DEGENERATION FORMULAE AND ITS APPLICATION TO LOCAL GW AND DT INVARIANTS

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1. Local GW invariants of blowups of Fano surfaces

Let S be a Fano surface and K_S its canonical bundle. For $\beta \in H_2(S, \mathbb{Z})$, denote by $\overline{\mathcal{M}}_{g,k}(S, \beta)$ the moduli space of k-pointed stable maps of degree β to S. Then the following diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,1}(S,\beta) & \xrightarrow{ev} & S \\ \rho \downarrow & & \\ \overline{\mathcal{M}}_{g,0}(S,\beta) & & \end{array}$$

defines the obstruction bundle $R^1\rho_* ev^* K_S$ whose fiber over a stable map $f : C \longrightarrow S$ is given by $H^1(C, f^*K_S)$.

One can define the local Gromov-Witten invariants of K_S by

$$K^S_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir}} e(R^1\rho_* ev^* K_S). \tag{1}$$

Denote by $Y_S = \mathbb{P}(K_S \oplus \mathcal{O})$ the projective bundle completion of the total space of the canonical bundle K_S. Then Y_S has two canonical sections S^+, S^- with normal bundle $N_{S^+|Y_S} \cong K_S$ and $N_{S^-|Y_S} \cong -K_S$ respectively. Then the normal bundle of a surface S^+ inside Y_S is negative. If the image of a stable map lies in S^+, it is not able to deform outside of S^+. This means if we denote also by β the image of a class $\beta \in H_2(S, \mathbb{Z})$ under the inclusion map $S \hookrightarrow Y_S$ via the section S^+, then one has $\overline{\mathcal{M}}_{g,0}(Y_S, \beta) = \overline{\mathcal{M}}_{g,0}(S, \beta)$. By the constructions of the virtual fundamental cycles, we have

$$[\overline{\mathcal{M}}_{g,0}(Y_S,\beta)]^{vir} = [\overline{\mathcal{M}}_{g,0}(S,\beta)]^{vir} \cap e(R^1\rho_* ev^* K_S). \tag{2}$$

Denote the Gromov-Witten invariant of Y_S of degree β by

$$n^{Y_S}_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,0}(Y_S,\beta)]^{vir}} 1. \tag{3}$$

Therefore, from (1) and (2), we have

$$K^S_{g,\beta} = n^{Y_S}_{g,\beta}. \tag{4}$$

Suppose that S and its blow-up $\tilde{S}$ at a point p_0 both are Fano. If we try to compare the local Gromov-Witten invariants of K_S and $K_{\tilde{S}}$, we first related the local invariants to the associated absolute Gromov-Witten invariants of the projective completions of K_S and $K_{\tilde{S}}$ via (4). Secondly we find a sequence of birational threefolds all of whose invariants are equal. In fact, the birational threefolds are the projective completion Y_S of K_S, the blow-up $\tilde{Y}_S$ of Y_S along the fiber over p_0, the projective completion $Y_{\tilde{S}}$ of $K_{\tilde{S}}$ and Z, a threefold dominating the last two, obtained by blowing them up along a specific section of the exceptional divisor in $\tilde{S}$. For each pair of spaces, a degeneration is constructed with the goal of comparing absolute invariants of one with relative invariants of the other. Then we prove that the virtual dimension of one of the moduli spaces of relative stable maps appearing in the degeneration formula is negative if there are nontrivial contacts with the relative divisors. Next a second application of the degeneration formula compares such relative invariants with the absolute invariants of the same space. This sequence of comparing results implies the following theroem:

Theorem 1.1. *Suppose that S and its blow-up $\tilde{S}$ of S at a smooth point p both are Fano surfaces. Let $\beta \in H_2(S, \mathbb{Z})$. Then for any genus g, we have*

$$K^S_{g,\beta} = K^{\tilde{S}}_{g,p!(\beta)}, \tag{5}$$

where $p : \tilde{S} \longrightarrow S$ is the natural projection of the blowup.

2. Donaldson-Thomas invariants of blowups of surfaces

Let S be a smooth surface and K_S its canonical bundle. Denote by $Y_S = \mathbb{P}(K_S \oplus \mathcal{O})$ the projective bundle completion of the total space of the canonical bundle K_S. The Donaldson-Thomas theory of Y_S is well defined in every rank. Let $\gamma_i \in H^*(Y_S)$, $i = 1, \cdots, r$. Denote by $\tilde{\tau}_{k_i}(\gamma_i)$ the associated descendent fields in Donaldson-Thomas theory. For $\beta \in H_2(Y_S, \mathbb{Z})$ and an integer $n \in \mathbb{Z}$, denote by $\langle \tilde{\tau}_{k_1}(\gamma_1), \cdots, \tilde{\tau}_{k_r}(\gamma_r)\rangle^{Y_S}_{n,\beta}$ the descendent Donaldson-Thomas invariant of Y_S. Denote by $Z'_{DT}(S; q)_\beta$ the reduced partition function for the Donaldson-Thomas theory of the local Calabi-Yau geometry of S.

Denote by $p : \tilde{S} \longrightarrow S$ the natural projection of the blow-up of S at a smooth point $p_0 \in S$. Let $\beta \in H_2(S, \mathbb{Z})$ and $p!(\beta) = PDp^*PD(\beta) \in H_2(\tilde{S}, \mathbb{Z})$. Similar to the case of local GW, we observed that the Donaldson-Thomas invariants of Y_S of degree β is equal to the Donaldson-Thomas invariants of $Y_{\tilde{S}}$ of degree $p!(\beta)$.

Theorem 2.1. *Suppose that S is a smooth surface and $\tilde{S}$ is the blown-up surface of S at a smooth point p. Let $\beta \in H_2(S, \mathbb{Z})$. Then we have*

$$Z'_{DT}(S; q)_\beta = Z'_{DT}(\tilde{S}; q)_{p!(\beta)}, \tag{6}$$

where $p : \tilde{S} \longrightarrow S$ is the natural projection of the blowup.

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