Generalized Feynman-Kac transformations, Fukushima's decomposition and stochastic calculus for nearly symmetric Markov processes

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Based on joint work with
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For more notations and references, the readers are suggested to refer to the references in the end and the references in them, we just give a summary of my presentations in this document.
Let J_0 be the maximum symmetric part of the jump measure of J in the Beurling-Deny formula of $(\mathcal{E}, D(\mathcal{E}))$, $J_1 = J - J_0$(cf.[HuMS06]). About the F-K transformation for nearly symmetric Markov process it is proved that

Theorem 1. *Assume that $J_1(E \times E - d) < \infty$, then the following are equivalent:*
(i) There exists $\alpha_0 \geq 0$ such that for every $f \in D(\mathcal{E})_b$,

$$Q^u_{\alpha_0}(f, f) \geq 0;$$

(ii)For every $t > 0$,

$$||P^u_t||_{2\to 2} \leq e^{\alpha_0 t}.$$

Furthermore, if one of the assertions holds, then $(P^u_t)_{t\geq 0}$ is strongly continuous on $L^2(E; m)$.

Theorem 2. *Let U be an open set of $\mathbf{R}^d$ and m be a positive Radon measure on U with* supp$[m] = U$. *Suppose that X is a right process which is associated with a (non-symmetric) Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(U; m)$ such that $C^\infty_0(U)$ is dense in $D(\mathcal{E})$. Then the conclusions of Theorem 1 remain valid without assuming that $J_1(E \times E - d) < \infty$.*

Let$A^{[u]}_t := \tilde{u}(X_t) - \tilde{u}(X_0) = M^{[u]}_t + N^{[u]}_t$. The aim of this part is to define an integral $\int_0^t v(X_s)dA^{[u]}_s$ for a suitable class of functions v on E. To this end, we need to define an integral $\int_0^t v(X_s)dN^{[u]}_s$. Note that in general $(N^{[u]}_t)_{t\geq 0}$ is not of bounded variation and hence $(A^{[u]}_t)_{t\geq 0}$ is not a semi-martingale.

Lemma 1. *Let $f \in D(\mathcal{E})$. Then there exist unique $f^* \in D(\mathcal{E})$ and $f^\triangle \in D(\mathcal{E})$ such that for any $g \in D(\mathcal{E})$,*

$$\mathcal{E}_1(f, g) = \tilde{\mathcal{E}}_1(f^*, g) \tag{1}$$

and

$$\tilde{\mathcal{E}}_1(f, g) = \mathcal{E}_1(f^\triangle, g). \tag{2}$$

Let $f, g \in D(\mathcal{E})$. We use $\tilde{\mu}_{<f,g>}$ to denote the mutual energy measure of f and g w.r.t. the symmetric Dirichlet form $(\tilde{\mathcal{E}}, D(\mathcal{E}))$. Suppose that $u \in D(\mathcal{E})$ and $v \in D(\mathcal{E})_b := D(\mathcal{E}) \cap \mathcal{B}_b(E)$. It is easy to see that there exists a unique element in $D(\mathcal{E})$, which is denoted by $\lambda(u, v)$, such that

$$\frac{1}{2}\int_E v d\tilde{\mu}_{<h,u^*>} = \tilde{\mathcal{E}}_1(\lambda(u, v), h), \quad \forall h \in D(\mathcal{E}). \tag{3}$$

About the Stochastic Calculus for nearly symmetric Markov Processes, it is proved that

Theorem 3. *Let $u \in D(\mathcal{E})$ and $v \in D(\mathcal{E})_b$. Then, for any $h \in D(\mathcal{E})_b$,*

$$\mathcal{E}(u, hv) = \mathcal{E}_1(\lambda(u,v)^{\triangle}, h) + \frac{1}{2}\int_E hd\tilde{\mu}_{<v,u^*>} + \int_E (u^* - u)hvdm. \tag{4}$$

Denote by A_c^+ the family of all positive CAFs (PCAFs in short) of X. Define

$$A_c^{+,f} := \{A \in A_c^+ \,|\, \text{the smooth measure}, \mu_A, \text{ cossreponding to } A \text{ is finite}\}$$

and

$$\mathcal{N}_c^* : \;=\; \{N_t^{[u]} + \int_0^t f(X_s)ds + A_t^{(1)} - A_t^{(2)} \,|\, u \in D(\mathcal{E}), f \in L^2(E;m) \text{ and } A^{(1)}, A^{(2)} \in A_c^{+,f}\}.$$

Similar to [S.Nakao 1985 Theorem 2.2], we can prove the following lemma.

Lemma 2. *If $C^{(1)}, C^{(2)} \in \mathcal{N}_c^*$ satisfying*

$$\lim_{t\downarrow 0}\frac{1}{t}E_{h\cdot m}[C_t^{(1)}] = \lim_{t\downarrow 0}\frac{1}{t}E_{h\cdot m}[C_t^{(2)}], \quad \forall h \in D(\mathcal{E})_b,$$

then $C^{(1)} = C^{(2)}$.

Definition 1. *Let $u \in D(\mathcal{E})$ and $v \in D(\mathcal{E})_b$. We define for $t \geq 0$,*

$$\begin{aligned}\int_0^t v(X_{s-})dN_s^{[u]} &:= \int_0^t v(X_s)dN_s^{[u]} \\ &:= \Gamma(u,v)_t - \frac{1}{2}G(u,v)_t - \int_0^t (u^* - u)v(X_s)ds.\end{aligned}$$

Proposition 1. *Let $u \in D(\mathcal{E})$ and $v \in D(\mathcal{E})_b$. Suppose that there exist $A^{(1)}, A^{(2)} \in A_c^+$ such that $N_t^{[u]} = A_t^{(1)} - A_t^{(2)}$ for $t < \zeta$, where ζ is the life time of X. Then*

$$\int_0^t v(X_s)dN_s^{[u]} = \int_0^t v(X_s)d(A_s^{(1)} - A_s^{(2)}) \text{ for } t < \zeta. \tag{5}$$

