Geometry and analysis of moduli spaces of curves

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1 Introduction and notations

Let Σ_g be a compact Riemann surface of genus g, and $\mathcal{M}_g$ be the moduli space of compact Riemann surfaces Σ_g. It is known that $\mathcal{M}_g$ is a noncompact quasi-projective variety and admits a compactication $\overline{\mathcal{M}_g}^{DM}$, the Deligne-Mumford compactification, obtained by adding stable Riemann surfaces.

It is also known that $\mathcal{M}_g$ is a complex orbifold and also admits many natural metrics, for example, the Teichmuller metric (a Finsler metric) and Weil-Petersson metric (a Kahler metric). It also admits many other metrics. Recently, a lot of work has been done on metrics of $\mathcal{M}_g$, for example, in a series of papers of Liu-Sun-Yau. Therefore, $\mathcal{M}_g$ should be an interesting space (orbifold) in geometry, topology and analysis, and the goal of this talk is to understand $\mathcal{M}_g$ from such points of view.

All results below work for the more general moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n punctures. For simplicity, we will concentrate on the moduli space $\mathcal{M}_g$ of the compact Riemann surfaces. We usually assume that $g \geq 2$ so that each Riemann surface Σ_g admits a canonical hyperbolic metric conformal to the complex structure, and Σ_g is considered as a hyperbolic surface.

One effective way to study $\mathcal{M}_g$ is to realize $\mathcal{M}_g$ as a quotient of the Teichmuller space $\mathcal{T}_g$ by the mapping class group Mod_g, $\mathcal{M}_g = \mathrm{Mod}_g\backslash\mathcal{T}_g$.

The action of Mod_g on $\mathcal{T}_g$ is also crucial for many problems about Mod_g, for example, classification of its elements, its cohomological properties, large scale geometric properties.

Let S_g be a compact oriented surface of genus g. A marked compact Riemann surface is a compact Riemann surface Σ_g together with a diffeomorphism $\varphi : \Sigma_g \to S_g$. Two marked Riemann surfaces (Σ_g, φ), (Σ_g', φ') are called *equivalent* if there exists a biholomorphic map $h : \Sigma_g \to \Sigma_g'$ such that the two maps $h \circ \varphi', \varphi : \Sigma_g \to S_g$ are homotopy equivalent. The Teichmuller space is the equivalence classes of marked Riemann surfaces $\mathcal{T}_g = \{(\Sigma_g, \varphi)\}/\sim$. It is known that $\mathcal{T}_g$ is a complex manifold diffeomorphic to $\mathbb{R}^{6g-6}$. In particular, it is contractible.

Let $\mathrm{Diff}^+(S_g)$ be the group of all orientation preserving diffeomorphisms of S_g, and $\mathrm{Diff}^0(S_g)$ the identity component (a normal subgroup). The quotient group $\mathrm{Diff}^+(S_g)/\mathrm{Diff}^0(S_g)$ is called the **mapping class group**. Mod_g acts on $\mathcal{T}_g$ by changing the markings, and the quotient $\mathrm{Mod}_g\backslash\mathcal{T}_g = \mathcal{M}_g$. Therefore, $\mathcal{M}_g$ is an orbifold.

When $g = 1$, $\mathcal{T}_g = \mathbf{H}^2 = \{x + iy \mid x \in \mathbb{R}, y > 0\}$, the Poincare upper half plane, $\mathrm{Mod}_g = SL(2, \mathbb{Z})$, and $\mathcal{M}_1 = SL(2, \mathbb{Z})\backslash\mathbf{H}^2$, the modular curve, which is a basic locally symmetric space. This suggests that $\mathcal{M}_g$ is an analogue of locally symmetric space $\Gamma\backslash X$, where $X = G/K$ is a symmetric space of noncompact type, and $\Gamma \subset G$ is an arithmetic subgroup.

This point of view has motivated many problems about $\mathcal{M}_g$ and Mod_g and their solutions.

2 Spectral theory of Weil-Petterson metric

For any point $p = (\Sigma_g, \varphi) \in \mathcal{T}_g$, its cotangent space $T_p^*\mathcal{T}_g$ is the space of holomorphic quadratic forms $Q(\Sigma_g)$. For $f, g \in Q(\Sigma)$, define an inner product $\langle f, g\rangle = \int_{\Sigma_g} f\bar{g}(ds^2)^{-1}$, where ds^2 is the area form of the hyperbolic metric on Σ_g.

This is the Weil-Petersson metric ω_{WP}. It is known that (1) ω_{WP} is invariant under Mod_g, (2) it is a Kahler metric and has negative sectional curvature, (3) it is incomplete but is geodesically convex (every two points are connected by a unique geodesic), (4) the voloume of $\mathcal{M}_g$ in ω_{WP} is finite, (5) the completion of $\mathcal{M}_g$ with respect to ω_{WP} is the Deligne-Mumford compactification $\overline{\mathcal{M}_g}^{DM}$.

