GRADIENT SHRINKING RICCI SOLITONS

HUAI-DONG CAO

Abstract. This is the summary of a talk given by the author at the Second Tsinghua-Sanya International Mathematics Forum in Sanya, China, December 19-21, 2011.

A complete Riemannian metric g_{ij} on a smooth manifold M^n is called a *gradient shrinking Ricci soliton* if there exists a smooth function f on M^n such that the Ricci tensor R_{ij} of the metric g_{ij} satisfies the equation

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}. \tag{1}$$

The function f is called a *potential function* of the gradient Ricci soliton. The concept of *Ricci solitons* was introduced by Hamilton in mid 80s'. In recent years, gradient shrinking Ricci solitons have become increasingly important in geometry and Ricci flow because they are

• *Natural extensions of Einstein metrics*: when f is a constant a gradient Ricci soliton is simply a Einstein metric;

• *Self-similar solutions to the Ricci flow*: if (M^n, g_{ij}, f) is a gradient Ricci soliton, then $g_{ij}(t) = (1-t)\varphi_t^* g_{ij}$ is a solution to Hamilton's Ricci flow

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t). \tag{2}$$

Here φ_t is the 1-parameter family of diffeomorphisms generated by $\frac{\nabla f}{1-t}$;

• *Singularity models of Ricci flow*: shrinking Ricci solitons arise as Type I singularity models;

• *Li-Yau-Hamilton inequality becomes equality on Ricci solitons*;

• *Critical points of Perelman's ν-entropy.*

Therefore it is important to classify gradient Ricci solitons and understand their geometry. This forms a significant part of the study of singularity formations of the Ricci flow, and has profound applications to geometry and topology.

The most basic examples include: (i) positive Einstein manifolds (M^n, g_{ij}) when f is a constant; (ii) the Gaussian (shrinking) soliton $(\mathbb{R}^n, \delta_{ij}, |x|^2/4)$; (iii) products $N^k \times \mathbb{R}^{n-k} (k \geq 2)$ for any positive Einstein manifold N^k.

In dimensions 2 and 3, Hamilton and Ivey respectively showed that the only compact shrinking solitons are quotients of the round spheres $\mathbb{S}^2$ and $\mathbb{S}^3$. By the work of Perelman, and the extensions by Ni-Wallach and Cao-Chen-Zhu, one has the following classification for 3-dimensional complete shrinking Ricci solitons.

Theorem 1 *A complete 3-dimensional gradient shrinking Ricci soliton is necessarily a quotient of either the round 3-sphere $\mathbb{S}^3$, or the Gaussian soliton $\mathbb{R}^3$, or the round cylinder $\mathbb{S}^2 \times \mathbb{R}$.*

However, for dimension $n \geq 4$, noncompact and non-product or compact non-Einstein shrinking solitons do exist. For example, in dimension $n = 4$, Koiso and the author independently constructed a $U(2)$-invariant gradient shrinking Kähler-Ricci soliton on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$, and Wang-Zhu constructed a toric Kähler shrinker on $\mathbb{C}P^2\#(-2\mathbb{C}P^2)$, while Feldman-Ilmanen-Knopf constructed a noncompact $U(2)$-invariant gradient shrinking Kähler-Ricci solitons on the tautological line bundle $\mathcal{O}(-1)$ of $\mathbb{C}P^1$, the blow-up of $\mathbb{C}^2$ at the origin. These are the only known examples of nontrivial (i.e., non-Einstein or non-product) complete shrinking Ricci solitons in dimension 4 so far.

The classifications of higher dimensional complete shrinking Ricci solitons under certain special curvature assumptions have been a hot topic in recent years and much progress has been made, e.g.,

• A. Naber proved that an 4-dimensional complete gradient shrinking soliton with *bounded and nonnegative curvature operator* $0 \leq Rm \leq C$ is necessarily a finite quotient of $\mathbb{S}^4$, or $\mathbb{R}^4$, or $\mathbb{S}^3 \times \mathbb{R}$, or $\mathbb{S}^2 \times \mathbb{R}^2$.

• The works of of Ni-Wallach and Z. H. Zhang imply that a complete n-dimensional ($n \geq 4$) gradient shrinking Ricci soliton *with vanishing Weyl tensor* $W_{ijkl} = 0$ is a quotient of either $\mathbb{S}^n$, or $\mathbb{R}^n$, or $\mathbb{S}^{n-1} \times \mathbb{R}$. (For different proofs, see Peterson-Wylie and Munteanu-Sesum, also the works X. Cao, B. Wang and Z. Zhang).

