BUNDLES OVER RATIONAL SURFACES AND WEIERSTRASS CUBICS

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This is a short summary of the author's presentation given at Sanya Forum on Dec. 21, 2011, and a complete and accurate version will be forthcoming as a formal paper. In this summary and the forthcoming paper, we study principal G-bundles over certain rational surfaces S, and describe the relation between the moduli space of the surfaces S, containing D as a possibly singular anti-canonical curve, and the moduli space of certain principal G-bundles over D.

This study is motivated by the following. Let D be a fixed elliptic curve, and G be a complex semi-simple Lie group. There is a well-known correspondence between the following two things. One is the (moduli) space of G-bundles over D, which we denote by $\mathcal{M}_D^G$. The other one is the (moduli) space of G-surfaces with anti-canonical curve D, which we denote by $\mathcal{S}_D^G$. And such a correspondence is induced by the restriction of a naturally defined G-bundle over a rational surface S to the anti-canonical curve D. Physically, this correspondence is a consequence of a duality between F-theory and heterotic string theory. In particular, the case where D is a smooth anti-canonical curve, S is a *del Pezzo* surface and G is an exceptional Lie group E_6, E_7 or E_8, is well-studied. In this case, Looijenga, Donagi, Friedman-Morgan-Witten and others showed that these two moduli spaces are in fact isomorphic, that is, $\mathcal{M}_D^G \cong \mathcal{S}_D^G$. Furthermore, both are isomorphic to a weighted projective space, by Looijenga's famous theorem. Such kind of correspondence was generalized by N.C. Leung and myself to complex semi-simple Lie groups of any type, simply laced or not, and was further partially generalized by N.C. Leung, M.Xu and myself to loop groups of $\widetilde{E_n}$-type. In the following, we consider the case where D is a singular curve of arithmetic genus one. We claim that when G is a complex semi-simple Lie group of any type, and D is a Weierstrass cubic, we still have an identification: $\mathcal{M}_D^G \cong \mathcal{S}_D^G$.

Firstly, we explain $\mathcal{M}_D^G$. A Weierstrass cubic D is a reduced irreducible curve of arithmetic genus 1, together with a point $o \in D_{reg}$ (where D_{reg} means the smooth locus of D). Note that D can be embedded into $\mathbb{CP}^2$ by the linear system $|3o|$ with the defining equation (the Weierstrass model) $y^2z = x^3 + a_2xz^2 + a_3z^3$. When $a_2 = a_3 = 0$, D (or (D, o)) has a cusp singularity, and is called cuspidal. When $a_2 \neq 0$ and $4a_2^3 + 27a_3^2 = 0$, D (or (D, o)) has a node singularity, and is called nodal. When $4a_2^3 + 27a_3^2 \neq 0$, D (or (D, o)) is smooth, that is, (D, o) is an elliptic curve with the identity element o. If D is nodal or cuspidal, let $\pi : \widetilde{D} \to D$ be the normalization. Thus $\widetilde{D} \cong \mathbb{CP}^1$. Note that the group $Pic^0(D)$

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is naturally identified with $\mathbb{C}$ or $\mathbb{C}^*$, if D is cuspidal or nodal, respectively, and it classifies the isomorphism classes of line bundles which are trivial when pulled back. Furthermore, by Burban, Friedman-Morgan and others, the space (denoted by $\mathcal{M}_D^G$) of the isomorphism classes of principal G-bundles $\mathcal{G}/D$, such that $\pi^*\mathcal{G}$ is trivial, is isomorphic to H/W (resp. $\mathfrak{h}/W$) if D is nodal (resp. if D is cuspidal).

Secondly, we explain $\mathcal{S}_D^G$ and sketch the proof. For simplicity, we assume that G is simply laced. Let S be the blow-up of $\mathbb{CP}^2$ at $n+1$ points. Fix a smooth rational curve C with $C^2 = -1, 0$ or 1. Let K be the canonical divisor. Roughly speaking, the sublattice $\langle K, C\rangle^\perp$ of the Picard lattice of S is a root lattice of type E_n, D_n or A_n, respectively. Such a pair (S, C) is called a G-surface. We claim that over any G-surface (S, C), there exists canonically a principal G-bundle over S, denoted by $\mathcal{G}$, such that the associated adjoint bundle is

$$\mathcal{O}_S^{\oplus n} \oplus \bigoplus_{\alpha \in \langle K,C\rangle^\perp, \alpha^2=-2} \mathcal{O}_S(\alpha).$$

Moreover, any Weierstrass cubic (D, o) can be embedded into such a G-surface (S, C) as an anti-canonical curve, such that $C \cap D = \{o\}$. The restriction of the principal G-bundle $\mathcal{G}$ to the anti-canonical curve D induces a homomorphism from the root lattice $\Lambda_r(G)$ into $Pic^0(D)$. Thus we have a morphism from $\mathcal{S}_D^G$, the moduli space of G-surfaces (S, C) with the anti-canonical curve D, into the space $Hom(\langle K, C\rangle^\perp, Pic^0(D))$. There is an ambiguity: we can have different choices of isomorphisms $\langle K, C\rangle^\perp \cong \Lambda_r(G)$. To eliminate the ambiguity, we have to first consider the 'G-marking' on the G-surface (S, C), which is, roughly speaking, a series of blowing down maps. Let $(\mathcal{S}_D^G)_m$ be the moduli space of marked G-surfaces (S, C) with the anti-canonical curve D. Then we can show that there is an isomorphism

$$(\mathcal{S}_D^G)_m \xrightarrow{\sim} Hom(\Lambda_r(G), Pic^0(D)).$$

And the latter is isomorphic to H or $\mathfrak{h}$ when D is nodal or cuspidal respectively, where H is a Cartan subgroup of G and $\mathfrak{h}$ is the Lie algebra of H. Moreover, we can show that for generic G-surface (S, C), the Weyl group $W(G)$ of G acts on the set of G-markings simply transitively. Therefore, for simply laced G, and for nodal or cuspidal D, we have

$$\mathcal{S}_D^G \cong \begin{cases} H/W \\ \mathfrak{h}/W \end{cases} \cong \mathcal{M}_D^G.$$

If G is non-simply laced, we reduce this case to the simply laced case, by taking a simply laced Lie group G', such that $G = (G')^\sigma$, where σ is the highest order generator of the outer automorphism group of G'. One can show that $\mathcal{M}_D^G$ is isomorphic to the connected component $(\mathcal{M}_D^{G'})_0^\sigma$ of the subvariety of $\mathcal{M}_D^{G'}$ fixed pointwise by the action of σ. One can also define the G-surfaces, and similarly prove that $\mathcal{S}_D^G \cong \mathcal{M}_D^G$. Thus we have the following main result.

Theorem. *Let G be a complex semi-simple Lie group of any type, and let D be nodal or cuspidal. Denote $\mathcal{S}_D^G$ the moduli space of G-surfaces (S, C) with the anti-canonical curve D, and denote $\mathcal{M}_D^G$ the moduli space of principal G-bundles $\mathcal{G}$ over D, such that $\pi^*\mathcal{G}$ is trivial. Then we have*

$$\mathcal{S}_D^G \cong \mathcal{M}_D^G.$$

As a corollary, we see that any principal G-bundle over the anti-canonical curve D with a trivial pull-back can extend to the whole surface S. Note that both of the above spaces are non-compact. It is interesting to consider the compactifications of these spaces.

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