Quantum Tunneling and Khovanov Homology

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In 1985 V. I. Jones discovered new knot invariant, the Jones polynomials, $J_q(K)$. The invariant came from braid group representations.

In 1989 E. Witten gave a three dimensional gauge theory construction of the knot invariant. The underlying gauge theory is the Chern-Simons gauge theory. The Jones polynomials are the expectation values of the Wilson loops of the Chern-Simons gauge theory.

In 2000, M. Khovanov associated to each link $L = \cup_i K_i$ a vector space K_L, such that

$$K_L = \oplus_{m,n} K^{m,n}$$

with two commuting operators F and P. The Jones polynomials can be constructed as Witten index:

$$J_q(L) = Tr_{K_L}(-1)^F q^P.$$

From this construction one has naturally that $J_q(L)$ are polynomials of integer coefficients.

A natural problem arises: Is there a field theory construction of the Khovanov homology?

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The recent work of Witten answers this question by using a series of field theories constructions and dualities.

To simplify one considers the three dimensional manifold $M = \mathbf{R}^3$. Let $V = \mathbf{R}^3 \times \mathbf{R}_+$. Let E be a principal bundle over V with gauge group G. Let A be a connection and ϕ be a g valued one forms. We then have the following equations:

$$(F - \phi \wedge \phi + t d_A \phi)^+ = 0,$$

$$(F - \phi \wedge \phi - t^{-1} d_A \phi)^- = 0,$$

$$d_A(*\phi) = 0.$$

where $P = \int_V TrF \wedge F$ is the instanton number, $t \in \mathbf{R} \cup \infty$. Let a_n be the number of solutions of the above equations with a fixed instatnton number $P = n$ with proper boundary conditions. We call this system of partial differential equations (*).

Then Witten claimed that

$$J_q(L) = \Sigma_n a_n q^n$$

is equal to Jones polynomial.

This claim can be justify by finding solutions of the PDE (*). By slicing knot into braids and considering (*) in three dimension one gets into Opers. The solutions correspond to the Bethe ansatz of the Gaudin model. The Bethe ansatz is also given by critical points of the Yang-Yang functional. This way the conformal blocks are constructed by solutions of the Bethe ansatz. One gets representations of the braid groups from this construction. For the case of $SU(2)$ one gets the skein relation as well. This way one recovers the Jones polynomials.

The Yang-Yang functional itself is obtained as superpotential of the effective theory. The results also shed lights on the underlying gauge theory.

To get Khovanov homology one goes to one dimension higher. Let $W = V \times \mathbf{R}, x_0 \in \mathbf{R}$. By replacing ϕ_y by D/Dx_0 one gets a system of PDE (**) in five dimensions. The solutions of equations (*) can be considered as time-independent solutions of (**) using x_0 as time variable.

The space of classical vacua consists of time independent solutions of (*). They are graded by the instanton number.

The time-dependent solutions of (**) whose past and future are solutions of (*) with difference one of the instanton numbers defines a differential of the above graded vector space of classical vacua. The time-dependent solutions gives quantum tunneling between classical solutions.

This defines a complex whose homology is the Khovanov homology.