• Fernández-López and García-Río and Munteanu-Sesum showed that n-dimensional complete gradient shrinking solitons *with harmonic Weyl tensor (i.e., Cotton tensor* $C_{ijk} = 0$*)* are necessarily finite quotients of the products $N^k \times \mathbb{R}^{n-k}$ of an Einstein manifold N^k with the Gaussian shrinking soliton $\mathbb{R}^{n-k}$.

• X. Chen and Y. Wang proved that 4-dimensional *half-conformally flat* gradient Ricci solitons are either Einstein or locally conformally flat.

Very recently, the author and his student Qiang Chen investigated an interesting class of complete gradient shrinking Ricci solitons, those with vanishing Bach tensor.

Theorem 2 (Cao-Chen) *Let (M^4, g_{ij}, f) be a complete Bach-flat gradient shrinking Ricci soliton. Then, (M^4, g_{ij}, f) is either*

(i) Einstein, or

(ii) locally conformally flat; more specifically, a finite quotient of either the Guassian shrinking soliton $\mathbb{R}^4$ or the round cylinder $\mathbb{S}^3 \times \mathbb{R}$.

More generally, for $n \geq 5$, we have:

Theorem 3 (Cao-Chen) *Let (M^n, g_{ij}, f) $(n \geq 5)$ be a complete Bach-flat gradient shrinking Ricci soliton. Then, (M^n, g_{ij}, f) is either*

(i) Einstein, or

(ii) a finite quotient of the Guassian shrinking soliton $\mathbb{R}^n$ or $N^{n-1} \times \mathbb{R}$, where N^{n-1} is an Einstein manifold of positive scalar curvature.

Note that on any Riemannian manifold (M^n, g_{ij}) $(n \geq 4)$ the Bach tensor, which is well-known in general relativity and conformal geometry, is defined by

$$B_{ij} = \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + \frac{1}{n-2}R_{kl}W_i{}^k{}_j{}^l. \tag{3}$$

Also, if (M^n, g_{ij}) is locally conformally flat (i.e., $W_{ikjl} = 0$) or Einstein, then (M^n, g_{ij}) is *Bach-flat*: $B_{ij} = 0$. The case when $n = 4$ is the most interesting since on any compact 4-manifold (M^4, g_{ij}), Bach-flat metrics are precisely the critical points of the *conformally invariant* functional $\mathcal{W}(g) = \int_M |W_g|^2 dV_g$, on the space of metrics. Here W_g denotes the Weyl tensor of g. Moreover, if (M^4, g_{ij}) is either *half conformally flat* (i.e., self-dual or anti-self-dual) or *locally conformal to an Einstein manifold*, then its Bach tensor vanishes. Thus, Theorem 2 provides one of the best statements of classifications so far, especially in the 4-D case.

By definition, a Ricci shrinker or Einstein manifold is called *linearly stable* if the second variation of Perelman's ν-entropy is non-positive. Now in dimension $n \geq 4$, so far no one knows how to completely classify Einstein manifolds of positive scalar curvature, let alone general gradient shrinking Ricci solitons. However, as far as applications of Ricci flow to topology is concerned, stable shrinking solitons or stable Einstein metrics are more important since they represent generic Type I singularity models, while unstable ones could be perturbed away. For this reason, it is important to classify stable shrinking Ricci solitons or stable Einstein metrics.

In 2004, Hamilton, Ilmanen and the author initiated the study of linear stability of Ricci solitons. They derived the second variation formula of the ν-entropy for positive Einstein manifolds and investigated the linear stability of certain Einstein manifolds. Among other results, they showed that in real dimension n=4, all positive Kähler-Einstein manifolds are unstable except $\mathbb{C}P^2$. More generally, they showed that all Kähler-Einstein manifolds with Hodge number $h^{1,1} > 1$ are unstable. They also showed that any product of two nonflat shrinkers $N = (N_1^{n_1}, g_1) \times (N_2^{n_2}, g_2)$ is linearly unstable.

An important open problem in 4-D is the following

Conjecture (Hamilton, 2004): The standard $\mathbb{S}^4$ and $\mathbb{C}P^2$ are the only 4-dimensional stable positive Einstein manifolds.

Concerning compact stable shrinkers, we have

Open Problem: Show that compact stable shrinking Ricci solitons are Einstein.

