# Recent Development and Techniques in Nevanlinna Theory and Complex Hyperbolicity: Talk at the "Second Conference of Tsignhua Sanya International Mathematics Forum", Dec. 20, 2011

Min Ru

My talk gives an overall review of the recent development in the study of Nevanlinna Theory and Complex Hyperbolicity, with the emphasis on the method of using the jet differentials.

A smooth complex projective variety $M$ is said to be (Brody) hyperbolic if every entire map $f : \mathbf{C} \to M$ is constant. Hyperbolicity problems have two aspects, the qualitative aspect and the quantitative aspect (known as Nevanlinna's theory). The relatively harder quantitative aspect is to get a defect relation. For example, by little Picard, $\mathbf{P}^1$ minus three points is hyperbolic, while Nevanlinna derived a defect relation $\sum_{j=1}^q \delta_f(a_j) \leq 2$ for $f : \mathbf{C} \to \mathbf{P}^1$ and distinct points $a_1, dots, a_q \in \mathbf{P}^1$. Note that if $0 \leq \delta_f(a) \leq 1$ and $\delta_f(a) = 1$ if $a \notin f(\mathbf{C})$, so Nevanlinna's defect indeed gives a quantitative description of Little Picard Theorem.

The hyperbolicity has strong relation with its number theory property.
**Lang's Conjecture**. *If $X$ is a projective variety defined over a number field $K$ and is hyperbolic, then $\#X(K) < \infty$ .*

**1. Nevanlinna Theory** Let $(L, h)$ be an Hermitian line bundle over a complex manifold $M$, $s \in H^0(M, L)$ and $D = \{s = 0\}$ be a divisor. Then, by Poincare-Lelong, since $\|s\|^2 = |s_\alpha|^2 h_\alpha$, in the sense of currents,

$$dd^c \log \|s\|^2 = -c(L) + [D].$$

Let $f: \mathbf{C} \to M$ be holomorphic with $f(\mathbf{C}) \not\subset D$. Then, by Stokes' theorem

$$\int_{r_0}^{r} \frac{dt}{t} \int_{|z|<r} f^* c_1(L) = \int_0^{2\pi} \log \frac{1}{\| f^* s(re^{i\theta}) \|} \frac{d\theta}{2\pi} + N(r, D) + O(1).$$

This give the **First Main Theorem**:

$$T_f(r, L) = m_f(r, D) + N_f(r, D) + O(1).$$

We know, for almost all divisors, $T_f(r, D) = N_f(r, D) + O(1)$, and we define the defect as

$$\delta_f(D) = 1 - \limsup \frac{N_f(r, D)}{T_f(r, D)} = \lim_{r \to \infty} \frac{m_f(r, D)}{T_f(r, D)}.$$

Then $\delta_f(D) = 0$ for all most all $D$, and it is then a task to control the total defect. Here are some known results:

- **Nevanlinna**: $\sum_{j=1}^{q} \delta_f(a_j) \leq 2$ for holo. $f: \mathbf{C} \to \mathbf{P}^1$.
- **Cartan, Ahlfors**: $f: \mathbf{C} \to \mathbf{P}^n$ is holomorphic and linearly non-degenerate. $D = H_1 + \cdots + H_q$ hyperplanes in general position, then $\delta_f(D) \leq n + 1$.
- **Ru** (Annals of Math. 2009): Replacing $H_j$ by hypersurfaces.
- **Siu-Yeung, 1996)**: $A$ =abelian variety, $D$ effective divisor, then $\delta_f(D) = 0$ for $f: \mathbf{C} \to A$ which is Zariski dense.