Theorem 4. *Let $v \in D(\mathcal{E})_b$ and $\{u_n\}_{n=0}^{\infty} \subset D(\mathcal{E})$ satisfying u_n converges to u_0 w.r.t. the $\tilde{\mathcal{E}}_1^{1/2}$-norm as $n \to \infty$. Then there exists a subsequence $\{n'\}$ such that for $\mathcal{E}$-q.e. $x \in E$,*

$$P_x(\lim_{n'\to\infty}\int_0^t v(X_s)dN_s^{[u_{n'}]} = \int_0^t v(X_s)dN_s^{[u_0]} \text{ uniformly on any finite interval of } t) = 1.$$

Proposition 2. *[Itô's formula] Let $u, v \in D(\mathcal{E})_b$. Then*

$$\int_0^t v(X_s)dN_s^{[u]} + \int_0^t u(X_s)dN_s^{[v]} = N_t^{[uv]} - \langle M^{[u]}, M^{[v]}\rangle_t, \quad t \geq 0. \tag{6}$$

Let $M_t^{[u],c}$ be the continuous part of $M_t^{[u]}$ and $\Delta u(X_t) := u(X_t) - u(X_{t-})$ for $t > 0$.

Theorem 5. *Let $u, v \in D(\mathcal{E})_b$. Then,*

$$\begin{aligned}uv(X_t) - uv(X_0) &= \int_0^t v(X_{s-})du(X_s) + \int_0^t u(X_{s-})dv(X_s) + \langle M^{[u],c}, M^{[v],c}\rangle \\ &\quad + \sum_{0<s\leq t}[\Delta uv(X_s) - v(X_{s-})\Delta u(X_s) - u(X_{s-})\Delta v(X_s)].\end{aligned}$$

Theorem 6. *Let $\Phi \in C^2(\mathbb{R}^n)$ and $u_1, \ldots, u_n \in D(\mathcal{E})_b$. Then,*

$$\begin{aligned} A_t^{[\Phi(u)]} &= \sum_{i=1}^n \int_0^t \Phi_i(u(X_{s-}))dA_s^{[u_i]} \\ &+ \frac{1}{2}\sum_{i,j=1}^n \Phi_{ij}(u(X_s))d\langle M^{[u_i],c}, M^{[u_j],c}\rangle \\ &+ \sum_{0<s\le t}\left[\Delta\Phi(u(X_s)) - \sum_{i=1}^n \Phi_i(u(X_{s-}))\Delta u_i(X_s)\right], \end{aligned}$$

where $u = (u_1, \ldots, u_n)$ and

$$\Phi_i(x) = \frac{\partial\Phi}{\partial x_i}(x), \quad \Phi_{ij}(x) = \frac{\partial^2\Phi}{\partial x_i \partial x_j}(x), \quad i, j = 1, \ldots, n.$$

Assumption 1. There exists $\{V_n\} \in \Theta$ such that, for each $n \in \mathbb{N}$, there exists a Dirichlet form $(\eta^{(n)}, D(\eta^{(n)}))$ on $L^2(V_n; m)$ and a constant $C_n > 1$ such that $D(\eta^{(n)}) = D(\mathcal{E})_{V_n}$ and for any $u \in D(\mathcal{E})_{V_n}$,

$$\frac{1}{C_n}\eta_1^{(n)}(u,u) \le \mathcal{E}_1(u,u) \le C_n\eta_1^{(n)}(u,u).$$

About Fukushima's decomposition for semi-Dirichlet forms, it is proved that

Theorem 7. *Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular local semi-Dirichlet form on $L^2(E; m)$ satisfying Assumption 1. Then, for any $u \in D(\mathcal{E})_{loc}$, there exist $M^{[u]} \in \dot{\mathcal{M}}_{loc}$ and $N^{[u]} \in \mathcal{N}_{c,loc}$ such that*

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \quad t \ge 0, \quad P_x\text{-}a.s. \text{ for } \mathcal{E}\text{-}q.e.\ x \in E.$$

Moreover, $M^{[u]} \in \mathcal{M}_{loc}^{[\![0,\zeta[\![}$. Decomposition (7) is unique up to the equivalence of local AFs.

About the transformation formula of local martingale additive functionals, we have the following results.

Theorem 8. *Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular local semi-Dirichlet form on $L^2(E; m)$ satisfying Assumption 1. Let $m \in \mathbb{N}$, $\Phi \in C^1(\mathbb{R}^m)$, and $u = (u_1, u_2, \ldots, u_m)$ with $u_i \in D(\mathcal{E})_{loc}$, $1 \le i \le m$. Then $\Phi(u) \in D(\mathcal{E})_{loc}$ and*

$$M^{[\Phi(u)],c} = \sum_{i=1}^m \Phi_{x_i}(u) \cdot M^{[u_i],c} \text{ on } [0,\zeta), \quad P_x\text{-}a.s. \text{ for } \mathcal{E}\text{-}q.e.\ x \in E.$$

Reference:

• Li Ma, Zhiming Ma, Wei Sun, Fukushima's decomposition for diffusions associated with semi-Dirichlet forms. Submitted to Stochasics and Dynamics.
• Li Ma, Wei Sun, On the generalized Feynman-Kac transformation for nearly symmetric Markov process. Journal of Theoretical Probability, DOI: 10.1007/s10959-010-0318-3.
• Chuanzhong Chen, Li Ma , Wei Sun, Stochastic Calculus for Markov Processes Associated with Non-symmetric Dirichlet Forms, preprint.

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Outline

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1. Dirichlet form theory
2. F-K transformation for nearly symmetric Markov process.