Recently, the author and his graduate student Meng Zhu have successfully extended the second variation formula of Cao-Hamilton-Ilmanen for Einstein manfiolds to compact shrinking Ricci solitons. Also, Hall and Murphy have extended the result of Cao-Hamilton-Ilmanne by showing that any compact gradient Kähler-Ricci solitons are unstable, provided their Hodge number $h^{1,1} > 1$. In particular, the Koiso-Cao soliton on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$ and the Wang-Zhu soliton on $\mathbb{C}P^2\#(-2\mathbb{C}P^2)$, the only known 4-D compact examples, are unstable.

Department of Mathematics, Lehigh University, Bethlehem, PA 18015
E-mail address: huc2@lehigh.edu

Gradient Shrinking Ricci Solitons

Huai-Dong Cao (Lehigh University)

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Ricci Solitons

A complete Riemannian manifold (M^n, g_{ij}) is called a *Ricci soliton* if there exists a smooth vector field X on M^n and a constant $\rho \in \mathbb{R}$ such that

$$2R_{ij} + \nabla_i X_j + \nabla_j X_i = \rho g_{ij}.$$

Gradient Ricci soliton: if $X = \nabla f$ for some $f \in C^\infty(M)$ so that

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}.$$

f: *potential function* of the Ricci soliton.

$\rho = 0$: *steady* soliton;

$\rho > 0$: *shrinking* soliton;

$\rho < 0$: *expanding* soliton.

$f = const.$: Einstein metrics.

Why study Ricci solitons?

- Natural extension of Einstein manifolds;

- Self-similar solutions to the Ricci flow: if (M^n, g_{ij}, f) is a gradient Ricci soliton, then

$$g_{ij}(t) = (1 - 2\rho t)\varphi_t^* g_{ij}$$

is a solution to Hamilton's Ricci flow

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t).$$

Here φ_t is the 1-parameter family of diffeomorphisms generated by $\nabla f/(1 - 2\rho t)$

- Singularity models of the Ricci flow:

  Shrinking Ricci solitons arise as Type I singularity models, while *steady* solitons as Type II singularity models.

- Li-Yau-Hamilton inequality becomes equality:

  Let X be the soliton vector field, and define

$$M_{ab} = \Delta R_{ab} - \frac{1}{2}\nabla_a\nabla_b R + 2R_{acbd}R_{cd} - R_{ac}R_{bc} - \rho R_{ab}$$
$$P_{abc} = \nabla_a R_{bc} - \nabla_b R_{ac}.$$

  Then on Ricci solitons the matrix Li-Yau-Hamilton inequality becomes equality:

$$Z_{ab} =: M_{ab} + (P_{cab} + P_{cba})X_c + R_{acbd}X_cX_d = 0$$

- Critical points of certain geometric functionals (e.g., Perelman's λ-entropy and ν-entropy).

  Given any Riemannian manifold (M^n, g), consider

  $$\lambda(g_{ij}) = \inf\{\mathcal{F}(g_{ij}, u) : u \in C^\infty(M), \int_M e^{-u} dV = 1\},$$

  where

  $$\mathcal{F}(g_{ij}, u) = \int_M (R + |\nabla u|^2) e^{-u} dV.$$

  Then, *the critical points of λ are precisely steady gradient Ricci solitons*:

  $$R_{ij} + \nabla_i \nabla_j f = 0.$$

Similarly, if we consider the ν-entropy

$$\nu(g_{ij}) = \inf\{\mathcal{W}(g,u,\tau) : u \in C^\infty(M), \tau > 0, (4\pi\tau)^{-\frac{n}{2}} \int e^{-u} dV = 1\}.$$

associated to Perelman's $\mathcal{W}$-functional

$$\mathcal{W}(g_{ij},u,\tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M [\tau(R + |\nabla u|^2) + u - n] e^{-u} dV,$$

then the critical points of ν-entropy are precisely gradient shrinking solitons:

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2\tau} g_{ij}.$$

Main Problems:

To classify, or to understand the geometry/topology of, gradient Ricci solitons.

Basic Examples of gradient Shrinking Ricci Solitions

- $\mathbb{S}^n/\Gamma$, *more generally any positive Einstein manifold* N^n*.*

- *Gaussian shrinking solitons* $(\mathbb{R}^n, \delta_{ij}, f(x) = |x|^2/4)$*:*

$$R_{ij} = 0, \quad \text{but} \quad f_{ij} = \frac{1}{2}\delta_{ij}.$$

- *Round cylinders* $N^k \times \mathbb{R}^{n-k}$ $(k \geq 2)$, where N^k is positive Einstein.

