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EYNARD-ORANTIN THEORY AND INTERSECTION NUMBERS

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The Eynard-Orantin recursion formalism from random matrix theory provides a unifying framework for moduli spaces and integrable systems. The first part of my talk reviews the Eynard-Orantin theory and its relations with the intersection numbers on moduli spaces of curves.

The second part of my talk is about the work (in progress) with Professor K. Liu and M. Mulase on asymptotics of intersection numbers.

We will follow Mirzakhani's notation in. For $\mathbf{d} = (d_1, \cdots, d_n)$ with d_i non-negative integers and $|\mathbf{d}| = d_1 + \cdots + d_n < 3g - 3 + n$, let $d_0 = 3g - 3 + n - |\mathbf{d}|$ and define

$$[\tau_{d_1} \cdots \tau_{d_n}]_{g,n} = \frac{\prod_{i=1}^n (2d_i+1)!! 2^{2|\mathbf{d}|} (2\pi^2)^{d_0}}{d_0!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_1^{d_0}, \tag{1}$$

where κ_1 is the first Mumford class on $\overline{\mathcal{M}}_{g,n}$. Note that $V_{g,n} = [\tau_0, \cdots \tau_0]_{g,n}$ is the Weil-Peterson volume of $\overline{\mathcal{M}}_{g,n}$. Mirzakhani's volume polynomial is given by

$$V_{g,n}(2L) = \sum_{|\mathbf{d}| \le 3g-3+n} [\tau_{d_1} \cdots \tau_{d_n}]_{g,n} \frac{L_1^{2d_1}}{(2d_1+1)!} \cdots \frac{L_n^{2d_n}}{(2d_n+1)!}.$$

Let $S_{g,n}$ be an oriented surface of genus g with n boundary components. Let $\mathcal{M}_{g,n}(L_1, \dots, L_n)$ be the moduli space of hyperbolic structures on $S_{g,n}$ with geodesic boundary components of length $L_1, \dots, L_n$. Then we know that the Weil-Petersson volume $\mathrm{Vol}(\mathcal{M}_{g,n}(L_1, \dots, L_n))$ equals $V_{g,n}(L_1, \dots, L_n)$. In particular, when $n = 1$, Mirzakhani's volume polynomial can be written as

$$V_g(2L) = \sum_{k=0}^{3g-2} \frac{a_{g,k}}{(2k+1)!} L^{2k},$$

where $a_{g,k} = [\tau_k]_{g,1}$ are rational multiples of powers of π

$$a_{g,k} = \frac{(2k+1)!! 2^{3g-2+2k} \pi^{6g-4-2k}}{(3g-2-k)!} \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^k \kappa_1^{3g-2-k}. \tag{2}$$

Here we list values of $a_{g,k}$, $0 \le k \le 3g - 2$ when $g \le 3$:

$$a_{1,0} = \frac{\pi^2}{12}, \quad a_{1,1} = \frac{1}{2}, \quad a_{2,0} = \frac{29\pi^8}{192}, \quad a_{2,1} = \frac{169\pi^6}{120}, \quad a_{2,2} = \frac{139\pi^4}{12},$$
$$a_{2,3} = \frac{203\pi^2}{3}, \quad a_{2,4} = 210, \quad a_{3,0} = \frac{9292841\pi^{14}}{4082400}, \quad a_{3,1} = \frac{8497697\pi^{12}}{388800},$$
$$a_{3,2} = \frac{8983379\pi^{10}}{45360}, \quad a_{3,3} = \frac{127189\pi^8}{81}, \quad a_{3,4} = \frac{94418\pi^6}{9},$$
$$a_{3,5} = \frac{166364\pi^4}{3}, \quad a_{3,6} = \frac{616616\pi^2}{3}, \quad a_{3,7} = 400400.$$

Let γ be a separating simple closed curve on S_g and $S_g(\gamma) = S_{g_1,1} \times S_{g_2,1}$ the surface obtained by cutting S_g along γ. Then for any $L > 0$, we have

$$\mathrm{Vol}(\mathcal{M}(S_g(\gamma), \ell_\gamma = L)) = V_{g_1}(L) \cdot V_{g_2}(L), \tag{3}$$

where $\mathcal{M}(S_g(\gamma), \ell_\gamma = L)$ is the moduli space of hyperbolic structures on $S_g(\gamma)$ with the length of γ equal to L.

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The celebrated Witten-Kontsevich theorem shows that the intersection numbers satisfy the KdV equation. On the other hand, Mirzakhani's remarkable recursion formula of Weil-Petersson volumes may be regarded as a deformation of the Witten-Kontsevich theorem.

There is a large amount of work on the computation of Weil-Petersson volumes. Recently, Mirzakhani proved some interesting estimates on the asymptotics of Weil-Petersson volumes and found important applications in the geometry of random hyperbolic surfaces. In particular, Mirzakhani proved the following asymptotic relations of the coefficients of the one-point volume polynomial.

Theorem 0.1 (Mirzakhani)**.** *For given $i \geq 0$.*

$$\lim_{g\to\infty} \frac{a_{g,i+1}}{a_{g,i}} = 1, \qquad \lim_{g\to\infty} \frac{a_{g,3g-2}}{a_{g,0}} = 0.$$

Mirzakhani asked what is the asymptotics of $a_{g,k}/a_{g,k+1}$ for an arbitrary k (which can grow with g). The following result gives a partial answer to Mirzakhani's question.

Theorem 0.2 (Liu-Xu)**.** *For any given $k \geq 0$, there is a large genus asymptotic expansion*

$$\frac{a_{g,3g-2-k}}{g^k a_{g,3g-2}} = \frac{\pi^{2k}}{5^k k!}\left(1 + \frac{b_1(k)}{g} + \frac{b_2(k)}{g^2} + \cdots\right), \tag{4}$$

In particular, $\forall k \geq 0$ we have

$$b_1(k) = \frac{k}{14}(k-8), \qquad b_2(k) = \frac{k}{10584}(27k^3 - 428k^2 + 2040k + 125).$$

Moreover, for any given $k \geq 0$, the series in the bracket of (4) *is a rational function of g.*

Partly in answering a question of Mirzakhani, Zograf made the following conjectural large genus asymptotic expansion based on numerical experiments.

Conjecture 0.3 (Zograf)**.** *For any fixed $n \geq 0$*

$$V_{g,n} \sim (4\pi^2)^{2g+n-3}(2g-3+n)!\frac{1}{\sqrt{g\pi}} \tag{5}$$

as $g \to \infty$.

In fact, Zograf conjectured more precisely

$$V_{g,n} = (4\pi^2)^{2g+n-3}(2g-3+n)!\frac{1}{\sqrt{g\pi}}\left(1 + \frac{c_n}{g} + O\left(\frac{1}{g^2}\right)\right), \tag{6}$$

where c_n is a constant depending only on n.

Note that the asymptotic behavior of $V_{g,n}$ for fixed g and large n has been determined by Manin and Zograf. Important progress on Zograf's conjecture has been made very recently by Mirzakhani and Zograf.

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