2|2-SUPER HERMITIAN FIELD THEORY AND JACOBI FORMS

QIN LI

In the 1980s, Segal suggests that conformal field theories(CFT) over a manifold M might provide cocycles for the elliptic cohomology of M. Stolz and Teichner have developed this idea by adding super symmetry into the picture. Their conjecture is that the space of certain 2 dimensional super symmetric QFT's give the spectrum of topological modular forms(TMF). One of the main evidence is that, they have shown in [ST3] that the partition function of a supersymmetric 2|1-dimensional Euclidean Field theory is a weak integral modular form. They defined the super symmetry using a rigid geometric structure, which they call the super Euclidean structure. It's an algebraic fact of this rigid geometry that leads to the key properties of the partition functions, including holomorphicity and integrality. The physics corresponding to this theorem is that the partition function of the non-linear super symmetric σ-model associated to a closed string manifold M is a modular form.

In physics, field theories with one more super symmetry, namely, $N = 2$ super conformal field theories have partition functions which depends on one more variable other than the modular parameter. Physicists call this extra variable $U(1)$ angle, which comes from the $U(1)$ current of the $N = 2$ super conformal algebra. The partition function of a $N = 2$ super symmetric σ-model whose target space M is a given Calabi-Yau manifold is an integral weak Jacobi form of weight 0 and index $d/2$, where $d = dim_{\mathbb{C}}M$. This is called the elliptic genus of M, given by the following formula.

$$\chi(M;\tau,z) = y^{d/2}\int_M ch(\mathbf{E}_{q,y})td(T_M), q = e^{2\pi i\tau}, y = e^{2\pi iz}$$

$\mathbf{E}_{q,y}$ is given by

$$\mathbf{E}_{q,y} = \bigotimes_{n=0}^{\infty}\bigwedge_{-y^{-1}q^n} TM^* \otimes \bigotimes_{n=1}^{\infty}\bigwedge_{-yq^n} TM \otimes \bigotimes_{n=1}^{\infty} S_{q^n}T_M^* \otimes \bigotimes_{n=1}^{\infty} S_{q^n}T_M$$

On the other hand, from a homotopy theoretic point of view, Ando, French and Ganter showed in [AFG] that the sigma orientation of [Ho] gives rise to a two-variable genus of SU-manifolds M, which is just the elliptic genus.

A natural question is, if there is a corresponding theory in the framework of Stolz and Teichner, from which we can see the two variable elliptic genus of Calabi-Yau manifolds. We give in this paper evidence that the answer is yes. As Stolz and Teichner did in [ST3], we define a rigid geometric structure, which we call super Hermitian structure. This is

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the complex analogue of super Euclidean structure. We find that the space of flat $U(1)$ bundles has a close relation to the super Hermitian structure, and hence give a geometric interpretation of the other variable other than the modular variable as the parameter space of flat $U(1)$ bundles on elliptic curves, i.e., the Jacobian. Unlike super symmetric $2|1$ theories, the partition function of a $2|2$ theory is not necessarily holomorphic with respect to the two variables. What physicists call the elliptic genus of an $N=2$ super conformal field theory is a holomorphic component of the full partition function of the theory. We show that super Hermitian field theories satisfying certain "partial holomoprhic" properties has the partition function holomorphic with respect to both variables and is in fact an integral weak Jacobi form.

References

[AS] O. Alvarez and I.M.Singer, *Beyond the elliptic genus*, Nuclear Physics B, Volume 633, Issue 3, 1 July 2002, 309-344.

[AFG] M. Ando, C.P. French, N. Ganter, *The Jacobi orientation and the two-variable elliptic genus*, available at arxiv:math.AT/0605554

[DMVV] R.Dijkgraaf, G.Moore, E.Verlinde and H.Verlinde, *Elliptic Genera of Symmetric Products and Second Quantized Strings*, Communications in Mathematical Physics , Volume 185, Number 1, 197-209

[Gr] V. Gritsenko, *Complex vector bundles and Jacobi forms*, Automorphic forms and L-functions (Kyoto, 1999). Surikaisekikenkyusho Kokyuroku No. 1103 (1999), 71–95

[Ho] M. Hopkins, *Topological modular forms, the Witten genus, and the theorem of the cube*, In Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zurich, 1994), pages 554C565, Basel, 1995. Birkhauser.

[KYY] T. Kawai, Y. Yamada, and S-K. Yang, *Elliptic genera and $N=2$ superconformal field theory*, Nuclear Physics B Volume 414, Issues 1-2, 14 February 1994, 191–212

[Se] G. Segal, *The definition of conformal field theory.* Topology, geometry and quantum field theory, 423 C 577, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004

[St] S. Stolz, *A conjecture concerning positive Ricci curvature and the Witten genus*, Math. Ann. 304(1996), **4**, 785–800.