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3. Stochastic Calculus for nearly symmetric Markov Processes
4. Fukushima's decomposition for semi-Dirichlet forms

Introduction

$(E, \mathcal{B}, m)$: σ-finite measure space

Let $(\mathcal{E}, D(\mathcal{E}))$ be a bilinear form on $L^2(E; m)$.

$$\tilde{\mathcal{E}}(u,v) := \frac{1}{2}(\mathcal{E}(u,v) + \mathcal{E}(v,u))$$

$$\check{\mathcal{E}}(u,v) := \frac{1}{2}(\mathcal{E}(u,v) - \mathcal{E}(v,u))$$

$$\tilde{\mathcal{E}}_1(u,v) := \tilde{\mathcal{E}}(u,v) + (u,v)_{L^2(E;dm)}$$

Coercive closed form

$(\mathcal{E}, D(\mathcal{E}))$ is called coercive closed form if $D(\mathcal{E})$ is dense in $L^2(E; m)$ and

(i) $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ is a positive definite closed form, i.e., $\tilde{\mathcal{E}}(u,u) \geq 0, \forall u \in D(\mathcal{E})$ and $(D(\mathcal{E}), \tilde{\mathcal{E}}_1)$ is a Hilbert space.

(ii) (coercive condition) There exists a constant $C > 0$ such that

$$|\mathcal{E}(u,v)| \leq C\mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2}.$$

Dirichlet form

- Semi-Dirichlet form: $(\mathcal{E}, D(\mathcal{E}))$ satisfies

$$u \in D(\mathcal{E}) \Rightarrow u^+ \wedge 1 \in D(\mathcal{E}) \text{ and } \mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0.$$

- Dirichlet form: both $(\mathcal{E}, D(\mathcal{E}))$ and its dual form $(\hat{\mathcal{E}}, D(\mathcal{E}))$ are semi-Dirichlet forms, where $\hat{\mathcal{E}}(u, v) := \mathcal{E}(v, u)$.

- Symmetric Dirichlet form: if $\mathcal{E}(u, v) = \mathcal{E}(v, u)$, $\forall u, v \in D(\mathcal{E})$.

Associated Markov processes

E: metrizable Lusin space, i.e., E is topologically isomorphic to a Borel subset of a Polish space.

$(\mathcal{E}, D(\mathcal{E}))$: a semi-Dirichlet form on $L^2(E; m)$.

■ A right (continuous strong Markov) process $\mathbf{M}$ on E is said to be *(properly) associated* with $(\mathcal{E}, D(\mathcal{E}))$ if and only if $P_t f$ is an ($\mathcal{E}$-quasi-continuous) m-version of $T_t f$ for all $f \in \mathcal{B}_b(E) \cap L^2(E; m)$ and all $t > 0$.

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■ There is a right process $\mathbf{M}$ on E associated with $(\mathcal{E}, D(\mathcal{E}))$, if and only if $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular.

Quasi-homeomorphism

- A semi-Dirichlet form is quasi-regular if and only if it is quasi-homeomorphic to a regular semi-Dirichlet form.

- If $(\mathcal{E}, D(\mathcal{E}))$ is a regular Dirichlet form (i.e., E is locally compact and $C_0(E) \cap D(\mathcal{E})$ is dense in both $D(\mathcal{E})$ and $C_0(E)$), then $(\mathcal{E}, D(\mathcal{E}))$ is associated with a Hunt process.

Advantages of DF theory

•The one-to-one correspondence between Dirichlet forms and Markov processes sets up a bridge between the classical potential theory and stochastic analysis, by which we can transfer between some analytic problems and stochastic problems.

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•The one-to-one correspondence between Dirichlet forms and Markov processes sets up a bridge between the classical potential theory and stochastic analysis, by which we can transfer between some analytic problems and stochastic problems.

•One advantage of the correspondence between Markov processes and Dirichlet forms is that some sample path properties of the Markov processes can be described by the associated Dirichlet forms. For example, the continuity of the sample paths of Markov processes is equivalent to the local property of Dirichlet forms.

Examples

Let S be a negative definite (i.e. $(Su, u) \leq 0$) symmetric (i.e. $(Su, v) = (Sv, u)$) linear operator on a Hilbert space $\mathcal{H}$, for $u, v \in D(\mathcal{S})$, define

$$\mathcal{E}(u.v) = (-Su, v).$$

Let $D(\mathcal{E})$ be the closure of $D(S)$ w.r.t. $\mathcal{E}_1^{\frac{1}{2}}$. If S is an Dirichlet operator (i.e. $(Su, (u-1)^+) \geq 0$ for any $u \in D(S)$), then $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form. Specially, if $S = \frac{1}{2}\Delta$ and $D(S) = C_0^\infty(R)$, then $\mathcal{E}(u, v) = \frac{1}{2}\int u'v'dx$, $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet form associated with Brown motion on R.

Examples

$E = U$: open set of $\mathbb{R}^d$, $m = dx$.

$$\begin{aligned}\mathcal{E}(f,g) &= \sum_{i,j=1}^{d}\int_U \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} a_{ij}dx + \sum_{i=1}^{d}\int_U \frac{\partial f}{\partial x_i} g b_i dx \\ &\quad + \sum_{i=1}^{d}\int_U f\frac{\partial g}{\partial x_i} d_i dx + \int_U fgc dx, \quad f,g \in C_0^\infty(U).\end{aligned}$$

Under suitable conditions, $(\mathcal{E}, C_0^\infty(U))$ is closable on $L^2(U;dx)$ and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a regular local semi-Dirichlet form on $L^2(U;dx)$. Therefore, there exists a diffusion associated with $(\mathcal{E}, D(\mathcal{E}))$.

Outline

Notations

- $(\mathcal{E}, D(\mathcal{E}))$: a non-symmetric DF on $L^2(E;m)$.
- X: is a right process associated with $(\mathcal{E}, D(\mathcal{E}))$.
- Fukushima's decomposition: for $u \in D(\mathcal{E})$,

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + N_t^u,$$

- $\tilde{u}$: a quasi-continuous version of u.
- M_t^u: a MAF with finite energy.
- N_t^u: a CAF of zero energy.
- $P_t^u f(x) = E_x[e^{N_t^u} f(X_t)]$.