Classification of 3-d Shrinking Ricci Solitons

By the works of Perelman (2003), Ni-Wallach (2008) and Cao-Chen-Zhu (2008), we have the following complete classification result in 3-D:

Theorem *Let* (M^3, g_{ij}, f) *be a 3-d gradient shrinking Ricci soliton. Then,* (M^3, g_{ij}, f) *is a quotient of either*

(i) $\mathbb{S}^3/\Gamma$, or

(ii) $\mathbb{R}^3$, or

(iii) $\mathbb{S}^2 \times \mathbb{R}$.

In particular, there is **no 3-D noncompact** *shrinking Ricci solitons with* $Rm > 0$

Remark:

- T. Ivey proved that any 3-d *compact* gradient shrinking Ricci soliton is $\mathbb{S}^3/\Gamma$.

- Perelman (2003) first proved the theorem under the additional assumption that (M^3, g_{ij}, f) has bounded and non-negative sectional curvature $0 \leq Rm \leq C$ by studying the geometry of the level sets of the potential function f.

- Ni-Wallach (2008) allowed $Rc \geq 0$ and

$$|Rm|(x) \leq Ce^{ar(x)}$$

  for some constants $C > 0$ and $a > 0$.

- Finally, Cao-Chen-Zhu (2008) were able to remove all the assumptions on the curvatures based on a key result of B.-L. Chen. In fact, $Rm \geq 0$ automatically and the scalar curvature R has at most quadratic growth.

Non-trivial 4-D Gradient Shrinking Ricci Solitions

- *Koiso-Cao soliton on* $\mathbb{C}P^2\#(-\mathbb{C}P^2)$

  In early 90's Koiso, and independently by myself, constructed a gradient shrinking Kähler-Ricci soliton metric on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$. It has $U(2)$ symmetry and positive Ricci curvature.

- *Wang-Zhu soliton on* $\mathbb{C}P^2\#2(-\mathbb{C}P^2)$

  In 2004, X. Wang and X.H. Zhu found a gradient Kähler-Ricci soliton on $\mathbb{C}P^2\#2(-\mathbb{C}P^2)$ which has $U(1)\times U(1)$ symmetry.

Remark: incidentally, D. Page constructed a (hermitian, non-Kähler) Einstein metric on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$ in 1979, and Chen-LeBrun-Webber on $\mathbb{C}P^2\#2(-\mathbb{C}P^2)$ in 2008.

- *Noncompact Feldman-Ilmanen-Knopf soliton on* $\hat{\mathbb{C}}^2$

  In 2003, Feldman-Ilmanen-Knopf found the first complete noncompact $U(2)$-invariant shrinking Kähler-Ricci soliton on the tautological line bundle $\mathcal{O}(-1)$ of $\mathbb{C}P^1$, or $\hat{\mathbb{C}}^2$, the blow-up of $\mathbb{C}^2$ at the origin.

These are the *only* known non-Einstein and non-product examples of 4-d (gradient) shrinking Ricci solitons so far.

(M^n, g_{ij}, f): complete gradient shrinking Ricci soliton.

A. Topological restrictions:

- Wylie (2008): $\pi_1(M) < \infty$.
- Fang-Man-Zhang (2008): M^n has finite topological type.

B. Asymptotic behavior of the potential function f:

- Cao-Zhou (2010): Asymptotic behavior like the Gaussian soliton,

  $$\frac{1}{4}(r(x) - C_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + C_2)^2,$$

  where $r(x) = d(x_0, x)$ is the distance function from $x_0 \in M$.

  Renmark:This was first proved by Perelman under the additional assumption of bounded Ricci curvature.

C. Volume growth of geodesic balls:

- Cao-Zhou (2010): the Bishop type volume upper estimate,

  $$\mathrm{Vol}(B_{x_0}(r)) \leq C_3 r^n.$$

- Munteanu-J. Wang (2011): Calabi-Yau type volume lower estimate,

  $$\mathrm{Vol}(B_{x_0}(r)) \geq C_4 r.$$

Remark: Cao-Zhu (2008) showed that $\mathrm{Vol}(M^n, g_{ij}) = \infty$.

Open Problem: What can one say about the asymptotoc curvature behavior? Is it true that $|Rm| \leq C$? Does the scalar curvature R decay in some average sense?

Theorem (Munteanu-M. Wang, 2010) Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton with bounded Ricic curvature $|Rc| \leq C$, then there exist constants $C' > 0$ and $a > 0$ such that

$$|Rm|(x) \leq C' r(x)^a.$$

Classifications under various curvature assumptions

I. 4-D with $0 \le Rm \le C$

A. Naber (2009) *Any complete shrinking Ricci soliton* (M^4, g_{ij}, f) *with* $0 \le Rm \le C$ *is a finite quotient of either*

(a) $\mathbb{S}^4$, or

(b) $\mathbb{R}^4$, or

(c) $\mathbb{S}^3 \times \mathbb{R}$, or

(d) $\mathbb{S}^2 \times \mathbb{R}^2$.

