GLUEING QUANTUM D MODULS OVER FLOPS

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ABSTRACT. In this note I summarize my talk at Sanya on December 19, 2011. The main theme is a glueing theorem of two quantum D modules $QH(X)$ and $QH(X')$ via analytic continuations over the Kähler moduli, where X and X' are local projective models related by an ordinary flop. This is a joint work with Yuan-Pin Lee and Hui-Wen Lin.

1. BF/GMT FOR TORIC BUNDLES

A general framework to determine $g = 0$ GW invariants is to go from certain localization data I to the generating function of one descendent: Let $\tau = \sum_\mu \tau^\mu T_\mu \in H(X)$, $g_{\mu\nu} = (T_\mu, T_\nu)$, $T^\mu = \sum g^{\mu\nu} T_\nu$.

$$J^X(\tau, z^{-1}) = 1 + \frac{\tau}{z} + \sum_{\beta \in NE(X), n, \mu} \frac{q^\beta}{n!} T_\mu \left\langle \frac{T^\mu}{z(z-\psi)}, \tau, \cdots, \tau \right\rangle_{0,n+1,\beta}.$$

Witten's *dilaton, string, and topological recursion relation* in 2D gravity had been reformulated by Givental into a symplectic space theory. Let $H := H(X)$, $\mathcal{H} := H[z, z^{-1}]\!]$, $\mathcal{H}_+ := H[z]$ and $\mathcal{H}_- := z^{-1}H[\![z^{-1}]\!]$. $\mathcal{H} \cong T^*\mathcal{H}_+$. Let $F_0(\mathbf{t})$ be the generating function of all descendent invariants. The one form dF_0 gives a section of $\pi : \mathcal{H} \to \mathcal{H}_+$. Givental's *Lagrangian cone* $\mathcal{L}$ is the graph of dF_0. Let $R = \mathbb{C}[\widehat{NE(X)}]$. Denote $a = \sum q^\beta a_\beta(z) \in R\{z\}$ if $a_\beta(z) \in \mathbb{C}[z]$.

Lemma 1.1. *$z\nabla J = (z\partial_\mu J^\nu)$ forms a matrix whose column vectors $z\partial_\mu J(\tau)$ generates the tangent space L_τ of the Lagrangian cone $\mathcal{L}$ as an $R\{z\}$-module.*

By TRR, $z\nabla J$ is the fundamental solution matrix of the Dubrovin connection

$$\nabla^z = d - \frac{1}{z} d\tau^\mu \otimes \sum_\mu T_\mu *_\tau$$

on $TH = H \times H$. Namely we have the quantum differential equation (QDE)

$$z\partial_\mu z\partial_\nu J = \sum \tilde{C}^\kappa_{\mu\nu}(\tau, q) z\partial_\kappa J.$$

Let $\bar{p} : X \to S$ be a smooth toric bundle with fiber divisor $D = \sum t^i D_i$. $H(X)$ is a free over $H(S)$ with finite generators $\{D^e := \prod_i D_i^{e_i}\}_{e \in \Lambda}$. Let $\bar{t} := \sum_s \bar{t}^s \bar{T}_s \in H(S)$. $H(X)$ has basis $\{T_{\mathbf{e}} = T_{(s,e)} = \bar{T}_s D^e\}_{\mathbf{e} \in \Lambda^+}$. Denote by $\partial_{\bar{T}_s} \equiv \partial_{\bar{t}^s}$ the $\bar{T}_s$ directional derivative on $H(S)$, $\partial^{\mathbf{e}} = \partial^{(s,e)} := \partial_{\bar{t}^s} \prod_i \partial_{t^i}^{e_i}$ and the *naive quantization*

$$\hat{T}_e \equiv \partial^{z\mathbf{e}} \equiv \partial^{z(s,e)} := z\partial_{\bar{t}^s} \prod_i z\partial_{t^i}^{e_i} = z^{|e|+1} \partial^{(s,e)}.$$

The $T_{\mathbf{e}}$ directional derivative is $\partial_{\mathbf{e}} = \partial_{T_{\mathbf{e}}}$. $\partial^{z\mathbf{e}}$ and $z\partial_{\mathbf{e}}$ are closely related.

