The asymptotic Hodge theory of vector bundles

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1 The Hodge Conjecture as a problem in geometric analysis

Conjecture 1.1. *(Hodge Conjecture) Let X be a smooth projective variety. Then every element of*

$$H^{p,p}_{\mathbb{Q}} := H^{2p}(X,\mathbb{Q}) \cap H^{p,p}(X)$$

is Poincare dual to a $\mathbb{Q}$ linear combination of complex subvarieties of X.

Let $Vect(X)$ denote isomorphism classes of smooth complex vector bundles on X, and let $Vect(X,d)$ denote the rank d bundles. The Chern character defines a surjective map

$$ch : Vect(X) \otimes \mathbb{Q} \twoheadrightarrow H^{\text{even}}(X,\mathbb{Q}).$$

In particular, every $z \in H^{2p}(X,\mathbb{Q})$ is a rational multiple of $ch_p(E(z))$, for some $E(z) \in Vect(X)$.

If X is a smooth projective variety and $E \in Vect(X)$ is holomorphic, then the Chern character classes, $ch_p(E)$, are of type (p,p) and Poincare dual to complex subvarieties. Recall that a connection A induces a holomorphic structure iff writing $d_A = \partial_A + \bar{\partial}_A$, and $F_A = d_A^2$,

$$\bar{\partial}_A^2 = 0 = F_A^{0,2}.$$

Hence, a fundamental question for Hodge theory is :

Question 1.2. If $z \in \oplus_p H^{p,p}_{\mathbb{Q}}$, can some $E(z)$ be endowed with a holomorphic structure?

An affirmative answer, of course, implies the Hodge conjecture.

One approach for an analytical study of the Hodge conjecture then is to seek special connections, A, on suitable candidate bundles $E(z)$ for which $F_A^{0,2} = 0$. Possible approaches include trying to minimize the Yang–Mills energy

$$YM(A) := \|F_A\|^2,$$

or perhaps the energy

$$Y'(A) = \|F_A^{0,2}\|^2,$$

and then trying to show minimizers are holomorphic. A major difficulty is understanding how the nonlinear nonlocal necessary condition that the Chern classes are of type (p,p) enters the analysis.

As a warm up, we first explore minima on manifolds where the (p,p) conditions are absorbed into the underlying geometry.

Theorem 1.3. *(Stern). Let M be a compact Calabi Yau 3-fold, with strict $SU(3)$ holonomy. Let $E \in Vect(M)$ with Y' minimizing connection A. Then (E, A) is holomorphic.*

For strict $SU(3)$ holonomy, all even cohomology classes are (p, p) and therefore the difficult necessary conditions are absorbed into the geometry. Of course the Hodge conjecture is known for 3-folds, but if not, this would have reduced the Hodge conjecture for Calabi-Yau's to a (hard) variational problem.

Next, motivated by work of Yau, Donaldson, Tian, Zelditch, Song, Phong, Sturm, and others, we turn to geometric quantization in order to seek better behaved energy functionals. We first encounter a long detour.

2 Non holomorphic Kodaira embeddings

Suppose F is a holomorphic vector bundle. A metric h_F on $F \otimes L^k$ determines a unitary projection $\Pi_{H^0} : L^2(F \otimes L^k) \to H^0(F \otimes L^k)$. Let $\pi_{H^0}(x, y)$ denote the Schwartz kernel for Π_{H^0}. We can use π_{H^0} to define a new metric on $F \otimes L^k$ by setting

$$h^{\Pi} := \frac{\chi(F \otimes L^k)}{rank(F) Volume(M)} h_F(\cdot, \pi_{H^0}^{-1}(x, x)\cdot).$$

This metric can also be obtained by pulling back the standard metric on the Universal bundle via an embedding in $Grass(rank(F), H^0(E \otimes L^k)^*)$ determined by a unitary basis of H^0.

Definition 2.1. The metric h_F is called *balanced* if $h^{\Pi} = h_F$.

Theorem 2.2. *(Wang) F is Gieseker poly-stable if and only if for k sufficiently large, $F \otimes L^k$ admits balanced metrics.*

If the balanced metrics converge as $k \to \infty$, they converge to a Hermitian Einstein metric (modulo conformal change).

