Mean curvature flow of submanifolds in higher codimensions

Entao Zhao *

Let $F_0 : M^n \to N^{n+d}$ be a smooth immersion from an n-dimensional Riemannian manifold without boundary to an $(n+d)$-dimensional Riemannian manifold. Consider a one-parameter family of smooth immersions $F : M \times [0,T) \to N$ satisfying

$$\begin{cases} \frac{\partial}{\partial t} F(x,t) = H(x,t), \\ F(x,0) = F_0(x), \end{cases} \tag{1}$$

where $H(x,t)$ is the mean curvature vector of $F_t(M)$ and $F_t(x) = F(x,t)$. We call $F : M \times [0,T) \to N$ the mean curvature flow with initial value $F_0 : M \to N$. The mean curvature flow is a (degenerate) quasilinear parabolic evolution equation. One can prove the short-time existence by using the Nash-Moser implicit function theorem as did in [5]. One can also used the so called De Turck trick to modify the mean curvature flow equation to a strongly parabolic equation, and the short-time existence follows from the standard parabolic theory.

The mean curvature flow was proposed by Mullins [13] to describe the formation of grain boundaries in annealing metals. In [4], Brakke introduced the motion of a submanifold by its mean curvature in arbitrary codimension and constructed a generalized varifold solution for all time. For the classical solution of the mean curvature flow, many works have been done on hypersurfaces. Huisken [6] showed that if the initial hypersurface in the Euclidean space is compact and uniformly convex, then the mean curvature flow converges to a round point in a finite time. Later, he generalized this convergence theorem to the mean curvature flow of hypersurfaces in a Riemannian manifold in [7]. He also studied in [8] the mean curvature flow of hypersurfaces satisfying a pinching condition in a sphere. For the mean curvature flow of submanifolds with higher codimensions, many previous results were obtained for submanifolds with low dimension or admitting some special structures, see [16, 17, 18, 19, 20] etc. for example. Recently, Andrews and Baker [1] proved a convergence theorem for the mean curvature flow of closed submanifolds satisfying a pinching condition in the Euclidean space. In [2], Baker obtained a convergence result for the mean curvature flow of submanifolds in a sphere, which generalizes the result in [8] to higher codimension case.

For an n-dimensional submanifold M in a Riemannian manifold, we denote by g the induced metric on M. Let A and H be the second fundamental form and the mean curvature vector of M, respectively. The tracefree second fundamental form $\mathring{A}$ is defined by $\mathring{A} = A - \frac{1}{n} g \otimes H$. Denote by $||\cdot||_p$ the L^p-norm of a function or a tensor field. Recently, we proved several convergence theorems for the mean curvature flow of higher codimension. Firstly, motivated by the topological sphere theorems for closed submanifolds with pinched L^n-norm of the tracefree second fundamental form in a space form obtained by Shiohama and Xu [14], we obtained the following convergence theorems under integral curvature pinching conditions.

Theorem 1 ([9])**.** *Let $F : M^n \to \mathbb{R}^{n+d}$ $(n \geq 3)$ be a smooth closed submanifold in an Euclidean space. For any fixed $1 < p < \infty$, there is a positive constant C_1 depending on n, p, the upper bounds V and Λ on the volume and the L^{n+2}-norm of the second fundamental form of the submanifold, such that if $||\mathring{A}||_p < C_1$, then the mean curvature flow with F as initial value has a unique solution $F : M \times [0,T) \to \mathbb{R}^{n+d}$ in a finite maximal time interval, and M_t converges uniformly to a point $x \in \mathbb{R}^{n+d}$ as $t \to T$. The rescaled maps $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ converge in C^∞-topology to a limiting embedding $\widetilde{F}_T$ such that $\widetilde{F}_T(M)$ is the unit n-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+d}$.*

Theorem 2 ([9])**.** *Let $F : M^n \to \mathbb{R}^{n+d}$ $(n \geq 3)$ be a smooth closed submanifold in an Euclidean space. For any fixed $n < p < \infty$, there is a positive constant C_2 depending on n, p, the upper bounds V and Λ on the volume and the L^{n+2}-norm of the mean curvature of the submanifold, such that if $||\mathring{A}||_p < C_2$, then the mean curvature flow with F as initial value has a unique solution $F : M \times [0,T) \to \mathbb{R}^{n+d}$ in a finite maximal time interval, and M_t converges*