Let $\bar{p}: X \to S$ be a split toric bundle quotient from $\bigoplus \mathscr{L}_\rho \to S$. The hypergeometric modification of J^S by the $\bar{p}$-fibration takes the form

$$I^X(\bar{t}, D, z, z^{-1}) := \sum_{\beta \in NE(X)} q^\beta e^{\frac{D}{z} + (D.\beta)} I_\beta^{X/S}(z, z^{-1}) J_{\beta_S}^S(\bar{t}, z^{-1}),$$

where $I_\beta^{X/S} = \prod_{\rho \in \triangle_1} 1 / \prod_{m=1}^{(D_\rho + \mathscr{L}_\rho).\beta} (D_\rho + \mathscr{L}_\rho + mz)$ comes from fiber localization, and the product is *directed* when $(D_\rho + \mathscr{L}_\rho).\beta \le -1$.

In general positive z powers may occur in I^X. I is defined only on the subspace

$$\hat{t} := \bar{t} + D \in H(S) \oplus \bigoplus\nolimits_i \mathbb{C} D_i \subset H(X).$$

Theorem 1.2 (J. Brown 2009). *$(-z)I^X(\hat{t}, -z)$ lies in the Lagrangian cone $\mathcal{L}$.*

Definition 1.3 (GMT). For each $\hat{t}$, say $zI(\hat{t})$ lies in L_τ of $\mathcal{L}$. The correspondence

$$\hat{t} \mapsto \tau(\hat{t}) \in H(X) \otimes R$$

is called the *generalized mirror transformation*.

Proposition 1.4 (BF). (1) *The GMT: $\tau = \tau(\hat{t})$ satisfies $\tau(\hat{t}, q = 0) = \hat{t}$.*

(2) *Under the basis $\{T_{\mathbf{e}}\}_{\mathbf{e} \in \Lambda^+}$, there exists an* invertible *$N \times N$ matrix-valued formal series $B(\tau, z)$, the Birkhoff factorization, such that*

$$\left(\partial^{z\mathbf{e}} I(\hat{t}, z, z^{-1})\right) = \left(z \nabla J(\tau, z^{-1})\right) B(\tau, z),$$

where $(\partial^{z\mathbf{e}} I)$ is the $N \times N$ matrix with $\partial^{z\mathbf{e}} I$ as column vectors. The first column vectors are I and J respectively (string equation).

2. Quantum Leray-Hirsch with naturality

Consider the *local model* of a split P^r flop $f : X \dashrightarrow X'$ with data (S, F, F'), where

$$F = \bigoplus_{i=0}^r L_i \quad \text{and} \quad F' = \bigoplus_{i=0}^r L_i'.$$

The contraction $\psi : X \to \bar{X}$ has exceptional loci $\bar{\psi} : Z = P_S(F) \to S$ with $N = N_{Z/X} = \bar{\psi}^* F' \otimes \mathscr{O}_Z(-1)$. Similarly $Z' \subset X'$, N'. $\bar{p} : X = P_Z(N \oplus \mathscr{O}) \xrightarrow{p} Z \xrightarrow{\bar{\psi}} S$ is a *double projective bundle*. For h, ξ being the *relative hyperplane classes*,

$$H(X) = H(S)[h, \xi] / (f_F, f_{N \oplus \mathscr{O}}),$$

$$f_F = \prod_{i=0}^r a_i := \prod (h + L_i), \qquad f_{N \oplus \mathscr{O}} = b_{r+1} \prod_{i=0}^r b_i := \xi \prod (\xi - h + L_i').$$

The graph correspondence $\mathscr{F} = [\bar{\Gamma}_f] \in A(X \times X')$ induces an isomorphism $\mathscr{F} : H(X) \cong H(X')$ as groups: $\mathscr{F} \bar{t} h^i \xi^j = \bar{t} (\mathscr{F} h)^i (\mathscr{F} \xi)^j = \bar{t} (\xi' - h')^i \xi'^j$ if $i \le r$. $\mathscr{F}$ also preserves the Poincaré pairing, but not the ring structure.

Theorem 2.1 (LLW 2010). *$\mathscr{F}$ induces an isomorphism of quantum rings $QH(X) \cong QH(X')$ under analytic continuations in the Kähler moduli formally defined by*

$$\mathscr{F} q^\beta = q^{\mathscr{F}\beta}, \qquad \beta \in NE(X).$$

Let γ, ℓ be the fiber line class in $X \to Z \to S$. Then $\mathscr{F}\gamma = \gamma' + \ell'$, but $\mathscr{F}\ell = -\ell' \notin NE(X')$. So analytic continuations are necessary. Any $\beta \in A_1(X)$ is of the form $\beta = \beta_S + d\ell + d_2\gamma$ where $\beta_S \in A_1(S)$ is identified with its *canonical lift* in $A_1(Z)$ with $(\beta_S.h) = 0 = (\beta_S.\xi)$. h, ξ are dual to ℓ, γ hence $\beta.h = d$, $\beta.\xi = d_2$.

