HYPERBOLIC MEAN CURVATURE FLOW

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Abstract. This note describes the hyperbolic mean curvature flow, some of the discoveries that have been done about it, and some unresolved questions.

Key words: Hyperbolic mean curvature flow, geometric partial differential equations, smooth solution, global existence, blow-up, singularities.

Classical differential geometry has been the study of curved spaces, shapes and structures of manifolds in which the time does not play a role. However, in the last few decades geometers have made great strides in understanding the shapes and structures of manifolds that evolve in time. There are many processes in the evolution of a manifold, but among them two successful examples are the Ricci flow (see Hamilton [8]) and the mean curvature flow (see [6], [9], [11], [12]). For the Ricci flow, there are many deep and outstanding works, for example, it can be used to solve successfully the Poincaré conjecture and geometrization conjectures. On the other hand, the mean curvature flow is related on the motion of surfaces or manifolds. Much more well-known motion of surfaces are those equating the velocity with some scalar multiple of the normal of the surface. The scalar can be the curvature, mean curvature or the inverse of the mean curvature with suitable sign attached. For the mean curvature flow, a beautiful theory has been developed by Hamilton, Huisken and other researchers (e.g., [6], [9], [11]), and some important applications have been obtained, for example, Huisken and Ilmanen developed a theory of weak solutions of the inverse mean curvature flow and used it to prove successfully the Riemannian Penrose inequality (see [12]).

The traditional geometric analysis has successfully applied the theory of elliptic and parabolic partial differential equations to differential geometry and physics (see [24]). There are three typical examples: the first one is the Yau's solution of the Calabi's conjecture (see [25]) and the Schoen-Yau's solution of the positive mass conjecture (see [22], [23]), the second one is the Hamilton's Ricci flow, and the third one is the mean curvature flow. On the other hand, since the hyperbolic equation or system is one of the most natural models in the nature, a natural and important question is if we can apply the theory of hyperbolic differential equations to solve some problems arising from differential geometry and theoretical physics (in particular, general relativity). However, up to now, only a few results on hyperbolic partial differential equations have been known.

The hyperbolic geometric flow, or say, the hyperbolic versions of the Ricci flow, introduced recently by K.-F. Liu and the author, is nothing but an attempt to answer the above question. The hyperbolic geometric flow is a very natural tool to understand the wave character of the metrics, wave phenomenon of the curvatures, the evolution of manifolds and their structures (see [14], [3], [4], [15], [17]), also see the survey paper [13].

In this note we are interested in the hyperbolic version of mean curvature flow, named the hyperbolic mean curvature flow (HMCF): let M be a Riemannian manifold and $X(\cdot,t): M \to \mathbb{R}^{n+1}$ be a smooth map, the hyperbolic mean curvature flow is described by

$$\frac{\partial^2 X}{\partial t^2} = H\vec{N} + \cdots \triangleq \mathcal{P}X + \mathcal{L}X, \tag{1}$$

where $X = X(t, x)$ stands for a family of hypersurfaces in $\mathbb{R}^n$, H is the mean curvature of X, N is the unit inner normal vector of X, $\mathcal{P}X$ denotes the leading terms with a hyperbolic operator $\mathcal{P}$, while $\mathcal{L}X$ stands for the lower order terms. Obviously, the HMCF is defined by a system of hyperbolic partial differential equations of second order, thus its initial data should include the initial hypersurface itself and the initial velocity. Therefore the HMCF has an advantage: it has a freedom to choose the initial velocity, and then one can, in some sense, control the evolution of the initial hypersurface.

The HMCF is very important in both mathematics and applications, and has attracted many mathematicians to study it (e.g., [1] and [26]). However, to the author's knowledge, up to now only a few results have been known.

Melting crystals of helium exhibits a phenomenon generally not found in other materials: oscillations of the solid-liquid interface in which atoms of the solid move only when they melt and enter the liquid (see [7] and references therein). Gurtin and Podio-Guidugli [7] developed a hyperbolic theory for the evolution of plane curves. Rostein, Brandon and Novick-Cohen [21] studied a hyperbolic theory by the mean curvature flow equation

$$v_t + \psi v = k, \tag{2}$$

where v_t is the normal acceleration of the interface, ψ is a constant. Moreover, a crystalline algorithm was developed by [21] for the motion of polygonal curves.

He, Kong and Liu [10] studied the following HMCF

$$\frac{\partial^2}{\partial t^2} X(u,t) = H(u,t)\vec{N}(u,t), \quad \forall\ u \in M, \quad \forall\ t \in [0,T), \tag{3}$$

where T stands for some positive constants. It is easy to check that the equation (3) is not strictly hyperbolic, however, by a trick of DeTurck [5], the authors of [10] obtained strictly hyperbolic system of partial differential equations, and based on this, they showed that this flow admits a unique short-time smooth solution and possesses the nonlinear stability defined on the Euclidean space with dimension larger than 4. Moreover, the nonlinear wave equations satisfied by curvatures are also derived in [10], these equations will play an important role in future study.

