Twisted Courant algebroids and Transgressions

Zhangju Liu
Department of Mathematics, Peking University

Abstract

The notion of twisted Courant algebroids by a closed 4-form was introduced by Hansen and Strobl in *First Class Constrained Systems and Twisting of Courant Algebroids by a Closed 4-form, arXiv:0904.0711v1* from their study of three dimensional sigma models with a Wess-Zumino term. In this talk, we show that twisted Courant algebroids give rise to Leibniz 2-algebras. In particular, every transitive twisted Courant algebroid can be constructed by twisting the standard Courant bracket with a 3-form coming from the transgression of the first Pontryagin class of the associated quadratic Lie algebroid.

A twisted Courant algebroid by a closed 4-form H is denoted by $(E,\{\cdot,\cdot\},\rho,\langle\cdot,\cdot\rangle,H)$, which E is a vector bundle over M with a fiber metric $\langle\cdot,\cdot\rangle$ (so one can identify E with E^*) ρ is a bundle map $\rho : E \longrightarrow TM$ (called anchor), $\{\cdot,\cdot\}$ is a bilinear operation (Dorfman bracket) on $\Gamma(E)$ and $H \in \Omega^4(M)$ is a closed 4-form such that for all $e, e_1, e_2, e_3 \in \Gamma(E)$,

$$\{e,e\} = \frac{1}{2}\rho^* d\langle e,e\rangle;$$

$$\rho(e_1)\langle e_2,e_3\rangle = \langle\{e_1,e_2\},e_3\rangle + \langle e_2,\{e_1,e_3\}\rangle;$$

$$\rho^* H(e_1,e_2,e_3,e) = \langle Jac(e_1,e_2,e_3),e\rangle,$$

where $Jac(e_1,e_2,e_3) = \{e_1,\{e_2,e_3\}\} - \{\{e_1,e_2\},e_3\} - \{e_2,\{e_1,e_3\}\}$. If H is zero, a twisted Courant algebroid $(E,\langle\cdot,\cdot\rangle,\{\cdot,\cdot\},0)$ is just a usual Courant algebroid.

It is known that every Courant algebroid gives rise to a 2-term L_∞-algebra (Lie 2-algebra). With the Dorfman bracket, the section space $(\Gamma(E),\{\cdot,\cdot\})$ is a Leibniz algebra. In *Twisted Courant algebroids and Leibniz 2-algebras, arXiv:1012.5515*, with Y.-H. Sheng, we we proved that every twisted Courant algebroid by a closed 4-form $(E,\langle\cdot,\cdot\rangle,\{\cdot,\cdot\},\rho,H)$ gives rise to a Leibniz 2-algebra, whose degree-1 part is $\Omega^1(M)$, degree-0 part is $\Gamma(E)$, differential is $\rho^* : \Omega^1(M) \longrightarrow \Gamma(E)$, the bilinear bracket operation l_2 is given by

$$\left\{\begin{array}{rcl} l_2(e_1,e_2) & \triangleq & \{e_1,e_2\}, \quad \forall\ e_1,e_2 \in \Gamma(E),\\ l_2(e,\xi) & \triangleq & L_{\rho(e)}\xi,\ \xi \in \Omega^1(M),\\ l_2(\xi,e) & \triangleq & -i_{\rho(e)}d\xi, \end{array}\right. \tag{1}$$

and the trilinear map l_3^H is given by $l_3^H(e_1,e_2,e_3) \triangleq i_{\rho(e_1)\wedge\rho(e_2)\wedge\rho(e_3)}H$.