**Griffiths' Conjecture**. *$f: \mathbf{C} \to X$ Zariski dense holomorphic map, $D$ s.n.c divisor on $X$, then*

$$\delta_f(D) \leq -\liminf_{r \to \infty} \frac{T_f(r, K_X)}{T_f(r, [D])}.$$

The key in the proofs is to use the derivatives(first and higher orders), in particular the following so-called

**Logarithmic Derivative Lemma** (or LDL): *If $f$ is a meromorphic function on $\mathbf{C}$, then*

$$\int_0^{2\pi} \log^+ \left| \frac{f^{(k)}}{f}(re^{i\theta}) \right| \frac{d\theta}{2\pi} \leq_{exc} O(\log(rT_f(r))), \qquad k \geq 1,$$

*where $\leq_{exc}$ means that the inequality holds for all $r \in (0, +\infty)$ except a set of finite Lebesgue measure.*

**2. Jet Bundles and Jet Differentials:** $X$ =Cpx analytic space with $\dim X = n$. Let $x \in X$ and consider a germ of holo. mappings $\phi : \triangle \subset \mathbf{C} \to X$ with $\phi(0) = x$. Two germs $\phi, \tilde{\phi}$ osculate to order $k$ if $\phi^{(i)}(0) = \tilde{\phi}^{(i)}(0)$ for $i = 0, \ldots, l$. Let $j_k(\phi)$ = equivalent class of $\phi$ and set $J_k(X)_x = \{j_k(\phi) \mid \phi : \triangle \to X, \phi(0) = x\}$ and $J_k(X) = \cup_{x \in X} J_k(X)_x$. Locally, $J_k(U) = U \times \mathbf{C}^{kn}$ with $j_k(\phi) = (x, \phi'(0), \ldots, \phi^{(k)}(0))$. There is a natural $\mathbf{C}^*$ action on $J_k(X) - \{0\}$ define as: for $\phi : \triangle \to X$ and $t \in \mathbf{C}^*$, $t \cdot j_k(\phi) := j_k(\phi_t)$ where $\phi_t(z) = \phi(tz)$. Denote by $\mathbf{P}(J_k(X)) := (J_k(X) - \{0\})/\mathbf{C}^*$. The fiber $\mathbf{P}(J_k(X)_x)$ of $\mathbf{P}(J_k(X)) \to X$ is a weighted projective space. A function on $\mathbf{P}(J_k(X)_x)$ is said to be homogeneous of (total) weight $m$ if $\phi(t \cdot w) = t^m \phi(w)$. By taking local coordinates on $X$ and allowing the coefficients of $\phi(w)$ to be holomorphic functions, we may define the sheaf $\mathcal{F}_{k,m}$ and a global section $\omega \in H^0(X, \mathcal{F}_{k,m})$ is called a $k$-jet differentials on $X$ of weight $m$ (assuming $k!|m$). When $k = 1$, $\omega$ is just a (symmetric tensor) of differential forms (1-forms) on $X$. There is a canonical isomorphism $H^0(X, \mathcal{F}_{k,m} \otimes S) = H^0(\mathbf{P}(J_k(X)), \mathcal{O}(m) \otimes \pi^* S)$ for any $S$=analytic sheaf.

**Theorem (LDL).** *Let $X$ =compact cpx mfd. and $f : \mathbf{C} \to X$ be holo. For $\omega \in H^0(X, \mathcal{F}_{k,m})$, write write $f^*\omega = \xi(d\zeta)^m$ on $\mathbf{C}$. Then either $f^*\omega \equiv 0$ or $\int_0^{2\pi} \log^+ |\xi(re^{i\theta})| \frac{d\theta}{2\pi} \leq_{exc} O(\log(rT_f(r)))$.*

If in addition that $\omega$ vanishes on an ample divisor $A$ of $X$, then vanishing on an ample divisor $A$ contributes to faster growth than $O(\log T_f(r) + \log r)$. Thus one can get $f^*\omega \equiv 0$. Thus

**Fundamental Vanishing Theorem (or Schwarz lemma)**: *Same assumption for $f$. If $\omega \in H^0(X, \mathcal{F}_{k,m} \otimes \mathcal{O}(-A))$ where $A$ is ample, then $f^*\omega \equiv 0$.*

It may be convenient to use $H^0(X, \mathcal{F}_{k,m} \otimes \mathcal{O}(-A)) = H^0(\mathbf{P}(J_k(X)), \mathcal{O}(m) \otimes \pi^*\mathcal{O}(-A))$. In particular, if $T^*$ is ample, then $X$ is hyperbolic(The result of Kobayashi). In general, the study of complex hyperbolicity is reduced to two steps: (1) The existence of $\omega$, (2) How to reduce from $f^*\omega \equiv 0$ to $P(f) \equiv 0$ for some polynomial, or equivalently, to study the locus locus of $H^0(\mathbf{P}(J_k(X)), \mathcal{O}(m) \otimes \pi^*\mathcal{O}(-A))$.

