D-MODULES AND ÉTALE FUNDAMENTAL GROUP IN CHAR. $p > 0$

HÉLÈNE ESNAULT AND XIAOTAO SUN

1. D-modules in positive characteristic

We fix the following notation

- k: an algebraically closed field of characteristic $p > 0$
- X: a smooth projective variety over k, with a fixed point

$$(a : \mathrm{Spec}(k) \to X) \in X(k).$$

- F: Frobenius morphism $F = F_X : X \to X$.
- π_1: $\pi_1 = \pi_1^{\text{ét}}(X, a)$ the étale fundamental group of X.
- $\mathcal{D}_X$: the sheaf of differential operators.

Definition 1.1. A D-module on X is a coherent $\mathcal{O}_X$-module E with a morphism

$$\nabla : \mathcal{D}_X \to \mathcal{E}nd_k(E)$$

of $\mathcal{O}_X$-algebras (i.e. E is a $\mathcal{D}_X$-module).

By a theorem of Katz, it is equivalent to the following definition

Definition 1.2. A D-module E on X is a sequence of bundles

$$\mathbb{E} = \{E := E_0, E_1, E_2, \cdots, \} = \{E_i\}_{i \in \mathbb{N}}$$

such that $E_i \cong F_X^* E_{i+1} E_i$, where $F_X : X \to X$ is the absolute Frobenius map: $F_X^* : \mathcal{O}_X \to \mathcal{O}_X, \;\; s \mapsto s^p$.

A morphism $\phi = \{\phi_i\}_{i \in \mathbb{N}} : \mathbb{E} \to \mathbb{E}'$ of D-modules consists of

$$\begin{array}{ccc} E_i & \xrightarrow{\phi_i} & E_i' \\ \cong \downarrow & & \downarrow \cong \\ F_X^* E_{i+1} & \xrightarrow{F_X^* \phi_{i+1}} & F_X^* E_{i+1}' \end{array}$$

2. Gieseker's conjecture

Example 2.1 (Giesker, 1973)**.** Let $\pi_1 = \pi_1^{\text{ét}}(X, a)$ be the étale fundamental group of X, for any continuous representation

$$\rho : \pi_1 \to GL(V)$$

the associated bundle V_ρ is a D-module on X. In fact,

$$\rho \longmapsto V_\rho$$

defines a faithful functor: $\mathbf{Rep}_k(\pi_1) \longrightarrow \mathbf{str}(X)$.

Theorem 2.2 (Giesker, 1973)**.** *(i) If every D-module on X is trivial, then π_1 is trivial.*
(ii) If all D-module are rank 1, then $[\pi_1, \pi_1]$ is a pro-p-group.
(iii) If every D-module is a direct sum of rank 1 D-modules, then π_1 is abelian with no p-power order quotient.

Conjecture 2.3 (Gieseker, 1973)**.** *The converses of above statements might be true.*

Theorem 2.4. *Let X be a smooth projective variety over an algebraically closed field k of characteristic $p > 0$, then:*
(i) Every D-module on X is trivial if and only if π_1 is trivial. (Esnault-Mehta: Invent. math., 2010)
(ii) All irreducible D-modules have rank 1 if and only if $[\pi_1, \pi_1]$ is a pro-p-group. (Esnault-Sun)
(iii) Every D-module is a direct sum of rank 1 D-modules if and only if π_1 is abelian with no non-trivial p-power quotient. (Esnault-Sun)

3. Extensions of stratified line bundles

Theorem 3.1. *Let X be a smooth projective connected variety, and π_1 be abelian without non-trivial p-power order quotient. Then any extension*

$$0 \to \mathbb{L} \to \mathbb{E} \to \mathbb{L}' \to 0$$

*in the category **str**(X) of D-modules is split when $\mathbb{L}$ and $\mathbb{L}'$ are rank 1 objects.*

Then, by definition, a nontrivial extension $0 \to \mathbb{L} \to \mathbb{E} \to \mathbb{I} \to 0$ in $\mathbf{str}(X)$, means that we have a set

$$\Sigma = \{\ e_i = (0 \to L_i \to E_i \to \mathcal{O}_X \to 0)\ \}_{i \in \mathbb{N}}$$

of non-trivial extensions such that $e_i \cong F^*(e_{i+1})$.