Known results for symmetric DF

■Known results about $(P_t^u)_{t\geq 0}$: For symmetric Markov process X, CMS(2007) showed that the semigroup $(P_t^u)_{t\geq 0}$ is strongly continuous on $L^2(E;m)$ if and only if the bilinear form $(Q^u, D(\mathcal{E})_b)$ is lower semi-bounded, where

$$Q^u(f,g) := \mathcal{E}(f,g) + \mathcal{E}(u, fg), \quad f,g \in D(\mathcal{E})_b := D(\mathcal{E}) \cap L^\infty(E;m). \tag{1}$$

CFKZ(2009) gave another proof for the equivalence of the strong continuity of $(P_t^u)_{t\geq 0}$ and the lower semi-boundedness of $(Q^u, D(\mathcal{E})_b)$.

Non-symmetric case

■Difficulties: Many useful tools of symmetric Dirichlet forms, e.g. time reversal and Lyons-Zheng decomposition, do not apply well to the non-symmetric Dirichlet forms setting.

■Method: We will combine the h-transform method of CMS(2007) and the localization method used in CFKZ(2009).

First result

Let J_0 be the maximum symmetric part of the jump measure of J in the Beurling-Deny formula of $(\mathcal{E}, D(\mathcal{E}))$, $J_1 = J - J_0$.

Theorem

Assume that $J_1(E \times E - d) < \infty$, then the following are equivalent:
(i) There exists $\alpha_0 \geq 0$ such that for every $f \in D(\mathcal{E})_b$,

$$Q^u_{\alpha_0}(f, f) \geq 0;$$

(ii)For every $t > 0$,

$$||P^u_t||_{2\to 2} \leq e^{\alpha_0 t}.$$

Furthermore, if one of the assertions holds, then $(P^u_t)_{t\geq 0}$ is strongly continuous on $L^2(E; m)$.

Second result

Theorem

Let U be an open set of $\mathbf{R}^d$ and m be a positive Radon measure on U with $\mathrm{supp}[m] = U$. Suppose that X is a right process which is associated with a (non-symmetric) Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(U; m)$ such that $C_0^\infty(U)$ is dense in $D(\mathcal{E})$. Then the conclusions of Theorem 1 remain valid without assuming that $J_1(E \times E - d) < \infty$.

Remark 1

Remark

If $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric DF, then the assumption of Theorem 1 is satisfied and $(P_t^u)_{t\geq 0}$ is symmetric on $L^2(E;m)$. If $(P_t^u)_{t\geq 0}$ is strongly continuous, then

$$\|P_t^u\|_2 \leq e^{\alpha_0 t}, \quad \forall t > 0.$$

holds (cf.[Remark 1.6(ii)]Z.Q. Chen 2009). Therefore, the following three assertions are equivalent to each other:

(i) $(Q^u, D(\mathcal{E})_b)$ is lower semi-bounded.

(ii) There exists a constant $\alpha_0 \geq 0$ such that $\|P_t^u\|_2 \leq e^{\alpha_0 t}$ for $t > 0$.

(iii) $(P_t^u)_{t\geq 0}$ is strongly continuous on $L^2(E;m)$.

Remark 2

Remark

Let μ be a signed smooth measure and $(A_t)_{t\geq 0}$ be the corresponding additive function. Define $\bar{P}_t^A f(x) = E_x[e^{-A_t} f(X_t)], \quad f \geq 0 \text{ and } t \geq 0$, and

$$\begin{cases} \mathcal{E}^\mu(f,g) := \mathcal{E}(f,g) + \int_E fg d\mu, \\ f,g \in D(\mathcal{E}^\mu) := \{w \in D(\mathcal{E}) \,|\, w \text{ is } \mid \mu \mid \text{square integrable}\}. \end{cases}$$

Then the similar results hold. This result generalizes the corresponding results of Ma(1991) and Chen(2007). Note that, similar to Theorems 1 and 2, it is not necessary to assume that the bilinear form $(\mathcal{E}^\mu, D(\mathcal{E}^\mu))$ satisfies the sector condition.

Outline

1. Dirichlet form theory
2. F-K transformation for nearly symmetric Markov process.
3. Stochastic Calculus for nearly symmetric Markov Processes
4. Fukushima's decomposition for semi-Dirichlet forms

Stochastic Calculus

Let

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Note that in general $(N_t^{[u]})_{t\geq 0}$ is not of bounded variation and hence $(A_t^{[u]})_{t\geq 0}$ is not a semi-martingale.

Stochastic Calculus

In [S.Nakao1985], Nakao defined stochastic integrals for CAFs and established the corresponding Itô's formula in the framework of symmetric Dirichlet forms.

Stochastic Calculus

In [S.Nakao1985], Nakao defined stochastic integrals for CAFs and established the corresponding Itô's formula in the framework of symmetric Dirichlet forms.

To define the stochastic integrals, he essentially used the assumption of symmetry and the formulae of Beurling-Deny and LeJan.

Stochastic Calculus

difficulties for non-symmetric case

Although both the Beurling-Deny decomposition and LeJan's transformation rule can be extended to the non-symmetric Dirichlet forms setting, the SPV integrability in the Beurling-Deny decomposition (cf. [HC06] and [MS11 Section 4]) and the unavailability of the mutual energy measure for the co-symmetric diffusion part of $(\mathcal{E}, D(\mathcal{E}))$ (cf. [HC10] and [MS11 Section 4]) make it difficult to directly extend Nakao's stochastic integrals to the non-symmetric Dirichlet forms setting.