Lemma 2.2 (Minimal lift). *Given a primitive $\beta_S \in NE(S)$, $\beta \in NE(X)$ if and only if*

$$d \geq -\mu \quad \text{and} \quad d_2 \geq -\nu,$$

where $\mu = \max_i\{(\beta_S.L_i)\}$, $\mu' = \max_i\{(\beta_S.L_i')\}$, and $\nu = \max\{\mu + \mu', 0\}$.

For general β_S, the above numerical condition defines $NE^I(X)$. The minimal one β_S^I is called the I-minimal lift. Back to $\bar{p} : X \to S$ where $D = t^1 h + t^2 \xi$ and $\bar{t} \in H(S)$. $I_\beta^{X/S} = I_\beta^{Z/S} I_\beta^{X/Z}$ is given by

$$\prod_{i=0}^{r} \frac{1}{\prod_{m=1}^{\beta.a_i} (a_i + mz)} \prod_{i=0}^{r} \frac{1}{\prod_{m=1}^{\beta.b_i} (b_i + mz)} \frac{1}{\prod_{m=1}^{\beta.\xi} (\xi + mz)}.$$

Although $I_\beta^{X/S}$ makes sense for any $\beta \in N_1(X)$, it is non-trivial only if $\beta \in NE^I(X)$.

Proposition 2.3 (Picard–Fuchs system on X/S). *$\Box_\ell I^X = 0$ and $\Box_\gamma I^X = 0$, where*

$$\Box_\ell = \prod_{j=0}^{r} z\partial_{a_j} - q^\ell e^{t^1} \prod_{j=0}^{r} z\partial_{b_j}, \qquad \Box_\gamma = z\partial_\xi \prod_{j=0}^{r} z\partial_{b_j} - q^\gamma e^{t^2}.$$

Proposition 2.4 ($\mathscr{F}$-invariance of PF ideal)**.**

$$\mathscr{F}\langle \Box_\ell^X, \Box_\gamma^X \rangle \cong \langle \Box_{\ell'}^{X'}, \Box_{\gamma'}^{X'} \rangle.$$

Theorem 2.5 (Quantum Leray–Hirsch).

(1) *(I-Lifting) The QDE on $QH(S)$ can be lifted to $H(X)$ as*

$$z\partial_i\, z\partial_j I = \sum_{k,\bar{\beta}} q^{\bar{\beta}^I} e^{(D.\bar{\beta}^I)} \bar{C}_{ij,\bar{\beta}}^k(\bar{t})\, z\partial_k D_{\bar{\beta}^I}(z) I,$$

where $D_{\bar{\beta}^I}(z)$ is an operator depending only on $\bar{\beta}^I$. Any other lifting is related to it modulo the Picard–Fuchs system.

(2) *Together with the Picard–Fuchs $\Box_\ell$ and $\Box_\gamma$, they determine a first order matrix system under the naive quantization basis:*

$$z\partial_a(\partial^{z\mathbf{e}} I) = (\partial^{z\mathbf{e}} I) C_a(z, q), \qquad \text{where } t^a = t^1,\ t^2 \text{ or } \bar{t}^i.$$

(3) *For $\bar{\beta} \in NE(S)$, its coefficients in C_a are polynomial in $q^\gamma e^{t^2}$, $q^\ell e^{t^1}$ and $\mathbf{f}(q^\ell e^{t^1})$, and formal in $\bar{t}$. Here $\mathbf{f}(q) := q/(1 - (-1)^{r+1} q)$ is the "origin of analytic continuation" satisfying $\mathbf{f}(q) + \mathbf{f}(q^{-1}) = (-1)^r$.*

(4) *The system is $\mathscr{F}$-invariant, though in general $\mathscr{F}\bar{\beta}^I \neq \bar{\beta}^{I'}$.*

Finally we construct a gauge transformation B to eliminate all the z dependence of C_a in the $\mathscr{F}$-invariant system $z\partial_a(\partial^{z\mathbf{e}} I) = (\partial^{z\mathbf{e}} I) C_a$. B is nothing more than the Birkhoff factorization matrix in $\partial^{z\mathbf{e}} I(\hat{t}) = (z\nabla J)(\tau) B(\tau)$ valid at the generalized mirror point $\tau = \tau(\hat{t})$. The above $\mathscr{F}$-invariance leads to $\mathscr{F}\tau = \tau'$ and $\mathscr{F}B(\tau) = B'(\tau')$, hence the glueing of Dubrovin connections under analytic continuations.

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