- If Σ is a finite set, then there is a $e_i \in \Sigma$ such that

$$(F_X^*)^a e_i = e_i$$

$$(F_X^*)^a (L_i \hookrightarrow E_i \twoheadrightarrow \mathcal{O}_X) \cong (L_i \hookrightarrow E_i \twoheadrightarrow \mathcal{O}_X)$$

Then the following Proposition implies a contradiction.

Proposition 3.2. *Let L be a line bundle on X with $(F_X^*)^a L = L$. Then*

$$(F_X^*)^a : H^1(X, L) \to H^1(X, L)$$

is nilpotent if π_1 is abelian without non-trivial p-power quotient.

- If Σ is an infinite set, we will show: There is a good reduction X_s of X over $\bar{\mathbb{F}}_p$, and a non-trivial extension (on X_s)

$$e = (0 \to L \to E \to \mathcal{O}_{X_s} \to 0)$$

  such that $(F_{X_s}^*)^a e = e$ for some integer $a > 0$.
- Since $\pi^{ét}(X, x_0) \twoheadrightarrow \pi^{ét}(X_s, (x_0)_s)$ is surjective, using above proposition to X_s, we still get a contradiction.

4. The Proof when $k = \bar{\mathbb{F}}_p$

- there is a reduced scheme M such that

$$M(k) = \{ \ e = (0 \to L \to E \to \mathcal{O}_X \to 0) \ \}$$

- there is a rational map $f : M \dashrightarrow M$ over k such that

$$f(e) = (0 \to F^*L \to F^*E \to \mathcal{O}_X \to 0) := F^*(e)$$

- Let $\Sigma(m) := \{ \ e_{i+m} \in \Sigma \ \}_{i \in \mathbb{N}} \subset \Sigma$, then

$$f(\Sigma(m)) = \Sigma(m-1), \quad f(e_{i+1}) = f(e_i)$$

- Let $Z \subset M$ be the Zariski closure of $\Sigma \subset M$, then

$$f = f|_Z : Z \dashrightarrow Z$$

  is a dominant rational map.
- Let $Z' \subset Z$ be the union of irreducible components Z_i with $dim(Z_i) > 0$. Thus there is an irreducible component Z_{i_0} and an integer $a_1 > 0$ such that

$$f^{a_1} : Z_{i_0} \dashrightarrow Z_{i_0}$$

  is a dominant rational map.

Theorem 4.1 (Corollary of twisted Lang-Weil estimate). *Let* $Y \subset \mathbb{A}^n_{\mathbb{F}_q}$ *be an affine variety,* $\Gamma \subset Y \times Y$ *be an irreducible subvariety over* $\overline{\mathbb{F}}_q$*. Assume the two projections* $\Gamma \to X$ *are dominant. Then, for any closed subvariety* $W \subsetneq Y$*, there exists*

$$x = (x_1, ..., x_n) \in Y(\overline{\mathbb{F}}_q), \quad x^{q^m} := (x_1^{q^m}, ..., x_n^{q^m})$$

such that $(x, x^{q^m}) \in \Gamma$ *and* $x \notin W$ *for* $m >> 0$*.*

- $Y := Z_{i_0} \subset \mathbb{A}^n_{\mathbb{F}_q}, \quad \Gamma = \overline{graph(f^{a_1})} \subset Y \times Y, \ W \subset Y$ where f^{a_1} is not well-defined. Let $f^{a_1} = (f_1, ..., f_n), \quad f_i \in \mathbb{F}_q(Y)$.
- $\exists\ x = (x_1, ..., x_n) \in Y(\overline{\mathbb{F}}_q)$ such that $x \in W, \ (x, x^{q^m}) \in \Gamma$.
- $f^{a_1}(x) = (x_1^{q^m}, ..., x_n^{q^m}), \ f^{a_1}(f^{a_1}(x)) = f^{a_1}(x^{q^m}) = f^{a_1}(x)^{q^m}$
  $\Rightarrow \ f^{2a_1}(x) = x^{2q^m}, ..., \ f^{a_2 a_1}(x) = x\,, \ (a_2 >> 0)$

References

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