We also would like to mention LeFloch and Smoczyk's work [19]. In [19], the authors studied the following geometric evolution equation of hyperbolic type which governs the evolution of a hypersurface moving in the direction of its mean curvature vector

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} X = eH(u,t)\vec{N} - \bigtriangledown e, \\ X(u,0) = X_0, \quad (X_t(u,0))^{\vec{T_0}} = 0, \end{cases} \tag{4}$$

where $\vec{T_0}$ stands for the unit tangential vector of the initial hypersurface X_0, $e \triangleq \frac{1}{2}\left(|X_t|^2 + n\right)$ is the local energy density and $\bigtriangledown e \triangleq \bigtriangledown^i e_i$, in which $e_i = \frac{\partial e}{\partial x^i}$. This flow stems from a geometrically natural action containing kinetic and internal energy terms. They proved that the normal hyperbolic mean curvature flow would blow up in finite time. In the case of graphs, they introduced a concept of weak solution suitably restricted by an entropy inequality and proved that the classical solution is unique in the larger class of entropy solutions. In the special case of one-dimensional graphs, a global-in-time existence result is established. Moreover, an existence theorem is established under the assumption that the BV norm of initial data is small.

For the plane curves, Kong, Liu and Wang [16] studied the following equation on $S^1 \times [0,T)$

$$\begin{cases} \dfrac{\partial^2 F}{\partial t^2}(z,t) = k(z,t)\vec{N}(z,t) + \rho(z,t)\vec{T}(z,t), \quad \forall\ (z,t) \in \mathbb{R} \times [0,T), \\ F(z,0) = F_0(z), \quad \dfrac{\partial F}{\partial t}(z,0) = h(z)\vec{N}_0, \end{cases} \tag{5}$$

where $F(z,t)$ is the unknown vector-valued function standing for the curve at time t, k is the mean curvature of F, $\vec{N}$ is the unit normal vector of F, the function ρ is defined by

$$\rho = -\left\langle \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t} \right\rangle \tag{6}$$

in which s is the arclength parameter, $\vec{T}$ stands for the unit tangent vector of F, F_0 denotes the initial curve, while h and $\vec{N}_0$ are the initial velocity and unit normal vector of initial curve F_0, respectively. It is easy to show that, if the initial velocity is normal to the initial curve, the flow described by (5) is always a normal one. By means of the supported function, the authors derived a beautiful hyperbolic Monge-Ampère equation. Based on this, they showed that there exists a class of initial velocities such that the solution of the corresponding Cauchy problem exists only in a finite time interval $[0, T_{\max})$ and when t goes to $T_{\max}$, the solution either converges to a point or develops shocks as well as other singularities. In the paper [18], a quasilinear wave equation was derived and studied for the motion of plane curves under the HMCF in (5). Based on this, they investigated the formation of singularities in the motion of these curves. In particular, the authors proved that the motion of periodic plane curves with small variation on one period and small initial velocity in general blows up and singularities develop in finite time. Some blow-up results have been obtained and the estimates on the life-span of the solutions are given.

Contrast to the hyperbolic mean curvature flows studied in [10], [16], [18] and [19], K.-S. Chou and W.-F. Wo [2] proposed a new hyperbolic curvature flow for convex hypersurfaces. This flow is most suited when the Gauss curvature is involved. The equation satisfied by the graph of the hypersurface under this flow gives rise to a new class of fully nonlinear Euclidean invariant hyperbolic equations.

Recently, Notz in his Ph.D thesis [20] introduced and studied a new geometric flow equation, which describes the motion of closed hypersurfaces in Riemannian manifolds. If the surface is spherical, this equation can be considered as an idealised mathematical model of a moving soap bubble. It can be obtained as an Euler-Lagrange equation of a suitable action integral. In addition to the kinetic energy this action integral contains terms for the surface tension and the inner pressure, which depends on the enclosed volume. The resulting Euler-Lagrange equation is a quasilinear degenerate hyperbolic partial differential equation of second order, which describes the motion of the surface extrinsically. The author showed the short time existence theorem, and proved a continuation criterion which gives a sufficient condition under which the solution can be extended to a larger time interval.

From the above discussion, we observe that, so far there have been many successes of elliptic and parabolic equations applied to mathematics and physics, but by now very few results on the applications of hyperbolic PDEs are known. In particular, the results on the HMCF is vastly fragmentary. However, we believe that the HMCF is a new and powerful tool to study some problems arising from geometry and physics. There are many interesting and fundamental problems, for example, the long-time existence and asymptotic behavior of solutions, formation of singularities for solutions as well as the physical and geometrical applications, being worthy study in future.

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