Open Problem: What about for $n \ge 5$?

II. $n \geq 4$ **and Rotationally Symmetric**

B. Kotschwar (2008) *For* $n \geq 4$*, rotationally symmetric complete gradient shrinking Ricci solitons defined on (a)* $\mathbb{S}^n$*, or (b)* $\mathbb{R}^n$*, or (c)* $\mathbb{S}^{n-1} \times \mathbb{R}$ *are the* **standard ones**.

Here by rotationally symmetric, we mean that g is of the form

$$g = dr^2 + \varphi^2(r)\, \bar{g}_{\mathbb{S}^{n-1}}$$

III. $n \geq 4$ **and locally conformally flat** $(W_{ijkl}$**=0)**.

From the works of Ni-Wallach (2008) and Z.-H. Zhang (2009), we have

Theorem (Z.-H. Zhang): *Any complete gradient shrinking Ricci soliton* (M^n, g_{ij}, f) $(n \geq 4)$ *with Weyl tensor* $W = 0$ *is a* **finite quotient** *of either*

(a) $\mathbb{S}^n$, or

(b) $\mathbb{R}^n$, or

(c) $\mathbb{S}^{n-1} \times \mathbb{R}$.

Remark

- Ni-Wallach (2008) assumed $Rc \geq 0$ and

  $$|Rm|(x) \leq Ce^{ar(x)}$$

  for some constants $C > 0$ and $a > 0$.

- Z.-H. Zhang (2009) was able to remove all the assumptions on the curvatures by showing that $Rm \geq 0$ automatically, and hence the scalar curvature R has at most quadratic growth.

- Eminenti-LaNave-Mantegazza proved the theorem for compact case; there are alternative proofs, e.g., by the works of Petersen-Wylie & Munteanu-Sesem.

IV. $n \geq 4$ **and harmonic Weyl (Cotton tensor** $C_{ijk} = 0$**).**

Theorem (Fernández-López & García-Río, and Munteanu & Sesum, 2010) *If* (M^n, g_{ij}, f) *(*$n \geq 4$*) is a complete gradient shrinking Ricci soliton with harmonic Weyl tensor (i.e., Cotten tensor* $C_{ijk} = 0$*), then it is of the form* $N^k \times \mathbb{R}^{n-k}$*, where* N *is Einstein.*

Remark:

- For $n \geq 3$, the Cotten tensor is
$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R).$$

- Morover, for $n \geq 4$ the Cotton tensor is, up to a factor, the divergence of the Weyl tensor:
$$C_{ijk} = -\frac{n-2}{n-3}\nabla_l W_{ijkl}.$$

- Clearly, for $n \geq 4$, $W = 0$ or Einstein implies $C_{ijk} = 0$.

- It is well-known that for any 3-manifold (M, g), $C_{ijk} = 0$ if and only if M^3 is locally conformally flat.

V. $n = 4$ and Half-conformally Flat ($W^+ = 0$ or $W^- = 0$)

Recall that for any 4-manifold M^4,

$$\Lambda^2(M) = \Lambda^2_+(M) + \Lambda^2_-(M), \quad \text{hence} \quad W = W^+ + W^- .$$

$W^+ = 0$ or $W^- = 0$: *half-conformally flat.*

Theorem (X.X. Chen and Y. Wang, Feb. 2011) *Any complete 4-d half-conformally flat gradient shrinking Ricci soliton (M^4, g_{ij}, f) $(n \geq 4)$ is either*

(a) Einstein, or

(b) locally conformally flat, hence a finite quotient of $\mathbb{R}^4$ or $\mathbb{S}^3 \times \mathbb{R}$.

VI. $n = 4$ and Bach-flat

The Bach tensor ($n = 4$):

$$B_{ij} = \nabla^k \nabla^l W_{ikjl} + \frac{1}{2} R_{kl} W_i{}^k{}_j{}^l.$$

Basic Facts:

(a) Half-conformally flat implies $B_{ij} = 0$;

(b) Local conformal to Einstein implies $B_{ij} = 0$;

(c) $B_{ij} = 0$ is the critical point of the conformally invariant functional

$$\mathcal{W}(g) = \int_M |W_g|^2 dV_g.$$

(d) B_{ij} is symmetric, traceless and divergence-free (such a symmetric 2-tensor is called a TT tensor in general relativity).