In the rest of the talk, I explained how to deal with step (2) in the case of (sub-varieties) of abelian variety (in the proof of Bloch's theorem) by Green-Griffiths, in the case of algebraic surface $X$ with $c_1^{(}X) > 2c_2(X)$ by Lu-Yau, as well as the recent techniques developed by Y. T. Siu. Finally we remark that J.P. Demailly (J.P. Demailly: "Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture", preprint, 2011,) has made a significant step towards to the solution in step 1. He proved, using the holomorphic Morse inequality, the existence of $\omega \in H^0(X, \mathcal{F}_{k,m} \otimes \mathcal{O}(-A))$ when $X$ is projective of general type. Thus, from Schwarz lemma, $f^*\omega \equiv 0$ on $\mathbf{C}$.

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$$\delta_f(D) = 1 - \limsup \frac{N_f(r, D)}{T_f(r, D)} = \liminf_{r \to \infty} \frac{m_f(r, D)}{T_f(r, D)}.$$

Then $\delta_f(D) = 0$ for all most all $D$, and it is then a task to control the total defect. Here are some known results:

- Nevanlinna: $\sum_{j=1}^{q} \delta_f(a_j) \leq 2$ for holo. $f : \mathbf{C} \to \mathbf{P}^1$.
- Cartan, Ahlfors: $f : \mathbf{C} \to \mathbf{P}^n$ is holomorphic and linearly non-degenerate. $D = H_1 + \cdots + H_q$ hyperplanes in general position, then $\delta_f(D) \leq n + 1$.
- Ru (Annals of Math. 2009): Replacing $H_j$ by hypersurfaces.
- Siu-Yeung, 1996): $A$ =abelian variety, $D$ effective divisor, then $\delta_f(D) = 0$ for $f : \mathbf{C} \to A$ which is Zariski dense.

Griffiths' Conjecture. *$f : \mathbf{C} \to X$ Zariski dense holomorphic map, $D$ s.n.c divisor on $X$, then*

$$\delta_f(D) \leq -\liminf_{r\to\infty} \frac{T_f(r, K_X)}{T_f(r, [D])}.$$

The key in the proofs is to use the derivatives(first and higher orders), in particular the following so-called Logarithmic Derivative Lemma (or LDL): *If f is a meromorphic function on* **C**, *then*

$$\int_0^{2\pi} \log^+ \left| \frac{f^{(k)}}{f}(re^{i\theta}) \right| \frac{d\theta}{2\pi} \leq_{exc} O(\log(rT_f(r))), \qquad k \geq 1,$$

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# Jet Bundles and Jet Differentials

$X$ =Cpx analytic space with $\dim X = n$. Let $x \in X$ and consider a germ of holo. mappings $\phi : \triangle \subset \mathbf{C} \to X$ with $\phi(0) = x$. Two germs $\phi, \tilde{\phi}$ osculate to order $k$ if $\phi^{(i)}(0) = \tilde{\phi}^{(i)}(0)$ for $i = 0, \ldots, l$. Let $j_k(\phi)$ = equivalent class of $\phi$ and set $J_k(X)_x = \{j_k(\phi) \mid \phi : \triangle \to X, \phi(0) = x\}$ and $J_k(X) = \cup_{x \in X} J_k(X)_x$. Locally, $J_k(U) = U \times \mathbf{C}^{kn}$ with $j_k(\phi) = (x, \phi'(0), \ldots, \phi^{(k)}(0))$. There is a natural $\mathbf{C}^*$ action on $J_k(X) - \{0\}$define as: for $\phi : \triangle \to X$ and $t \in \mathbf{C}^*$, $t \cdot j_k(\phi) := j_k(\phi_t)$ where $\phi_t(z) = \phi(tz)$. Denote by $\mathbf{P}(J_k(X)) := (J_k(X) - \{0\})/\mathbf{C}^*$.