Stochastic Calculus

Lemma

Let $f \in D(\mathcal{E})$*. Then there exist unique* $f^* \in D(\mathcal{E})$ *and* $f^{\triangle} \in D(\mathcal{E})$ *such that for any* $g \in D(\mathcal{E})$,

$$\mathcal{E}_1(f, g) = \tilde{\mathcal{E}}_1(f^*, g) \tag{2}$$

and

$$\tilde{\mathcal{E}}_1(f, g) = \mathcal{E}_1(f^{\triangle}, g). \tag{3}$$

Stochastic Calculus

Let $f, g \in D(\mathcal{E})$. We use $\tilde{\mu}_{<f,g>}$ to denote the mutual energy measure of f and g w.r.t. the symmetric Dirichlet form $(\tilde{\mathcal{E}}, D(\mathcal{E}))$. Suppose that $u \in D(\mathcal{E})$ and $v \in D(\mathcal{E})_b := D(\mathcal{E}) \cap \mathcal{B}_b(E)$. It is easy to see that there exists a unique element in $D(\mathcal{E})$, which is denoted by $\lambda(u, v)$, such that

$$\frac{1}{2}\int_E v d\tilde{\mu}_{<h,u^*>} = \tilde{\mathcal{E}}_1(\lambda(u,v), h), \quad \forall h \in D(\mathcal{E}). \tag{4}$$

Stochastic Calculus

Theorem

Let $u \in D(\mathcal{E})$ *and* $v \in D(\mathcal{E})_b$. *Then, for any* $h \in D(\mathcal{E})_b$,

$$\mathcal{E}(u, hv) = \mathcal{E}_1(\lambda(u, v)^{\triangle}, h) + \frac{1}{2}\int_E h d\tilde{\mu}_{<v,u^*>} + \int_E (u^* - u)hvdm. \quad (5)$$

Stochastic Calculus

Denote by A_c^+ the family of all positive CAFs (PCAFs in short) of X. Define

$$\begin{aligned} A_c^{+,f} : &= \{A \in A_c^+ \,|\, \text{the smooth measure,} \\ &\quad \mu_A, \text{ cossreponding to } A \text{ is finite}\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}_c^* : &= \{N_t^{[u]} + \int_0^t f(X_s)ds + A_t^{(1)} - A_t^{(2)} \,|\, u \in D(\mathcal{E}), \\ &\quad f \in L^2(E;m) \text{ and } A^{(1)}, A^{(2)} \in A_c^{+,f}\}. \end{aligned}$$

Stochastic Calculus

Similar to [S.Nakao 1985 Theorem 2.2], we can prove the following lemma.

Lemma

If $C^{(1)}, C^{(2)} \in \mathcal{N}_c^*$ *satisfying*

$$\lim_{t \downarrow 0} \frac{1}{t} E_{h \cdot m}[C_t^{(1)}] = \lim_{t \downarrow 0} \frac{1}{t} E_{h \cdot m}[C_t^{(2)}], \quad \forall h \in D(\mathcal{E})_b,$$

then $C^{(1)} = C^{(2)}$.

Stochastic Calculus

Let $u \in D(\mathcal{E})$ and $v \in D(\mathcal{E})_b$. Then $\tilde{\mu}_{<v,u^*>}$ is a signed smooth measure w.r.t. $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ and hence $(\mathcal{E}, D(\mathcal{E}))$.

We use $G(u, v)$ to denote the unique element in $A_c^+ - A_c^+$ that is corresponding to $\tilde{\mu}_{<v,u^*>}$ under the Revuz correspondence between smooth measures of $(\mathcal{E}, D(\mathcal{E}))$ and PCAFs of X (cf. [MR92 theorem VI.2.4]).

Stochastic Calculus

To simplify notation, we define

$$\Gamma(u,v)_t := N_t^{[\lambda(u,v)^{\triangle}]} - \int_0^t \lambda(u,v)^{\triangle}(X_s)ds, \quad t \geq 0.$$

Stochastic Calculus

To simplify notation, we define

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Definition

Let $u \in D(\mathcal{E})$ *and* $v \in D(\mathcal{E})_b$. *We define for* $t \geq 0$,

$$\begin{aligned}\int_0^t v(X_{s-})dN_s^{[u]} &:= \int_0^t v(X_s)dN_s^{[u]} \\ &:= \Gamma(u,v)_t - \frac{1}{2}G(u,v)_t - \int_0^t (u^* - u)v(X_s)ds.\end{aligned}$$

Stochastic Calculus

Remark

Let $u \in D(\mathcal{E})$ and $v \in D(\mathcal{E})_b$. Then one can check that $\int_0^t v(X_s)dN_s^{[u]} \in \mathcal{N}_c^*$. By Definition 1, (**??**), [**?**, Theorem 3.4], [**?**, Theorem 5.8(iii)] and (5), we obtain that

$$\lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}\left[\int_0^t v(X_s)dN_s^{[u]}\right] = -\mathcal{E}(u, hv). \tag{6}$$

Therefore, by Lemma 2, $\int_0^t v(X_s)dN_s^{[u]}$ is the unique AF $(C_t)_{t\geq 0}$ in $\mathcal{N}_c^*$ that satisfies $\lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}[C_t] = -\mathcal{E}(u, hv)$.

Stochastic Calculus

Proposition

Let $u \in D(\mathcal{E})$ and $v \in D(\mathcal{E})_b$. Suppose that there exist $A^{(1)}, A^{(2)} \in A_c^+$ such that $N_t^{[u]} = A_t^{(1)} - A_t^{(2)}$ for $t < \zeta$, where ζ is the life time of X. Then

$$\int_0^t v(X_s)dN_s^{[u]} = \int_0^t v(X_s)d(A_s^{(1)} - A_s^{(2)}) \text{ for } t < \zeta. \tag{7}$$

Stochastic Calculus

Theorem

Let $v \in D(\mathcal{E})_b$ and $\{u_n\}_{n=0}^{\infty} \subset D(\mathcal{E})$ satisfying u_n converges to u_0 w.r.t. the $\tilde{\mathcal{E}}_1^{1/2}$-norm as $n \to \infty$. Then there exists a subsequence $\{n'\}$ such that for $\mathcal{E}$-$q.e.$ $x \in E$,

$$P_x\Big(\lim_{n' \to \infty} \int_0^t v(X_s)dN_s^{[u_{n'}]} = \int_0^t v(X_s)dN_s^{[u_0]} \text{ uniformly on any finite interval of } t\Big) = 1.$$

Itô's formula

Proposition

Let $u, v \in D(\mathcal{E})_b$. *Then*

$$\int_0^t v(X_s)dN_s^{[u]} + \int_0^t u(X_s)dN_s^{[v]} = N_t^{[uv]} - \langle M^{[u]}, M^{[v]}\rangle_t, \quad t \geq 0. \tag{8}$$