Before studying the spectral theory of $(\mathcal{M}_g, \omega_{WP})$, we recall some basic facts about self-adjoint extension of the Laplace operator. For any smooth Riemannian manifold M, the Green formula implies that its Laplacian Δ is a symmetric operator with the domain $C_0^\infty(M)$.

A basic fact is that if M is a complete Riemannian manifold, then Δ admits a unique self-adjoint extension to $L^2(M)$, i.e., Δ is essentially self-adjoint. The assumption of completeness of M is important.

Another important fact is that if M is a compact Riemannian manifold, then Δ has only a discrete spectrum, and its counting function satisfies the Weyl asymptotic law. Specifically, let $\lambda_1 \leq \lambda_2 \leq \cdots$ be its eigenvalues. Define the counting function $N(\lambda) = |\{\lambda_i \leq \lambda\}|$. Then as $\lambda \to +\infty$, $N(\lambda) \sim c\, vol(M)\lambda^{\frac{n}{2}}$, c only depends on the dimension. In the above result, the compactness of M is crucial. But for $(\mathcal{M}_g, \omega_{WP})$, both the completeness and compactnese conditions fail.

Theorem 2.1 (Ji-Mazzeo-Muller-Vasy) . *(1) The Laplacian Δ_{WP} of the W-P metric of $\mathcal{M}_g$ is essentially self-adjoint. (2) Δ_{WP} has only a discrete spectrum and its counting function satisfies the Weyl asymptotic law.*

In some sense, $(\mathcal{M}_g, \omega_{WP})$ behaves like a compact Riemannian manifold. We do have a compact orbifold, the Deligne-Mumford compactification $\overline{\mathcal{M}_g}^{DM}$, but the metric ω_{WP} is singular near the boundary divisors.

3 Simplicial volume of $\mathcal{M}_g$

The notion of simplicial volume was introduced by Gromov. Let M^n be a compact oriented manifold. Then it has a fundamental class $[M] \in H_n(M, \mathbb{Z})$. Denote the image of $[M]$ under the map $H_n(M, \mathbb{Z}) \to H_n(M, \mathbb{R})$ also by $[M]$. For every n-cycle $c = \sum_\sigma a_\sigma \sigma$, define its ℓ_1-norm $|c|_1 = \sum_\sigma |a_\sigma|$. The **simplicial volume** of M is defined by

$$|M| = \inf\{|c|_1 \mid c \text{ is an } \mathbb{R}\text{-cycle representing } [M]\}.$$

It could be zero. If M is an oriented noncompact manifold, then there is a fundamental class $[M]^{lf}$ in the locally finite homology group $H_n^{lf}(M, \mathbb{Z})$ and also in $H_n^{lf}(M, \mathbb{R})$. Using locally finite cycles, we can also define the simplicial volume $|M|$ of M as before. In this case $|M|$ could be ∞.

If M is an orbifold and admits a finite smooth cover, $N \to M$, then we can define $|M|_{orb} = |N|/d$, where d is the degree of the covering $N \to M$. It is independent of smooth covers N. One can show $|M|_{orb} \geq |M|$.

Motivations for simplicial volume include: (1) when the manifold is of odd dimension, the Euler characteristic is zero, it gives a potentially nonzero homotopy invariant, (2) it also gives a lower bound for the minimal volume of a manifold, $min - vol(M) \geq c(n)|M|$, $n = \dim M$, where $c(n)$ is a positive constant, and $min - vol(M)$ is defined by $min - vol(M) = \inf\{vol(M, g) \mid g \text{ is complete}, -1 \leq K_g \leq 1\}$.

In general, it is not easy to compute $|M|$. A natural problem is to **decide whether it vanishes or not.** Here is a brief summary of known results. (1) (Thurston-Gromov) If M is a complete hyperbolic manifold of finite area (or the curvature is negatively pinched), then $|M| > 0$. (2) (LaFont-Schmidt, conjectured by Gromov). If M is a compact locally symmetric space of noncompact type, then $|M| > 0$. (3) (Löh-Sauer) If M is a locally symmetric space of $\mathbb{Q}$-rank at least 3 (hence noncompact), then $|M| = 0$.

For some locally symmetric spaces of $\mathbb{Q}$-rank 1, $|M| > 0$. But no result is known for $\mathbb{Q}$-rank 2 locally symmetric spaces.

Given the dictionary between $\mathcal{M}_g$ and $\Gamma\backslash X$, the next result is natural.