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The fiber $\mathbf{P}(J_k(X)_x)$ of $\mathbf{P}(J_k(X)) \to X$ is a weighted projective space. A function on $\mathbf{P}(J_k(X)_x)$ is said to be homogeneous of (total) weight $m$ if $\phi(t \cdot w) = t^m \phi(w)$. By taking local coordinates on $X$ and allowing the coefficients of $\phi(w)$ to be holomorphic functions, we may define the sheaf $\mathcal{F}_{k,m}$ and a global section $\omega \in H^0(X, \mathcal{F}_{k,m})$ is called a $k$-jet differentials on $X$ of weight $m$ (assuming $k!|m$). When $k = 1$, $\omega$ is just a (symmetric tensor) of differential forms (1-forms) on $X$. There is a canonical isomorphism $H^0(X, \mathcal{F}_{k,m} \otimes S) = H^0(\mathbf{P}(J_k(X)), \mathcal{O}(m) \otimes \pi^* S)$ for any $S$=analytic sheaf.

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One can prove that *for sufficiently large l*, *the base locus of the linear system* $|\mathcal{O}(lm) \otimes \pi^*\mathcal{O}(-A))|$ *is contained in* $B_{k,m}$, *which is the union of the base locus of* $|\mathcal{O}(m)|$ *on* $\mathbf{P}(J_k(X))$ *and the set* $\{j \in \mathbf{P}(J_k(X)) \mid \dim(\phi_m^{-1}(\phi_m(j)) \geq 1\}$, *where* $\phi_m : \mathbf{P}(J_k(X)) \to \mathbf{P}^N$ *is the map defined by the linear system* $|\mathcal{O}(m)|$ *on* $\mathbf{P}(J_k(X))$.

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A Unified Version of SMT (Yamanoi) *$X$=smooth variety. $D = D_1 + \cdots + D_s$ divisor. Suppose that there is $N > 0$, a line bundle $L$ over $X$ and a morphism $\psi : J_N(X) \to L^*$ such that $D_1^{(N)} + \cdots + D_s^{(N)} \subset \psi^* \mathbf{0}$, where $\mathbf{0}$ is the zero section of $L^*$. Then for every holomorphic $f : \mathbf{C} \to X$ with a non-degeneracy condition $j_N(f)(\mathbf{C}) \not\subset \psi^* \mathbf{0}$ where $\mathbf{0}$ is the zero section of $L^*$, we have $\sum_{j=1}^s m(r, f, D_j) + N_{ram,f}(r) + T(r, f, L) \leq_{exc} O(\log(rT_f(r)))$.*

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Application: Result of Siu-Yeung: $f : \mathbf{C} \to A$ *such that* $f(\mathbf{C})$ *is not contained in D. Then* $m(r, f, D) \leq_{exc} O(\log(rT_f(r)))$.

Outline of the proof: Let $r_k : J_k(A) = A \times \mathbf{C}^{k \dim A} \to \mathbf{C}^{k \dim A}$. We claim that there is a positive integer $k$ such that $r_k(j_k(f)(0)) \notin r_k(J_k(D)) \subset \mathbf{C}^{k \dim A}$. Hence, for $k$ big enough, there exists a hyperplane $H \subset \mathbf{C}^{k \dim A}$ such that $r_k(j_k(f))(0) \notin H$, $r_k(D^{(k)})|_{J_k(A)} \subset H$. Let $\psi' : \mathbf{C}^{k \dim A} \to \mathbf{C}$ with $\psi'(0) = H$. Then $\psi := id_A \times \psi' : J_k(A) \to A \times \mathbf{C} = K_A^*$ satisfies the condition in the main theorem. The main Theorem thus implies the result.

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The unified version also implies Schwarz Lemma: *Assume that there is a $k$-jet differential $\omega$ on $X$ vanishes on an ample divisor $A$. The $f^*\omega$ is zero indentically on* $\mathbf{C}$.

The proof is as follows: If not. Note that $\omega$ induces a morphism $J_k(X) \to \mathbf{C}$. The fact that $\omega$ vanishes on an ample divisor $A$ means that $\phi : J_k(X) \to L_H^*$. $f^*\omega$ is not zero indentically on $\mathbf{C}$ implies that $j_k(f)(\mathbf{C}) \not\subset \psi^*\mathbf{0}$ where $\mathbf{0}$ is the zero section of $L_A^*$. Hence, from the above theorem,

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