Itô's formula

Corollary

Let $u \in D(\mathcal{E})_b$ and $\{v_n\}_{n=0}^{\infty} \subset D(\mathcal{E})_b$ satisfying v_n converges to v_0 w.r.t. the $\|\cdot\|_\infty$-norm and the $\tilde{\mathcal{E}}_1^{1/2}$-norm as $n \to \infty$. Then there exists a subsequence $\{n'\}$ such that for $\mathcal{E}$-$q.e.$ $x \in E$,

$$P_x(\lim_{n' \to \infty} \int_0^t v_{n'}(X_s) dN_s^{[u]} = \int_0^t v_0(X_s) dN_s^{[u]} \text{ uniformly on any finite interval of } t) = 1.$$

Itô's formula

Let $M_t^{[u],c}$ be the continuous part of $M_t^{[u]}$ and $\Delta u(X_t) := u(X_t) - u(X_{t-})$ for $t > 0$.

Theorem

Let $u, v \in D(\mathcal{E})_b$. *Then,*

$$\begin{aligned} & uv(X_t) - uv(X_0) \\ = \ & \int_0^t v(X_{s-})du(X_s) + \int_0^t u(X_{s-})dv(X_s) + \langle M^{[u],c}, M^{[v],c}\rangle \\ & \quad + \sum_{0<s\leq t} [\Delta uv(X_s) - v(X_{s-})\Delta u(X_s) - u(X_{s-})\Delta v(X_s)]. \end{aligned}$$

Theorem

Let $\Phi \in C^2(\mathbb{R}^n)$ *and* $u_1, \ldots, u_n \in D(\mathcal{E})_b$. *Then,*

$$
\begin{aligned}
A_t^{[\Phi(u)]} &= \sum_{i=1}^{n} \int_0^t \Phi_i(u(X_{s-}))dA_s^{[u_i]} \\
&+ \frac{1}{2}\sum_{i,j=1}^{n} \Phi_{ij}(u(X_s))d\langle M^{[u_i],c}, M^{[u_j],c}\rangle \\
&+ \sum_{0<s\leq t}\left[\Delta\Phi(u(X_s)) - \sum_{i=1}^{n}\Phi_i(u(X_{s-}))\Delta u_i(X_s)\right],
\end{aligned}
$$

where $u = (u_1, \ldots, u_n)$ *and*

$$
\Phi_i(x) = \frac{\partial \Phi}{\partial x_i}(x), \quad \Phi_{ij}(x) = \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x), \quad i, j = 1, \ldots, n.
$$

Outline

Question

■ Consider the following form

$$\mathcal{E}(u,v)=\int_0^1 u'v'dx+\int_0^1 bu'vdx \quad u,v\in D(\mathcal{E})=H_0^{1,2}(0,1),$$

where $b(x)=\sqrt{x}$ and $H_0^{1,2}(0,1)$ is the (1,2)-Sobolev space on $(0,1)$ with Dirichlet boundary condition, then by MR(1995) we know $(\mathcal{E},D(\mathcal{E}))$ is a semi-Dirichlet form instead of a Dirichlet form. Dose Fukushima's decomposition hold for such a semi-Dirichlet form?

The importance of Fukushima's decomposition

■ It is well known that Doob-Meyer decomposition and Itô's formula are essential in the study of stochastic dynamics. In the framework of Dirichlet forms, the celebrated Fukushima's decomposition and the corresponding transformation formula play the roles of Doob-Meyer decomposition and Itô's formula, which are available for a large class of processes that are not semi-martingales.

History of Fukushima's decomposition

■ The classical decomposition of Fukushima was originally established for regular symmetric Dirichlet forms ([Fu79] and [Fu 94 Theorem 5.2.2]). Later it was extended to the non-symmetric and quasi-regular cases, respectively ([Oshima Theorem 5.1.3] and [MR92 Theorem VI.2.5]).

Difficulties for semi-DF

■ There is a big difference between DF and semi-DF. For DF, $D(\mathcal{E})_b$ is an algebra, while this is not true for semi-DF. Also, there is a pair of Markov processes associated with a DF, but there is only one Markov process associated with a semi-DF.

Difficulties for semi-DF

■ There is a big difference between DF and semi-DF. For DF, $D(\mathcal{E})_b$ is an algebra, while this is not true for semi-DF. Also, there is a pair of Markov processes associated with a DF, but there is only one Markov process associated with a semi-DF.

■The assumption of the existence of dual Markov process plays a crucial role in all the Fukushima-type decompositions known up to now. In fact,without that assumption, the usual definition of energy of AFs

$$e(A) := \lim_{t\to 0} \frac{1}{2t} E_m[A_t^2].$$

is questionable.

Our method

To tackle this difficulty, we employ the notion of local AFs introduced in [Fu94]. Then we introduce a localization method to obtain Fukshima's decomposition for a class of diffusions associated with semi-Dirichlet forms.

Part process

Let V be a quasi-open subset of E.

Denote by $X^V = (X_t^V)_{t\geq 0}$ the part process of X on V.

Denote by $(\mathcal{E}^V, D(\mathcal{E})_V)$ the part form of $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(V; m)$.

Denote by $(G_\alpha^V)_{\alpha\geq 0}$ and $(\hat{G}_\alpha^V)_{\alpha\geq 0}$ the resolvent and co-resolvent associated with $(\mathcal{E}^V, D(\mathcal{E})_V)$, respectively.