When the closed 4-form H is exact, i.e. $H = dh$ for some 3-form h, a twisted Courant algebroid by 4-form H can be obtained by twisting the bracket of a Courant algebroid. Let $(E,\langle\cdot,\cdot\rangle,\{\cdot,\cdot\},\rho)$ be a Courant algebroid, consider the deformation of the Dorfman bracket:

$$\{e_1,e_2\}_\epsilon = \{e_1,e_2\} + \epsilon\rho^*(i_{\rho(e_1)\wedge\rho(e_2)}h).$$

It is easy to see that $(E, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}_\epsilon, \rho, \ \epsilon dh)$ is a twisted Courant algebroid by the closed 4-form ϵdh. A twisted Courant algebroid $(E, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}, \rho, H)$ is said to be *exact* if there is the following exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \longrightarrow 0.$$

In this case, as a vector bundle, $E \cong TM \oplus T^*M =: \mathcal{T}$ and one can transfer all the structures from E to $\mathcal{T}$. The anchor ρ and the fiber metric $\langle\cdot,\cdot\rangle$ are given by

$$\begin{array}{rcl} \rho(X+\xi) & = & X, \\ \langle X+\xi, Y+\eta\rangle & = & \xi(Y)+\eta(X). \end{array}$$

The bracket $\{\cdot,\cdot\}$ is given by

$$\{X+\xi, Y+\eta\} = [X,Y] + L_X\eta - i_Y d\xi + h(X,Y,\cdot),$$

for some 3-form $h \in \Omega^3(M)$. We denote this bracket by $\{\cdot,\cdot\}_h$ below. In fact, every exact twisted Courant algebroid is isomorphic to the twisted Courant algebroid $(\mathcal{T}, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}_h, \rho, dh)$ given above, i.e., the closed 4-form of an exact twisted Courant algebroid must be exact.

A *transitive* twisted Courant algebroid means that $\rho(E) = TM$. As a vector bundle, E is isomorphic to $TM \oplus \mathcal{G} \oplus T^*M$ where $\mathcal{G}$ is a bundle of quadratic Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle^{\mathfrak{g}})$ with ad-invariant pseudo-metric $\langle\cdot,\cdot\rangle^{\mathcal{G}}$. By such a decomposition, the Dorfman bracket of two vector fields is as follows:

$$\{X,Y\} = [X,Y] + R(X,Y) + h(X,Y,\cdot)$$

where h is a 3-form and $R : \wedge^2 TM \longrightarrow \mathcal{G}$ is the curvature. The Jacobi identity is controlled by

$$H = dh - \langle R \wedge R\rangle .$$

The cohomology class $[\langle R\wedge R\rangle] \in H^4(M)$ is called the **first Pontryagin class** of the associated quadratic Lie algebroid $TM \oplus \mathcal{G} \cong E/\rho^*(T^*M)$. That is, a transitive twisted Courant algebroid comes from twisting a Courant algebroid by a 3-form if and only if the first Pontryagin class of the associated quadratic Lie algebroid is trivial. This provides a way to find nontrivial examples of twisted Courant algebroids.

Let G be a Lie group with its Lie algebra $\mathfrak{g}$ and P a principal G-bundle over M. The Atiyah algebroid of the principal fiber bundle P is

$$0 \to \mathcal{G} \to TP/G \to TM \to 0.$$

The bundle $TP/G \cong TM \oplus \mathcal{G}$ by a connection θ with its curvature R. Now suppose that $\langle\cdot,\cdot\rangle$ is an invariant inner product on $\mathfrak{g}$. In this case, we have the first Pontryagin class as well as its transgression are given as follows:

$$[\langle R\wedge R\rangle] \in H^4(M), \quad \langle \theta \wedge R\rangle \in \Omega^3(P)$$

such that $d\langle\theta\wedge R\rangle = \pi^*\langle R\wedge R\rangle$ where $\pi : P \to M$ is the projection. With X.-M. Xu, we proved that the twisted Courant algebroid structure on $TM \oplus \mathcal{G} \oplus T^*M$ by the 4-form $\langle R\wedge R\rangle$ can be reduced from the standard Courant algebroid on $TP \oplus T^*P$ by use of the 3-form $\langle\theta\wedge R\rangle$ to twist the Courant bracket.