = -\partial_z(r\psi_1)$, $u^z = \frac{1}{r}\partial_r(r^2\psi_1)$. Moreover, they proved the corresponding non-blowup criterion of Beale-Kato-Majda type as well as a non-blowup criterion of Prodi-Serrin type for the model. In a subsequent paper, they proved a new partial regularity result for the model which is an analogue of the Caffarelli-Kohn-Nirenberg theory for the full Navier-Stokes equations.

Despite the striking similarity at the theoretical level between the 3D model and the Navier-Stokes equations, the former seems to have a very different behavior from the full Navier-Stokes equations. The authors presented numerical evidence which supports that the 3D model may develop a potential finite time singularity. They further studied the mechanism that leads to these singular events in the 3D model. On the other hand, the Navier-Stokes equations with the same initial data seems to have a completely different behavior.

One of the main results of our work is that we prove rigorously the finite time singularity formation of this 3D model for a class of initial boundary value problems with smooth initial data of finite energy. In our analysis, we focus on the inviscid version of the 3D model and consider the initial boundary value problem of the generalized 3D model which has the following form (we drop the subscript 1 and substitute (3) into (2)):

$$\begin{aligned} u_t &= 2u\psi_z, && (4)\\ -\Delta\psi_t &= \left(u^2\right)_z, && (5) \end{aligned}$$

where Δ is a n-dimensional Laplace operator with $(\mathbf{x}, z) \equiv (x_1, x_2, ..., x_{n-1}, z)$. Our results in this paper apply to any dimension greater than or equal to two ($n \geq 2$). To simplify our presentation, we only present our analysis for $n = 3$. We consider the generalized 3D model in both a bounded domain and in a semi-infinite domain with a mixed Dirichlet Robin boundary condition. The main result is the following theorem.

Theorem 1 *Let $\Omega_\mathbf{x} = (0, a) \times (0, a)$, $\Omega = \Omega_\mathbf{x} \times (0, b)$ and $\Gamma = \{(\mathbf{x}, z) \mid \mathbf{x} \in \Omega_\mathbf{x},\ z = 0\}$. Assume that the initial condition $u_0 > 0$ for $(\mathbf{x}, z) \in \Omega$, $u_0|_{\partial\Omega} = 0$, $u_0 \in H^2(\Omega)$, $\psi_0 \in H^3(\Omega)$ and satisfies (6). Moreover, we assume that ψ satisfies the following mixed Dirichlet Robin boundary condition:*

$$\psi|_{\partial\Omega\backslash\Gamma} = 0, \quad (\psi_z + \beta\psi)|_\Gamma = 0, \qquad (6)$$

with $\beta > \frac{\sqrt{2}\pi}{a}\left(\frac{1+e^{-2\pi b/a}}{1-e^{-2\pi b/a}}\right)$. Define $\phi(x_1, x_2, z) = \left(\frac{e^{-\alpha(z-b)}+e^{\alpha(z-b)}}{2}\right)\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)$ where α satisfies $0 < \alpha < \sqrt{2}\pi/a$ and $2\left(\frac{\pi}{a}\right)^2 \frac{e^{\alpha b}-e^{-\alpha b}}{\alpha(e^{\alpha b}+e^{-\alpha b})} = \beta$. If u_0 and ψ_0 satisfy the following condition:

$$\int_\Omega (\log u_0)\phi d\mathbf{x}dz > 0, \quad \int_\Omega \psi_{0z}\phi d\mathbf{x}dz > 0, \qquad (7)$$

then the solution of the 3D inviscid model (4)-(5) will develop a finite time singularity in the H^2 norm.

The analysis of the finite time singularity for the 3D model is rather subtle. The main technical difficulty is that this is a multi-dimensional nonlinear nonlocal system. Currently, there is no systematic method of analysis to study singularity formation of a nonlinear nonlocal system. The key issue is under what condition the solution u has a strong alignment with the solution ψ_z dynamically. If u and ψ_z have a strong alignment for long enough time, then the right hand side of the u-equation would develop a quadratic nonlinearity dynamically, which would lead to a finite time blowup. Note that ψ is coupled to u in a nonlinear and nonlocal fashion. It is not clear whether u and ψ_z will develop such a nonlinear alignment dynamically. As a matter of fact, not all initial boundary conditions of the 3D model would lead to finite time blowup. One of the interesting results we obtain in this paper is that we prove the global regularity of the 3D inviscid model for a class of small initial data with an appropriate boundary condition. We would like to point out that since there is no viscosity in the 3D inviscid model, such global regularity result is still interesting even though some smallness condition is imposed on the initial data. We note that there is currently no corresponding global regularity result for the incompressible 3D Euler equation even with small initial data.

We also study singularity formation of the 3D model with $\beta = 0$ in (6). This case is interesting because the smooth solution of the corresponding 3D model satisfies an energy identity. In this case, we can establish a finite time blowup under an additional condition :

$$\int_0^a \int_0^a (\psi - \psi_0)|_\Gamma \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) d\mathbf{x} < c_0 \int_\Omega \psi_{0z} \phi \mathrm{d}\mathbf{x} \mathrm{d}z,$$

as long as the solution remains regular, where $c_0 > 0$ depends only on the size of the domain.

We would like to point out that the study of [1, 2] is based on a reduced model for some special flow geometry. One should not conclude that convection term could lead to depletion of singularity of the Navier-Stokes equations in general. It is possible that convection term may act as a destabilizing term for a different flow geometry. One of the main findings of [1, 2] and the present paper is that convection term carries important physical information that should not be neglected in our analysis of the Navier-Stokes equations. Since the behavior of the 3D model is very different from that of the Navier-Stokes equations, it is important to develop a method of analysis that could take into account the physical significance of convection term in an essential way.

References

[1] T. Y. Hou and C. Li, *Dynamic stability of the 3D axi-symmetric Navier-Stokes equations with swirl*, Comm. Pure Appl. Math. **61** (2008), no. 5, 661–697.

[2] T. Y. Hou and Z. Lei, *On the stabilizing effect of convection in 3D incompressible Flows*, Comm. Pure Appl. Math. **62** (2009), no. 4, 501–564.