Title: Counting lattice points via Einstein metrics

In this talk, I report my recent joint work with Ma Chit. The original question is a combinatorial problem which asks the growth of the number of lattice points is a lattice polytope P after scaling. In large scale, it is roughly given by the volume of the scaled polytope. It is known that this growth function $p_P(k)$ is a polynomial in the scaling factor k, called the Ehrhard polynomial.

Given a lattice polytope P, there is toric variety X_P together with an ample line bundle associated with it. Furthermore, the number of lattice points in P equals the dimension of the space $H^0(X_P, L_P)$ of holomorphic sections of L_P. More generally, $p_P(k) = \#(kP) = \dim H^0\left(X_p, L_P^{\otimes k}\right)$ for any positive integer k. Indeed all higher cohomology group vanish and therefore $p_P(k)$ equals the holomorphic Euler characteristic $\chi\left(X_p, L_P^{\otimes k}\right)$ which then can be computed via the Hirzebruch-Riemann-Roch formula. From this, we can also see that the leading order coefficient of $p_P(k)$ is the volume of P.

When P is a reflexive polytope, then the corresponding toric variety X_P is Fano and $L_P = -K_P$ the anti-canonical line bundle. In this case, using the Riemann-Roch formula, we see that the subleading order coefficient of $p_P(k)$ is also proportional to the volume of P. We main result is a lower bound of the next order coefficient of $p_P(k)$. Using the Riemann-Roch formula again, this coefficient can be expressed in term of the integrals of $c_1(X)^n$ (which is the volume of X_P) and $c_1(X)^{n-2} c_2(X)$.

Since $c_1(X_P) > 0$ by the reflexive assumption on P, we can ask whether X_P admits a Kahler-Einstein metric. When it does, there is a Chern number inequality by Yau which says that the integral of $c_1(X)^{n-2} c_2(X)$ can be bounded by the integral of $c_1(X)^n$ and hence we are done. However, in general, X_P does not admit any Kahler-Einstein metric. Indeed a theorem of Wang-Zhu says that X_P admits Kahler-Einstein metric if and only if the polytope P is balanced, namely the center of mass of P is the origin.

For a general reflexive polytope P, we can *balanced* it in one higher dimension. Namely there is a unique hyperplane whose normal vector lies in the translated dual polytope which minimizes the $(n+1)$-dimensional volume of the cone of P cut by this hyperplane. In this situation there is associated Sasaki manifold, an odd dimensional analog of Kahler manifold, which admits a Sasaki-Einstein metric by the work of Futaki-Ono-Wang. And there is also an analog of Yau inequality in this setting.

By perturbing the hyperplane a little bit to make it rational, then we could again relate these Chern numbers with the holomorphic Euler characteristic of a certain toric variety of one higher dimension, which is an orbifold in general. In order to apply our earlier method, we need to use the Kawasaki Riemann-Roch formula for orbifolds. We identify all these orbifold contributions, thus obtain a lower bound on the growth function of lattice point counting function associated to the cone of P.