Fast numerical algorithms for harmonic maps and Riemann mappings

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Summary

In this talk, we present two fast numerical algorithms of planar parameterizations for 3D meshes referred to as harmonic maps and Riemann mappings, respectively.

1. An algorithm listed in [1] implements a harmonic map that maps a simply connected mesh with a single boundary to the unit disk. The algorithm begins with formulating an energy optimization problem and the harmonic map is the key to the optimization problem. It is straightforward to show that the harmonic map is the key to a linear system derived from the optimization problem. We had applied many existing linear solvers to find the harmonic map. We observed that it is time consuming to get an accurate enough solution. Consequently we hope to develop a robust and efficient linear solver for this specific large scale linear system. We note that in [2] several formulations of optimization problems that induce different types of linear systems are derived. We focus on the linear systems with coefficient matrices being M-matrices or symmetric positive definite ones. We solve these linear systems by the preconditioned conjugate gradient (PCG) method with the Symmetric Successive Over-Relaxation (SSOR) preconditioner. The SSOR preconditioner is parameterized by a parameter ω. Choosing an optimal value of ω will lead to faster convergence. However the cost of calculating an optimal value of ω may be too expensive. Based on the Lanczos decomposition, we present three efficient strategies to estimate a near optimal value. Numerical examples reveal that these strategies are all viable methods to estimate the parameter value. Moreover, calculating the Lanczos decomposition not only aids in estimating the parameter value but also gives a better initial in solving the linear system via PCG.

2. The planar parameterization by a harmonic map may result in severe angle deformations. To solve this problem, we turn to the implementation of the Riemann mapping in [1]. The Riemann mapping algorithm

computes a *conformal* mapping from a genus zero mesh with a single boundary to the unit disk. The Riemann mapping algorithm can be separated into three steps: (a) double covering, (b) spherical conformal mapping and (c) Mobius transform and stereographic projection. The most challenge part is calculating a spherical conformal mapping. Finding the spherical conformal mapping is associated with a nonlinear heat diffusion process. When using an explicit Euler method to devise an iteration formula to solve the nonlinear heat diffusion process, the iterations may take too many steps for a satisfactory solution. In some cases the iterations even fail to converge. Subsequently we may turn to apply an implicit Euler method to devise an iteration formula. However solving such nonlinear equations is really a challenging task. To conquer this question, we propose a novel quasi-implicit Euler method. The method aims to combine the pros and avoid the cons of the explicit Euler and implicit Euler methods. Numerical experiments reveal that the quasi-implicit Euler method is much superior to the explicit Euler method. Finally, we present some examples where the meshes not only from benchmarks but from a 3D scanner.

References

[1] Gu, X.D and Yau, S.T.: Computational Conformal Geometry, Higher education press, 2008.

[2] Wei, M.-Q., Pang, M.-Y. and Fan, C-L.: Survey on Planar Parameterization of Triangular Meshes, Proceedings of the 2010 International Conference on Measuring Technology and Mechatronics Automation, 3 (2010), 702–705.