*Center of Mathematical Sciences, Zhejiang University, Hangzhou, 310027, People's Republic of China, zhaoet@cms.zju.edu.cn

uniformly to a point $x \in \mathbb{R}^{n+d}$ *as* $t \to T$. *The rescaled maps* $\widetilde{F}_t = \frac{F_t - x}{\sqrt{2n(T-t)}}$ *converge in* C^∞*-topology to a limiting embedding* $\widetilde{F}_T$ *such that* $\widetilde{F}_T(M)$ *is the unit n-sphere in some* $(n+1)$*-dimensional subspace of* $\mathbb{R}^{n+d}$.

If the ambient space is the unit sphere, we proved the following convergence theorem for the mean curvature flow of submanifolds with pinched $L^p(p > n)$-norm of the second fundamental form.

Theorem 3 ([11])**.** *Let* $F : M^n \to \mathbb{S}^{n+d}$ $(n \geq 3)$ *be a smooth closed submanifold in a unit sphere. For any fixed* $n < p < \infty$*, there is a positive constant* C_3 *depending on n and p, such that if* $||A||_p < C_3$*, then the mean curvature flow with F as initial value has a unique solution* $F : M \times [0,T) \to \mathbb{S}^{n+d}$*, and either* $T < \infty$ *and* M_t *converges to a round point as* $t \to T$*, or* $T = \infty$ *and* M_t *converges to a totally geodesic sphere in* $\mathbb{S}^{n+d}$ *as* $t \to \infty$.

Secondly, motivated by the convergence theorems in [1, 4], we get the following convergence theorem for the mean curvature flow in a hyperbolic space, i.e., a complete simply connected space form with negatively constant sectional curvature.

Theorem 4 ([10])**.** *Let* $F : M^n \to \mathbb{F}^{n+d}(c)$ *be a smooth closed submanifold in a hyperbolic space with constant curvature* $c < 0$. *Assume F satisfies*

$$|A|^2 \leq \begin{cases} \frac{4}{3n}|H|^2 + \frac{n}{2}c, & n = 2,3, \\ \frac{1}{n-1}|H|^2 + 2c, & n \geq 4. \end{cases} \tag{2}$$

Then the mean curvature flow with F as initial value has a unique solution on a finite maximal time interval $[0,T)$*, and* M_t *converges to a round point as* $t \to T$.

Finally, we obtained a convergence theorem for the mean curvature flow in a Riemannian manifold, which may be considered as an extension of Huisken's theorem in [8].

Theorem 5 ([12])**.** *Let* $F : M^n \to N^{n+d}$ *be an n-dimensional smooth closed submanifold in an* $(n+d)$*-dimensional smooth complete Riemannian manifold. Suppose the sectional curvature* K_N*, the first covariant derivative* $\bar{\nabla}\bar{R}$ *of the Riemannian curvature tensor, and the injectivity radius* $\mathrm{inj}(N)$ *of the ambient space N satisfy* $-K_1 \leq K_N \leq K_2$, $|\bar{\nabla}\bar{R}| \leq L$, $\mathrm{inj}(N) \geq i_N$ *for nonnegative constants* K_1, K_2, *L and positive constant* i_N. *There is an explicitly given nonnegative constant* b_0 *depending on n, d,* K_1, K_2 *and L such that if F satisfies*

$$|A|^2 < \begin{cases} \frac{4}{3n}|H|^2 - b_0, & n = 2,3, \\ \frac{1}{n-1}|H|^2 - b_0, & n \geq 4, \end{cases} \tag{3}$$

then the mean curvature flow with F as initial value has a unique solution on a finite maximal time interval $[0,T)$*, and* M_t *converges to a round point as* $t \to T$.

Remark 6. The constant $b_0 = 0$ when $N = \mathbb{R}^{n+d}$. From Proposition 7 of [1] we see that, under their initial curvature pinching condition, (3) is satisfied after a short time interval. Hence our Theorem 5 may also be considered as a generalization of the convergence theorem in [1].

From convergence theorems we obtain several differentiable sphere theorem for submanifolds. We note that some similar differentiable sphere theorems for submanifolds have been proved in [3, 21, 22] by using different methods.

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