Because we wish to consider nonholomorphic E, it is natural to replace H^0 with $H^0(D_{A(k)}) :=$ kernel $D_{A(k)}$, $D_{A(k)} := \bar{\partial}_{A(k)} + \bar{\partial}^*_{A(k)}$. An element $\sigma \in H^0(D_{A(k)})$ decomposes into

$$\sigma = \sigma^0 + \sigma^2 + \cdots + \sigma^{2[\frac{m}{2}]}.$$

Distracted from our original goal, we wonder whether this grading on differential forms induces a Hodge-like grading on vector bundles.

Definition 2.3. We say $E \in S^q_V \mathrm{Vect}(M)$ if E admits a connection A s.t. for some polarization, for k sufficiently large, and all $s \in \ker(D_{A(k)})$

$$s^{2j} = 0, \ \forall j > q. \tag{2.4}$$

We say $E \in IS^q_V \mathrm{Vect}(M)$ if E admits a connection A which satisfies (2.4) for *every* polarization.

Let $S^p_H H^k(M, \mathbb{C}) = \oplus_{j \geq p} H^{j,k-j}(M)$ denote the Hodge filtration.

Conjecture 2.5.

$$\mathrm{ch_p(S^q_V Vect(M)) \subset (S^{p-q}_H \cap \bar{S}^{p-q}_H) H^{2p}(M, \mathbb{C})}.$$

This is clearly true for $q = 0$. For line bundles, the conjecture holds for all q.

Theorem 2.6. *(Charbonneau-Stern)*

$$\mathrm{ch_p(S_V^q Vect(M,1)) \subset (S_H^{p-q} \cap \bar{S}_H^{p-q})H^{2p}(M,\mathbb{C})}.$$

We can also show the following:

Theorem 2.7. *(Charbonneau-Stern) For $p < 7$,*

$$\mathrm{ch_p(IS_V^1 Vect(M)) \subset (S_H^{p-1} \cap \bar{S}_H^{p-1})H^{2p}(M,\mathbb{C})}.$$

$$\mathrm{ch_p(S_V^q Vect(M)) \subset (S_H^{p-q} \cap \bar{S}_H^{p-q})H^{2p}(M,\mathbb{C}),\ \ \forall p < q+3}.$$

When E is holomorphic (i.e. $E \in S_V^0\mathrm{Vect}(M)$), its Chern classes are Poincaré dual to rational linear combinations of projective subvarieties. Grothendieck's generalized Hodge conjecture suggests the following:

Question 2.8. If $E \in S_V^q\mathrm{Vect}(M,r)$, then is $c_p(E)$ Poincare dual to cycles supported in a finite union of codimension $p-q$ subvarieties?

(We have no evidence supporting a positive answer to this question.)

Return now to our original question, before we got side tracked: can we find in the asymptotics of $Ker(D_A)$ better energy functionals for analyzing holomorphicity?

Denote by Π_A the orthogonal L_2 projection onto $H^0(D_A)$. Let $\{s_a\}_a$ denote an L_2-unitary basis of $H^0(D_A)$. Natural quantities to consider:
$\Pi_A(x,x)$, $\sum_a \|s_a^2\|^2$, $\sum_a \|\bar{\partial}_A s_a\|^2$, and $\sum_a \|s_a^{2j}\|^2$. Observe $\sum_a \|s_a^2\|^2$ and $\sum_a \|\bar{\partial}_A s_a\|^2$ vanish in the holomorphic case and therefore seem relevant to our pursuit. The $\sum_a \|s_a^{2j}\|^2$ asymptotics are useful in proving the preceding filtration theorems. Compute the leading order asymptotics for these quantities to find:
(i) $\Pi_A(x,x) = \frac{k^m}{2^m\pi^m} Id_E + k^{m-1}(\frac{2im\phi_A}{4\pi} + \frac{\kappa Id_E}{8\pi}) + O(k^{m-2})$. $\kappa =$ scalar curvature. (See Ma et al).
(ii) $\sum_a \|s_a^2\|^2 = \frac{k^{m-2}}{2^{m-2}\pi^m}\|F_A^{0,2}\|^2 + O(k^{m-3})$.
(iii) $\sum_a \|\bar{\partial}_A s_a\|^2 = c_3 k^{m-1}\|F_A^{0,2}\|^2 + O(k^{m-2})$.
(iv) $\sum_a \|s_a^{2j}\|^2 = \frac{k^{m-2j}}{2^{m-2j}\pi^m(j!)^2}\|(F_A^{0,2})^j\|^2 + O(k^{m-2j-1})$.
So, to leading order both (ii) and (iii) lead us back to Y', the energy we started with. In some sense this is unavoidable since we have a holomorphic structure iff $F_A^{0,2}$ vanishes. On the other hand it does suggest our next energy functional.