Part energy function

Fix $\phi \in L^2(E;m)$ with $0 < \phi \leq 1$ m-$a.e.$ Define $\bar{h} := \hat{G}_1\phi$, $\bar{h}^V := \hat{h}|_V \wedge \hat{G}_1^V\phi$. Then $\bar{h}^V \in D(\mathcal{E})_V$ and $\bar{h}^V$ is 1-co-excessive. For any AF $A = (A_t)_{t\geq 0}$ of $(X_t^V)_{t\geq 0}$, we define

$$e^V(A) := \lim_{t\downarrow 0} \frac{1}{2t} E_{\bar{h}^V\cdot m}(A_t^2).$$

Other notations

Define

$$\dot{\mathcal{M}}^V := \{M \,|\, M \text{ is an AF of } X^V,\ E_x(M_t^2) < \infty, E_x(M_t) = 0 \text{ for all } t \geq 0 \text{ and } \mathcal{E}\text{-}q.e.\ x \in V, e^V(M) < \infty\},$$

$$\mathcal{N}_c^V := \{N \,|\, N \text{ is a CAF of } X^V, E_x(|N_t|) < \infty \text{ for all } t \geq 0 \text{ and } \mathcal{E}\text{-}q.e.\ x \in V, e^V(N) = 0\},$$

Other notations

Define

$$\begin{aligned}\dot{\mathcal{M}}_{loc} \quad := \quad & \{M \,|\, M \text{ is a local AF of } \mathbf{M},\ \exists\ \{V_n\}, \{E_n\} \in \Theta \\ & \text{and } \{M^n \,|\, M^n \in \dot{\mathcal{M}}^{V_n}\} \text{ such that } E_n \subset V_n, \\ & M_{t\wedge\tau_{E_n}} = M^n_{t\wedge\tau_{E_n}},\ t \geq 0,\ n \in \mathbb{N}\}\end{aligned}$$

$$\begin{aligned}\mathcal{N}_{c,loc} \quad := \quad & \{N \,|\, N \text{ is a local AF of } \mathbf{M},\ \exists\ \{V_n\}, \{E_n\} \in \Theta \\ & \text{and } \{N^n \,|\, N^n \in \mathcal{N}_c^{V_n}\} \text{ such that } E_n \subset V_n, \\ & N_{t\wedge\tau_{E_n}} = N^n_{t\wedge\tau_{E_n}},\ t \geq 0,\ n \in \mathbb{N}\}.\end{aligned}$$

Other notations

Define

$$\Theta := \{\{V_n\} \,|\, V_n \text{ is } \mathcal{E}\text{-quasi-open}, \ V_n \subset V_{n+1} \ \mathcal{E}\text{-}q.e., \\ \text{for all } n \in \mathbb{N}, \ \text{and } E = \cup_{n=1}^{\infty} V_n \ \mathcal{E}\text{-}q.e.\},$$

$$D(\mathcal{E})_{loc} := \{u \,|\, \exists \ \{V_n\} \in \Theta \text{ and } \{u_n\} \subset D(\mathcal{E}) \\ \text{such that } \ u = u_n \ m\text{-}a.e. \text{ on } V_n\}.$$

Assumption

Assumption

There exists $\{V_n\} \in \Theta$ such that, for each $n \in \mathbb{N}$, there exists aDirichlet form $(\eta^{(n)}, D(\eta^{(n)}))$ on $L^2(V_n; m)$ and a constant $C_n > 1$ such that $D(\eta^{(n)}) = D(\mathcal{E})_{V_n}$ and for any $u \in D(\mathcal{E})_{V_n}$,

$$\frac{1}{C_n}\eta_1^{(n)}(u,u) \leq \mathcal{E}_1(u,u) \leq C_n\eta_1^{(n)}(u,u).$$

Some Lemmas

We fix a $\{V_n\} \in \Theta$ satisfying Assumption 1. Without loss of generality, we assume that $\widetilde{\hat{h}}$ is bounded on each V_n, otherwise we may replace V_n by $V_n \cap \{\widetilde{\hat{h}} < n\}$. To simplify notations, we write

$$\bar{h}_n := \bar{h}^{V_n}, \quad \text{and} \quad D(\mathcal{E})_{V_n,b} := \mathcal{B}_b(E) \cap D(\mathcal{E})_{V_n}.$$

Some Lemmas

Lemma

$\dot{\mathcal{M}}^{V_n}$ is a real Hilbert space with inner product e^{V_n}. Moreover, if $\{M_l\} \subset \dot{\mathcal{M}}^{V_n}$ is e^{V_n}-Cauchy, then there exist a unique $M \in \dot{\mathcal{M}}^{V_n}$ and a subsequence $\{l_k\}$ such that $\lim_{k\to\infty} e^{V_n}(M_{l_k} - M) = 0$ and for $\mathcal{E}$-$q.e.$ $x \in V_n$,

$$P_x(\lim_{k\to\infty} M_{l_k}(t) = M(t) \text{ uniformly on each compact interval of } [0,\infty)) =$$

Some Lemmas

Lemma

Let $u \in D(\mathcal{E})_{V_n,b}$. *Then there exist unique* $M^{n,[u]} \in \dot{\mathcal{M}}^{V_n}$ *and* $N^{n,[u]} \in \mathcal{N}_c^{V_n}$ *such that* for $\mathcal{E}$-$q.e.$ $x \in V_n$,

$$\tilde{u}(X_t^{V_n}) - \tilde{u}(X_0^{V_n}) = M_t^{n,[u]} + N_t^{n,[u]}, \quad t \geq 0, \quad P_x\text{-}a.s.$$

Some Lemmas

Define $E_n = \{x \in E \mid \widetilde{h_n}(x) > \frac{1}{n}\}$, where $h_n := G_1^{V_n}\phi$. Define $f_n = n\widetilde{h_n} \wedge 1$. Then $u_n f_n \in D(\mathcal{E})_{V_n,b}$. For $n \in \mathbb{N}$, denote by $\{\mathcal{F}_t^n\}$ the minimum completed admissible filtration of X^{V_n}. For $n < l$, $\mathcal{F}_t^n \subset \mathcal{F}_t^l \subset \mathcal{F}_t$. Since $E_n \subset V_n$, τ_{E_n} is an $\{\mathcal{F}_t^n\}$-stopping time.