4-D classification of Bach-flat shrinking solitons

Theorem (H.D. Cao and Qiang Chen, May, 2011) Any complete 4-dimensional Bach-flat gradient shrinking Ricci soliton (M^4, g_{ij}, f) $(n \geq 4)$ is either

(a) Einstein, or

(b) locally conformally flat, hence a finite quotient of $\mathbb{R}^4$ or $\mathbb{S}^3 \times \mathbb{R}$.

Remark: A theorem of Hitchin (1970s) states that a compact 4-d half-conformally flat Einstein manifold (of positive scalar curvature) is $\mathbb{S}^4$ or $\mathbb{C}P^2$. Thus, one has

Corollary *If (M^4, g_{ij}, f) is a compact half-conformally flat gradient shrinking Ricci soliton, then (M^4, g_{ij}) is isometric to the standard $\mathbb{S}^4$ or $\mathbb{C}P^2$.*

Key Steps of the proof:

- Explore a crucial new covariant 3-tensor D_{ijk} we introduced in 2009;

- Show that Bach-flatness implies $D_{ijk} = 0$.

  1. We can link the B_{ij} with D_{ijk} and C_{ijk}:

  $$B_{ij} = -\frac{1}{n-2}(\nabla_k D_{ikj} + \frac{n-3}{n-2} C_{jli} \nabla_l f)$$

  2. Show that $B_{ij} = 0$ implies that $D_{ijk} = 0$ by a suitable integration by parts argument. For example, assuming M is compact, we can show

  $$\int_M B_{ij} \nabla_i f \nabla_j f e^{-f} dV = -\frac{1}{2} \int_M |D_{ikj}|^2 e^{-f} dV.$$

- Show that $D_{ijk} = 0$ implies $W = 0$ for $n = 4$.

The New Covariant 3-tensor D_{ijk}

For any gradient Ricci soliton with $R_{ij} + \nabla_i\nabla_j f = \rho g_{ij}$, we introduced a covariant 3-tensor:

$$D_{ijk} = \frac{1}{n-2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) + \frac{1}{2(n-1)(n-2)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) + \frac{R}{(n-1)(n-2)}(g_{ik}\nabla_j f - g_{jk}\nabla_i f).$$

An equivalent expression:

$$D_{ijk} = \frac{1}{n-2}(A_{jk}\nabla_i f - A_{ik}\nabla_j f) + \frac{1}{(n-1)(n-2)}(g_{jk}E_{il} - g_{ik}E_{jl})\nabla_l f,$$

$A_{ij} = R_{ij} - \frac{R}{2(n-1)}g_{ij}$ (the Schouten tensor) and

$E_{ij} = R_{ij} - \frac{R}{2}g_{ij}$ (the Einstein tensor).

Basic Properties of D_{ijk}:

- D_{ijk} is closely tied to the Cotton tensro and the Weyl tensor;

Proposition 1.

$$D_{ijk} = C_{ijk} + W_{ijkl}\nabla_l f,$$

- A key identity:

Proposition 2. *At any point $p \in M^n$ where $\nabla f(p) \neq 0$, we have*

$$|D_{ijk}|^2 = \frac{2|\nabla f|^4}{(n-2)^2}|h_{ab} - \frac{H}{n-1}g_{ab}|^2 + \frac{1}{2(n-1)(n-2)}|\nabla_a R|^2,$$

where h_{ab} and H are the second fundamental form and the mean curvature of the level surface $\Sigma = \{f = f(p)\}$.

Geometry of the Level Surfaces when $D_{ijk} = 0$

Proposition 3. *Let (M^n, g, f) $(n \geq 3)$ be any complete gradient Ricci soliton with $D_{ijk} = 0$, and let c be a regular value of f and $\Sigma_c = \{f = c\}$ be the level surface of f. Set $e_1 = \nabla f/|\nabla f|$ and pick any orthonormal frame $e_2, \cdots, e_n$ tangent to the level surface Σ_c, Then:*

(a) $|\nabla f|^2$ and the scalar curvature R of (M^n, g_{ij}, f) are constant on Σ_c;

(b) $R_{1a} = 0$ for any $a \geq 2$, hence $e_1 = \nabla f/|\nabla f|$ is an eigenvector of Rc;

(c) the second fundamental form h_{ab} of Σ_c is of the form $h_{ab} = \frac{H}{n-1} g_{ab}$;

(d) the mean curvature H is constant on Σ_c;