3 Dynamical Systems and Balanced Connections

Fix a metric on E. Let $V \subset C^\infty(M, E\otimes L^k)$. Let Π_V denote orthogonal projection onto V.

Definition 3.1. We call V *admissible* if
(i) $dim(V) = \chi(E\otimes L^k)$, and
(ii) V spans $E \otimes L^k$ at every point. Equivalently $\pi_V(x,x)$ is invertible.

Definition 3.2. We call a sequence of admissible subspaces $\{V_n\}_n$ *stable* if there exists $C > 0$ such that $|\pi_{V_n}^{-1}| \leq C,\ \forall n$.

This notion of stability is a smooth analog of Gieseker stability.

Given a unitary L_2 basis $\{s_a\}_a$ for V, a hermitian metric h for $E\otimes L^k$, and a connection A for E, let

$$e_k^{0,1}(A,V,h)(x) := \sum_a |\bar{\partial}_A s_a|_h^2(x),$$

and

$$e_k^{1,0}(A,V,h)(x) := \sum_a |\partial_A s_a|_h^2(x).$$

Define the functional

$$E_k^{0,1}(A,V,h) := \int e_k^{0,1}(A,V,h)dv.$$

If E admits a holomorphic structure, then $E_k^{0,1}$ is zero for some A, V, k, and any h.

Given a connection A, let $V(A)$ denote the $\chi(E\otimes L^k)$ dimensional subspace of $C^\infty(M, E\otimes L^k)$ which minimizes $E^{0,1}$. This is spanned by the eigenspaces with small eigenvalues. (It is unique if k is large relative to F_A).

Definition 3.3. We call a connection *balanced* if it is induced by pulling back the canonical connection on the universal quotient bundle over $Grass(rank(E), V^*(A))$, via an embedding induced by a unitary basis of $V(A)$.

Proposition 3.4. *(Stern): Suppose that $(A, V(A))$ is a critical point for $E_k^{0,1}$. Then A is a balanced connection.*

Question 3.5. What is the corresponding notion of stability?

What happens when we try to minimize $E_k^{0,1}$ when E does not satisfy the Chern class constraints? Possibilities:
(i) $E_k^{0,1}$ is bounded below away from 0.
(ii) Every minimizing sequence $\{V_n\}_n$ for $E_k^{0,1}$ is unstable.
(iii) Every stable minimizing sequence $\{V_n\}_n$ for $E_k^{0,1}$ develops singularities.

Theorem 3.6. *(Stern) If $c_1(E) \notin F^1H^2 \cap \bar{F}^1H^2$, the energy, $E^{0,1}$ remains bounded below on stable sequences.*

We get our first insight into the interplay between the topological obstructions and the nature of singularity formation:

Theorem 3.7. *(Stern) Suppose that $ch_p(E) \notin S_H^{p-j+1}H^{2p}(M) \cap \bar{S}_H^{p-j+1}H^{2p}$. Let $\{V_n\}_n$ be a stable sequence of subspaces and $\{A_n\}_n$ be a sequence of smooth hermitian connections. Let $q \geq 1$. If $e^{0,1}(A_n, V_n) \overset{L_{\frac{q(p-j)}{2}}}{\to} 0$, then*

$$\|e^{1,0}(B_n, V_n)\|_{L_{\frac{q(p+j)}{2(q-1)}}} \to \infty,$$

for **any** *sequence of hermitian connections $\{B_n\}_n$. If $e^{0,1}(A_n, V_n) \overset{L_\infty}{\to} 0$, then*

$$\|e^{1,0}(B_n, V_n)\|_{L_{\frac{p+j}{2}}} \to \infty,$$

for any sequence of hermitian connections $\{B_n\}_n$.