Lemma

For $n < l$, we have $M_{t\wedge\tau_{E_n}}^{n,[u_n f_n]} = M_{t\wedge\tau_{E_n}}^{l,[u_l f_l]}$ and $N_{t\wedge\tau_{E_n}}^{n,[u_n f_n]} = N_{t\wedge\tau_{E_n}}^{l,[u_l f_l]}$, $t \geq 0$, P_x-a.s. for $\mathcal{E}$-q.e. $x \in V_n$.

Fifth result

Theorem

Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular local semi-Dirichlet form on $L^2(E;m)$ satisfying Assumption 1. Then, for any $u \in D(\mathcal{E})_{loc}$, there exist $M^{[u]} \in \dot{\mathcal{M}}_{loc}$ and $N^{[u]} \in \mathcal{N}_{c,loc}$ such that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \quad t \geq 0, \quad P_x\text{-}a.s. \ \text{ for } \mathcal{E}\text{-}q.e.\ x \in E. \tag{9}$$

Moreover, $M^{[u]} \in \mathcal{M}_{loc}^{[\![0,\zeta[\![}$. Decomposition (9) is unique up to the equivalence of local AFs.

Sixth result

About the transformation formula of local martingale additive functionals, we have the following results.

Theorem

Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular local semi-Dirichlet form on $L^2(E;m)$ satisfying Assumption 1. Let $m \in \mathbb{N}$, $\Phi \in C^1(\mathbb{R}^m)$, and $u = (u_1, u_2, \ldots, u_m)$ with $u_i \in D(\mathcal{E})_{loc}$, $1 \le i \le m$. Then $\Phi(u) \in D(\mathcal{E})_{loc}$ and

$$M^{[\Phi(u)],c} = \sum_{i=1}^{m} \Phi_{x_i}(u) \cdot M^{[u_i],c} \text{ on } [0, \zeta), \quad P_x\text{-}a.s. \text{ for } \mathcal{E}\text{-}q.e.\ x \in E.$$

One important Lemma

Fix a $\{V_n\} \in \Theta$ satisfying Assumption 1 and $\tilde{\hat{h}}$ is bounded on each V_n. For $u, v \in D(\mathcal{E})_{V_n,b}$, we denote by $\mu^{n,c}_{<u>}$ the Revuz measure of $< M^{n,[u],c} >$. Define

$$\mu^{n,c}_{<u,v>} := \frac{1}{2}(\mu^{n,c}_{<u+v>} - \mu^{n,c}_{<u>} - \mu^{n,c}_{<v>}).$$

Lemma

Let $u, v, w \in D(\mathcal{E})_{V_n,b}$. *Then*

$$d\mu^{n,c}_{<uv,w>} = \tilde{u} d\mu^{n,c}_{<v,w>} + \tilde{v} d\mu^{n,c}_{<u,w>}. \tag{10}$$

Example 1

Example

Consider the following bilinear form

$$\mathcal{E}(u,v)=\int_0^1 u'v'dx+\int_0^1 x^2u'(x)v(x)dx, \quad u,v\in D(\mathcal{E}):=H_0^{1,2}(0,1).$$

Then $(\mathcal{E},D(\mathcal{E}))$ is a regular local semi-Dirichlet form (but not a Dirichlet form) on $L^2((0,1);dx)$. Note that any $u\in D(\mathcal{E})$ is bounded. Then we obtain Fukushima's decomposition, $u(X_t)-u(X_0)=M_t^{[u]}+N_t^{[u]}$, where X is the diffusion process associated with $(\mathcal{E},D(\mathcal{E}))$, $M^{[u]}$ is an MAF of finite energy and $N^{[u]}$ is a CAF of zero energy.

Example 2

Example

Recall in the beginning we consider the following semi-Dirichlet form

$$\mathcal{E}(u,v) = \int_0^1 u'v'dx + \int_0^1 \sqrt{x}u'(x)v(x)dx, \quad u,v \in D(\mathcal{E}) := H_0^{1,2}(0,1).$$

Let $u \in D(\mathcal{E})_{loc}$. Then we obtain Fukushima's decomposition by Theorem 4.2.

Example 3

Let $d \geq 3$, U be an open subset of $\mathbb{R}^d$, $\sigma, \rho \in L^1_{loc}(U; dx)$, $\sigma, \rho > 0$ dx-$a.e.$ For $u, v \in C_0^\infty(U)$, we define

$$\mathcal{E}_\rho(u, v) = \sum_{i,j=1}^{d} \int_U \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \rho dx.$$

Assume that $(\mathcal{E}_\rho, C_0^\infty(U))$ is closable on $L^2(U; \sigma dx)$.
Let $a_{ij}, b_i, d_i \in L^1_{loc}(U; dx)$, $1 \leq i, j \leq d$. For $u, v \in C_0^\infty(U)$, we define

$$\begin{aligned} \mathcal{E}(u, v) &= \sum_{i,j=1}^{d} \int_U \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} a_{ij} dx + \sum_{i=1}^{d} \int_U \frac{\partial u}{\partial x_i} v b_i dx \\ &\quad + \sum_{i=1}^{d} \int_U u \frac{\partial v}{\partial x_i} d_i dx + \int_U uvc dx. \end{aligned}$$

Example 3

Under some conditions, there exists $\alpha > 0$ such that the closure $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ of $(\mathcal{E}_\alpha, C_0^\infty(U))$, is a regular local semi-Dirichlet form on $L^2(U; dx)$. Define $\eta_\alpha(u,u) := \mathcal{E}_\alpha(u,u) - \int \langle \nabla u, \underline{\beta} \rangle u dx$ for $u \in D(\mathcal{E}_\alpha)$. Then $(\eta_\alpha, D(\mathcal{E})_\alpha)$ is a Dirichlet form and there exists $C > 1$ such that for any $u \in D(\mathcal{E}_\alpha)$,

$$\frac{1}{C}\eta_\alpha(u,u) \le \mathcal{E}_\alpha(u,u) \le C\eta_\alpha(u,u).$$

Let X be the diffusion process associated with $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$. Then, by Theorem 4.2, Fukushima's decomposition holds for any $u \in D(\mathcal{E})_{loc}$. Moreover, the transformation formula holds for local MAFs.