(e) on Σ_c, the Ricci tensor of (M^n, g_{ij}, f) either has a unique eigenvalue λ, or has two distinct eigenvalues λ and μ of multiplicity 1 and $n-1$ respectively. In either case, $e_1 = \nabla f/|\nabla f|$ is an eigenvector of λ

The Bach tensor in other dimensions $n \neq 4$

$$B_{ij} = \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + \frac{1}{n-2}R_{kl}W_i{}^k{}_j{}^l.$$

In terms of the Cotton tensor C_{ijk} we also have

$$B_{ij} = \frac{1}{n-2}(\nabla_k C_{kij} + R_{kl}W_i{}^k{}_j{}^l).$$

In particular, even for $n = 3$, we can define the Bach tensor as

$$B_{ij} = \nabla_k C_{kij}\,.$$

Fact: For $n \geq 5$, $W = 0$ or Einstein imply Bach-flat.

Classification of Bach-flat shrinking solitons ($n \geq 5$)

Theorem (Cao & Q. Chen, May, 2011) *Any complete Bach-flat gradient shrinking Ricci soliton* (M^n, g_{ij}, f) *($n \geq 5$) is either*

(a) Einstein, or

(b) a finite quotient of either $\mathbb{R}^n$ *or* $\mathbb{N}^{n-1} \times \mathbb{R}$*, where* $\mathbb{N}^{n-1}$ *is Einstein.*

Known Examples of 4-D Einstein Metrics of $R > 0$

- $\mathbb{S}^4$;
- $\mathbb{C}P^2$;
- $\mathbb{S}^2 \times \mathbb{S}^2$;
- Page metric (Einstein and hermitian) on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$;
- Einstein and hermitian metric on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$ (X. Chen, le Brun, and Weber);
- Kähler-Einsten metric on $\mathbb{C}P^2\#(-k\mathbb{C}P^2)$ $(3 \leq k \leq 8)$.

Remark: To include known compact 4-D shrinkers:

- Koiso-Cao soliton (Kähler, non-Einstein) on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$;
- Wang-Zhu soliton on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$.

Obstruction to the Existence of 4-D Einstein Metrics

Hitchin-Thorpe Inequality: For any compact Einstein 4-manifold (M^4, g_{ij}), we have

$$2\chi(M) \geq 3|\tau(M)|,$$

where $\chi(M)$ is the Euler characteristic and $\tau(M)$ is the signature.

Question: Does the above Hitchin-Thorpe inequality hold for 4-D compact shrinkers?

Stable Shrinking Ricci Solitons

Perelman introduced the $\mathcal{W}$-functional

$$\mathcal{W}(g_{ij}, u, \tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M [\tau(R + |\nabla u|^2) + f - n]e^{-u} dV,$$

The associated ν-entropy is defined by

$$\nu(g_{ij}) = \inf\{\mathcal{W}(g, u, \tau) : u \in C^\infty(M), \tau > 0, (4\pi\tau)^{-\frac{n}{2}} \int e^{-u} dV = 1\}.$$

It turns out that

$$\nu(g_{ij}) = \mathcal{W}(g, f, \tau)$$

with

$$\tau(-2\Delta f + |Df|^2 - R) - f + n + \nu = 0,$$

and

$$(4\pi\tau)^{-\frac{n}{2}} \int f e^{-f} = \frac{n}{2} + \nu.$$

First variation of ν-entropy

For any symmetric 2-tensor $h = h_{ij}$, consider variations $g_{ij}(s) = g_{ij} + sh_{ij}$. Then, the first variation $\delta_g \nu(h)$ of the ν-entropy is given by

$$\begin{aligned}\frac{d}{ds}\nu(g_{ij}(s)) =&(4\pi\tau)^{-\frac{n}{2}} \int -\tau < h, Rc + \nabla^2 f - \frac{1}{2\tau} g > e^{-f} dV \\ =&(4\pi\tau)^{-\frac{n}{2}} \int -\tau h_{ij}(R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}) e^{-f} dV.\end{aligned}$$

Thus, critical points are precisely gradient shrinking solitons:

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2\tau} g_{ij}.$$

Second variation of ν-entropy

Theorem **(Cao-Hamilton-Ilmanen, 2004, Cao and M. Zhu, 2010)** For any symmetric 2-tensor $h = h_{ij}$, consider variations $g_{ij}(s) = g_{ij} + sh_{ij}$. Then the second variation $\delta_g^2 \nu(h,h)$ is given by

$$\frac{d^2}{ds^2}\bigg|_{s=0} \nu(g(s)) = \frac{\tau}{(4\pi\tau)^{n/2}} \int_M < Nh, h > e^{-f} dV,$$

where the stability operator N is given by

$$Nh := \frac{1}{2}\Delta_f h + Rm(h,\cdot) + \operatorname{div}_f^\dagger \operatorname{div}_f h + \frac{1}{2}\nabla^2 v_h - Rc\, \frac{\int_M < Rc, h > e^{-f}}{\int_M Re^{-f}},$$

and v_h is the unique solution of

$$\Delta_f v_h + \frac{v_h}{2\tau} = \operatorname{div}_f \operatorname{div}_f h, \qquad \int_M v_h e^{-f} = 0.$$

$$\Delta_f h := \Delta h - \nabla_p f \cdot \nabla_p h,$$

$$Rm(h, \cdot) := R_{ijkl} h_{jl},$$

$$\operatorname{div} \omega := \nabla_i \omega_i, \qquad (\operatorname{div} h)_i := \nabla_j h_{ji}.$$

$$\operatorname{div}_f h := e^f \operatorname{div}(e^{-f} h) = \operatorname{div} h - h \nabla f,$$

i.e.,

$$(\operatorname{div}_f \ h)_i = \nabla_j h_{ij} - h_{ij} \nabla_j f.$$

We also define $\operatorname{div}_f^\dagger$ on 1-forms (and similarly on functions) by

$$(\operatorname{div}_f^\dagger \omega)_{ij} = -(\nabla_i \omega_j + \nabla_j \omega_i)/2 = -(1/2) L_{\omega^\#} g_{ij}$$

so that

$$\int_M e^{-f} < \operatorname{div}_f^\dagger \omega, h > dV = \int_M e^{-f} < \omega, \operatorname{div}_f h > dV.$$

Here $\omega^\#$ is the vector field dual to ω. Clearly, $\operatorname{div}_f^\dagger$ is just the adjoint of div with respect to the weighted L^2-inner product

$$(\cdot, \cdot)_f = \int_M < \cdot, \cdot > e^{-f} dV.$$

Stable/Unstable Shrinking Ricci Solitons

$$\frac{d^2}{ds^2}\bigg|_{s=0} \nu(g(s)) = \frac{\tau}{(4\pi\tau)^{n/2}} \int_M < Nh, h > e^{-f} dV \leq 0.$$

- S^n is stable. In fact, it is *geometrically stable* (i.e. nearby metrics are attracted to it up to scale and gauge);

  The following facts were observed by Cao-Hamilton-Ilmanen (2004):

- Complex projective space CP^m is stable (based on the work of Goldschmidt in 2004);
- Any product of two Einstein manifolds $M = M_1^{n_1} \times M_2^{n_2}$ is unstable. The destabilizing direction $h = g_1/n_1 - g_2/n_2$ corresponds to a growing discrepancy in the size of the factors;

- Any compact Kähler-Einstein manifold X^n of positive scalar curvature with $\dim H^{1,1}(X) \geq 2$ is unstable. (Indeed, if σ is a harmonic 2-form perpendicular to the Kähler form, then the corresponding metric perturbation h is a destabilizing direction);

Remark: Hall-Murphy (2010) showed that any compact Kähler-Ricci soliton X^n with $\dim H^{1,1}(X) \geq 2$ is unstable.

Shrinking Solitons	**Type**	**Stability**
S^4	Einstein	Stable
CP^2	Einstein	Stable
$S^2 \times S^2$	Einstein product	Unstable
$CP^2\#(-CP^2)$	Kähler-Ricci soltion	Unstable
$CP^2\#(-CP^2)$	Einstein	Unstable
$CP^2\#2{-}CP^2$	Kähler-Ricci soltion	Unstable
$CP^2\#2{-}CP^2$	Einstein	Unstable?
$CP^2\#3(-CP^2)$	Kähler-Einstein	Unstable
$CP^2\#4(-CP^2)$	Kähler-Einstein	Unstable
$CP^2\#5(-CP^2)$	Kähler-Einstein	Unstable
$CP^2\#6(-CP^2)$	Kähler-Einstein	Unstable
$CP^2\#7(-CP^2)$	Kähler-Einstein	Unstable
$CP^2\#8(-CP^2)$	Kähler-Einstein	Unstable

Main Conjectures

Conjecture 1: 4-D compact stable gradient shrinking solitons must be Einstein.

Conjecture 2: 4-D compact positive Einstein manifolds are either $\mathbb{S}^4$ or $\mathbb{C}P^2$.