Theorem 3.1 *The obifold simplicial volume of $\mathcal{M}_g$ vanishes, $|\mathcal{M}_g|_{orb} = 0$ if and only if $g \geq 2$.*

A simple known corollary is that *when $g \geq 2$, $\mathcal{M}_g$ does not admit a negatively pinched complete Riemannian metric.* A widely believed conjecture is that $\mathcal{M}_g$ does not admit a complete nonpositively curved Riemannian metric.

Ingredients needed for the proof include a Borel-Serre compactification of $\mathcal{T}_g$, an analogue of Solomon-Tits theorem for curve complex of the surface S_g, and standard tools in simplicial volume.

4 Duality property of the mapping class group

The Borel-Serre type compactification and the Solomon-Tits theorem for curve complex of the surface S_g has applications to cohomology properties of Mod_g.

A discrete group Γ is called a *Poincare duality group of dimension n* if for every $\mathbb{Z}\Gamma$-module A, there is an isomorphism $H^i(\Gamma, A) \cong H_{n-i}(\Gamma, A)$. Γ is called a *generalized Poincare duality group of dimension n* if there is a $\mathbb{Z}\Gamma$-module D, called the *dualizing module*, such that for every $\mathbb{Z}\Gamma$-module A, there is an isomorphism $H^i(\Gamma, A) \cong H_{n-i}(\Gamma, D \otimes A)$. If D is trivial, then Γ is a Poincare duality group.

Duality groups must be torsion-free. By pasing to torsion-free finite index subgroups, we can define **virtual** (generalized) Poincare duality groups.

Theorem 4.1 (Harer) Mod_g *is a virtual generalized Poincare duality group of dimension* $4g - 5$.

Theorem 4.2 (Ivanov-Ji) Mod_g *is a not virtual Poincare duality group.*

These results were motivated by Borel-Serre results on duality properties of arithmetic groups.

5 Spines of Teichmuller spaces

A universal space for proper actions of a discrete group Γ, denoted by $\underline{E}\Gamma$, is a space where Γ acts properly, and for every finite subgroup $F \subset \Gamma$, the set of fixed poin set $\underline{E}\Gamma^F$ is nonempty and contractible.

$\underline{E}\Gamma$ exists and is unique up to homotopy equivalence. If Γ is torsion-free, then $\underline{E}\Gamma$ is equal to $E\Gamma$, the universal covering space of $B\Gamma$, the classifying space of Γ.

If Γ contains torsion elements, then $\dim B\Gamma = \infty$ for every model. Hence $\dim E\Gamma = \infty$. But under suitable conditions on Γ, one can find models of $\underline{E}\Gamma$ of finite dimension.

For various applications such as Baum-Connes conjecture, Novikov conjecture, one wants models of $\underline{E}\Gamma$ such that $\Gamma\backslash\underline{E}\Gamma$ is compact, or a finite CW-complex. which is called a *cofinite model.* One also wants to have small $\dim \underline{E}\Gamma$. The smallest possible dimension is the virtual cohomology dimension of Γ.

Proposition 5.1 *$\mathcal{T}_g$ is a universal space for proper actions of* Mod_g.

This is a nontrivial result. The noemptiness of fixed point set of a finite subgroup is the positive solution by Kerckhoff. The noemptiness and contractibility can also be proved by using the negative curvature and geodesic convexity of the Weil-Petersson metric, due to Wolpert.

But $\mathcal{T}_g$ is **not a cofinite model** of $\underline{E}\Gamma$ since $\mathcal{M}_g$ is noncompact. For small $\varepsilon > 0$, define the thick part $\mathcal{T}_g(\varepsilon)$ consisiting of hyperbolic surfaces without geodesics of length shorter than ε.

It can be shown that $\mathcal{T}_g(\varepsilon)$ is a manifold with corners, stable under Mod_g with a compact quotient.

Theorem 5.2 (Ji-Wolpert) *There is a Mod_g-equivariant deformation retraction of $\mathcal{T}_g$ to $\mathcal{T}_g(\varepsilon)$. Hence $\mathcal{T}_g(\varepsilon)$ is a cofinite model of $\mathcal{T}_g(\varepsilon)$.*

A natural problem is to find equivariant deformation retractions of $\mathcal{T}_g$ of dimension as small as possible. These are called *spines* of $\mathcal{T}_g$. The smallest possible dimension is $4g-5$, the virtual cohomological dimension of Mod_g.

Thurston circulated a preprint outlining a construction of a spine of $\mathcal{T}_g$ in 1985. There seems to be several difficulties. We note that when $n > 0$, i.e., Riemann surfaces have punctures, an explicit spine of $\mathcal{T}_{g,n}$ of the smallest possible dimension is known, due to Mumford, Thurston, Penner, Bowditch-Epstein.

