ESTIMATES OF HEAT KERNELS FOR NON-LOCAL REGULAR DIRICHLET FORMS

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We are concerned with heat kernel estimates for a class of non-local regular Dirichlet forms. Let (M, d, μ) be a metric measure space, and let $(\mathcal{E}, \mathcal{F})$ be a regular, conservative *Dirichlet form* in $L^2(M, \mu)$. Let $P_t = e^{t\mathcal{L}}$ be the heat semigroup. The purpose of this paper is to give various equivalent conditions for upper bounds of the heat kernels of non-local type.

Let α, β be fixed positive numbers, and let C denote positive constant that can be different at different occurrences. Consider the following conditions that may be true or not:

$(\mathbf{V}_{\leq})$: *Upper α-regularity*: For all $x \in M$ and all $r > 0$,

$$V(x, r) \leq Cr^{\alpha}.$$

$(\mathbf{DUE})$: *On-diagonal upper estimate*: The heat kernel p_t exists and satisfies the on-diagonal upper estimate

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}},$$

for all $t > 0$ and μ-almost all $x, y \in M$.

$(\mathbf{UE})$: *Upper estimate of non-local type*: The heat kernel p_t exists and satisfies the off-diagonal upper estimate

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}$$

for all $t > 0$ and μ-almost all $x, y \in M$.

Recall that by a theorem of Beurling and Deny, any regular conservative Dirichlet form admits a decomposition

$$\mathcal{E}(u, v) = \mathcal{E}^{(L)}(u, v) + \mathcal{E}^{(J)}(u, v) \tag{0.1}$$

where $\mathcal{E}^{(L)}$ is a *local part* and

$$\mathcal{E}^{(J)}(u, v) = \int_{M \times M \setminus \mathrm{diag}} (u(x) - u(y))(v(x) - v(y))\, dj(x, y) \tag{0.2}$$

is a *jump part* with a jump measure j defined on $M \times M \setminus \mathrm{diag}$. In our setting the jump measure j will be assumed to have a density with

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respect to $\mu \times \mu$, which will be denoted by $J(x,y)$, and so the jump part $\mathcal{E}^{(J)}$ becomes

$$\mathcal{E}^{(J)}(u,v) = \int_{M\times M} (u(x)-u(y))(v(x)-v(y))\, J(x,y)d\mu(y)d\mu(x). \tag{0.3}$$

Let us further consider the following conditions:

($\mathbf{J}_{\le}$) : The jump density exists and admits the estimate

$$J(x,y) \le Cd(x,y)^{-(\alpha+\beta)},$$

for μ-almost all $x, y \in M$.

(**UE**Φ) : The heat kernel p_t exists and there exist $C, \alpha > 0, \beta > 0$ such that

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right) \tag{0.4}$$

for all $t > 0$ and μ-almost all $x, y \in M$, where $\Phi : [0,\infty) \to [0,\infty)$ is continuous, non-increasing function such that

$$\int_0^\infty s^{\alpha-1}\Phi(s)ds < \infty. \tag{0.5}$$

(**S**) : *Survival estimate.* There exist constants $\varepsilon, \delta \in (0,1)$ and $\beta > 0$ such that, for all balls $B = B(x_0, r)$ and for all $t^{1/\beta} \le \delta r$,

$$1 - P_t^B \mathbf{1}_B(x) \le \varepsilon \text{ for } \mu\text{-almost all } x \in \frac{1}{4}B, \tag{0.6}$$

where $\lambda B = B(x_0, \lambda r)$.

(**T**) : *Tail estimate.* There exist constants $\varepsilon, \delta \in (0,1)$ and $\beta > 0$ such that, for all balls $B = B(x_0, r)$ and for all $t^{1/\beta} \le \delta r$,

$$P_t \mathbf{1}_{B^c}(x) \le \varepsilon \text{ for } \mu\text{-almost all } x \in \frac{1}{4}B.$$

($\mathbf{T}_{strong}$) : *Strong tail estimate.* There exist constants $c > 0$ and $\beta > 0$ such that, for all balls $B = B(x_0, r)$ and for all $t > 0$,

$$P_t \mathbf{1}_{B^c}(x) \le \frac{ct}{r^\beta} \text{ for } \mu\text{-almost all } x \in \frac{1}{4}B.$$

Here are our main results.

Theorem 0.1. *Let (M, d, μ) be a metric measure space with precompact balls, and let $(\mathcal{E}, \mathcal{F})$ be a regular conservative Dirichlet form in $L^2(M,\mu)$ with jump density J. Then the following implication holds:*

$$(V_{\le}) + (DUE) + (J_{\le}) + (S) \Rightarrow (UE). \tag{0.7}$$

Theorem 0.2. *Let (M, d, μ) be a metric measure space with precompact balls, and let $(\mathcal{E}, \mathcal{F})$ be a regular conservative Dirichlet form in*

$L^2(M,\mu)$ with jump density J. If $(V_{\leq})$ holds, then the following equivalences are true:

$$\begin{aligned}(UE) &\Leftrightarrow (UE\Phi) + (J_{\leq}) \\ &\Leftrightarrow (DUE) + (J_{\leq}) + (T) \\ &\Leftrightarrow (DUE) + (J_{\leq}) + (S) \\ &\Leftrightarrow (DUE) + (J_{\leq}) + (T_{strong}). \qquad (0.8)\end{aligned}$$

Remark 0.3. *The upper estimate (UE) is best possible for non-local forms in the following sense: if the heat kernel p_t satisfies the estimate*

$$p_t(x,y) \leq \frac{1}{t^{\alpha/\beta}}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right)$$

for all $t > 0$ and μ-almost all $x, y \in M$, where Φ is a continuous decreasing function on $[0, +\infty)$ then necessarily

$$\Phi(s) \geq c(1+s)^{-(\alpha+\beta)}$$

for some $c > 0$ (see [2, Lemma 3.1]*).*

References

[1] A. Grigor'yan, J. Hu and K.-S. Lau, Estimates of heat kernels for non-local regular Dirichlet forms, preprint.

[2] A. Grigor'yan and T. Kumagai, On the dichotomy in the heat kernel two sided estimates. In: Analysis on Graphs and its Applications (P. Exner et al. (eds.)), Proc. of Symposia in Pure Math. **77**, pp. 199–210, Amer. Math. Soc. 2008.

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