Define $R \subset \mathcal{T}_g$ to consist of hyperbolic surfaces where at least two shortest closed geodesics intersect. By the collar theorem for hyperbolic surfaces, this a closed real-analytic subset stable under Mod_g with a compact quotient.

Theorem 5.3 *There is an equivariant deformation retraction of $\mathcal{T}_g$ to R. In particular, R is a cofinite model of $\underline{E}\mathrm{Mod}_g$.*

The spine R is of positive codimension. But $\mathcal{T}_g(\varepsilon)$ is of codimension 0. This is the first spine of positive codimension. Based on work in progress, we can get a spine of at least codimension 2.

The idea of the construction and proof came from symmetric space $SL(n,\mathbb{R})/SO(n)$ of unimodular lattices of $\mathbb{R}^n$, where there is a well-rounded deformation retraction to well-rounded lattices.

Given a lattice $\Lambda \subset \mathbb{R}^n$, define $m(\Lambda) = \inf\{||v|| \mid v \in \Lambda - \{0\}\}$, and $M(\Lambda) = \{v \in \Lambda \mid ||v|| = m(\Lambda)\}$, the set of shoretest vector vectors. If $M(\Lambda)$ spans Λ, Λ is called a **well-rounded** lattice.

The *deformation procedure to well-rounded lattices* is given by *successive scaling up shortest vectors at the same time,* i.e., scaling up the linear subspace spanned by $M(\Lambda)$ and scaling down the orthogonal complement to keep the lattice unimodular, until a well-rounded lattice is reached.

The procedure to deform $\mathcal{T}_g$ to the spine R is by simultaneously increasing the length of all systoles (shortest geodesics) using gradient flow of the length functions until they intersect.

In the result of Ji-Wolpert, the deformation procedure depends on a partition of unity and is not *intrinsic or canonical.* The above procedure gives a canonical deformation retraction of $\mathcal{T}_g$ to its thick part $\mathcal{T}_g(\varepsilon)$.

6 L^p-cohomology of $\mathcal{M}_g$

The L^2-cohomology of a Riemannian manifold (M, ds^2), $H^i_{(2)}(M)$, is defined by the complex of L^2-differential forms. If M is compact, then the L^2-cohomology is the de Rham cohomology. Similarly, L^p-cohomology can be similarly defined. Note that L^p-cohomology groups only depend on the quasi-isometry class of the metric.

Let $\Gamma\backslash X$ be a noncompact arithmetic locally symmetric varieties (i.e., arithmetic locally Hermitian symmetric space). It has a Baily-Borel compactification, $\overline{\Gamma\backslash X}^{BB}$, which is a normal projective variety. It admits the intersection cohomology group with respect to the middle perversity $IH^i(\overline{\Gamma\backslash X}^{BB})$.

The following is a positive solution to Zucker conjecture.

Theorem 6.1 (Looijenga, Saper-Stern) *The L^2-cohomology of $\Gamma\backslash X$ is canonically isomorphic to the intersection cohomology of $\overline{\Gamma\backslash X}^{BB}$: $H^i_{(p)}(\Gamma\backslash X) \cong IH^i(\overline{\Gamma\backslash X}^{BB})$.*

The singularities of $\overline{\Gamma\backslash X}^{BB}$ are big. One topological resolution is given by the reductive Borel-Serre compactification $\overline{\Gamma\backslash X}^{RBS}$, which is defined for all arithmetic locally symmetric spaces $\Gamma\backslash X$.

Theorem 6.2 (Zucker) *For $p \gg 0$, the L^p-cohomology $H^i_{(p)}(\Gamma\backslash X)$ is isomorphic to $H^i(\overline{\Gamma\backslash X}^{RBS})$.*

Recall that the Deligne-Mumford compactification $\overline{\mathcal{M}_g}^{DM}$ is an orbifold.

Theorem 6.3 (Ji-Zucker) *(1) For any Riemannian metric quasi-isometric to the Teichmuller metric, and any $p < +\infty$, $H^i_{(p)}(\mathcal{M}_g) \cong IH^i(\overline{\mathcal{M}_g}^{DM}) = H^i(\overline{\mathcal{M}_g}^{DM})$. (2) For any metric quasi-isometric to Weil-Petersson metric, when $p \geq 4/3$, $H^i_{(p)}(\mathcal{M}_g) \cong H^i(\overline{\mathcal{M}_g}^{DM})$, when $p < 4/3$, $H^i_{(p)}(\mathcal{M}_g) \cong H^i(\mathcal{M}_g)$.*

This result shows a rank-1 phenomenon of $\mathcal{M}_g$, since when the rank of a Hermitian locally symmetric space $\Gamma\backslash X > 1$, $\overline{\Gamma\backslash X}^{RBS}$ is different from $\overline{\Gamma\backslash X}^{BB}$.