[ST1] S. Stolz and P. Teichner, *What is an elliptic object?* Topology, geometry and quantum field theory, 247–343, London Math. Soc. Lecture Note Ser., **308**, Cambridge Univ. Press, Cambridge, 2004.

[ST2] S. Stolz and P. Teichner, *Super symmetric field theories and integral modular forms*, preprint

[ST3] S. Stolz and P. Teichner, *Super symmetric field theories and generalized cohomology*, preprint

[ST4] S. Stolz and P. Teichner, *Traces in monoidal categories*, preprint

[Wi1] E. Witten,*Phases of N=2 theories in two dimensions*, Mirror symmetry, II, 143–211, Amer. Math. Soc., Providence, RI, 1997

[Wi2] E. Witten,*On the Landau-Ginzburg description of N=2 minimal models*, Chen Ning Yang, 429–444, Int. Press, Cambridge, MA, 1995

[Wi3] E. Witten, *Phases of N=2 theories in two dimensions*, Nuclear Phys. B 403 (1993), no. 1-2, 159-222

[Wi4] E. Witten, *Two-dimensional models with (0,2) supersymmetry: perturbative aspects*, Adv. Theor. Math. Phys. **11** (2007), no. 1, 1-63

Supersymmetric field theories and Modular forms

Qin Li

Sanya, Dec, 2011

Definition of Witten genus

- The Witten genus $\phi_W(M)$ of a closed orientable n-dimensional manifold M is the power series in q defined by

$$\phi_W(M) := \langle \hat{A}(TM) ch(\bigotimes_{m=1}^{\infty} S_{q^m}((TM - n) \otimes \mathbb{C})), [M] \rangle$$

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- In particular , if M is spin, by Index Theorem, the Witten genus can alternatively be described as

$$Index(D(M) \otimes \bigotimes_{m=1}^{\infty} S_{q^m}((TM - n) \otimes \mathbb{C})$$

Witten genus

- When M is spin, the coefficients of q^m must be integers
- (Zagier) $\phi_W(M)$ is the q-expansion of a modular form of weight $\frac{n}{2}$ if the first Pontrjagin class $p_1(M) = 0$
- $\phi_w(M_1 \coprod M_2) = \phi_w(M_1) + \phi_w(M_2)$
  $\phi_w(M_1 \times M_2) = \phi_w(M_1) \cdot \phi_w(M_2)$

Summary

$\phi_W(M)$ is an *integral* modular form if

1. M is spin

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2. $p_1(M) = 0$

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  1. the partition function of a 2-dim super-symmetric quantum field theory, which is known as *non-linear σ-model*
  2. or as the S^1-equivariant index of a Dirac-like operator on the free loop space of M
- The characteristic class $p_1(M)$ is the obstruction to quantization.
- Unfortunately, neither of these have been rigourously defined in mathematics.

Topological interpretation

Hopkins and Miller developed a generalized cohomology theory TMF^*, whose coefficient groups $TMF^*(pt)$ are rationally isomorphic to the ring of integral modular forms. More precisely, there is a graded ring homomorphism

$$TMF^*(pt) \to MF^* = \bigoplus_{n \in \mathbb{Z}} MF^n$$

where MF^n is the abelian group of integral modular forms of weight $-\frac{n}{2}$

Theorem

There is a topological push-forward map $\pi_* : TMF^0(M) \to TMF^{-k}(X)$, *if* $\pi : M \to X$ *is a fiber bundle of k-dimensional string manifolds.*

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1. The image of 1 in $TMF^{-k}(M)$ under the push-forward map
2. There is a ring spectra map

$$\sigma : MString \to TMF$$

which is called the σ-orientation. And associated homomorphism of homotopy groups

$$\sigma_* : \pi_k(MString) \to \pi_k(TMF)$$

$$\phi_w(M) = \sigma_*([M])$$

TMF vs. elliptic cohomology

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  TMF is the global section of the structure sheaf of E_∞-rings over the derived moduli stack of elliptic curves

Geometric interpretation?

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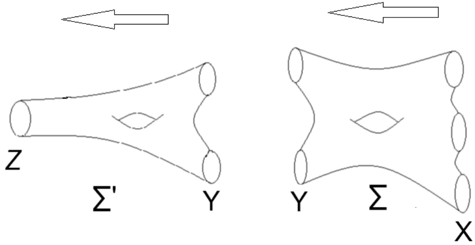
- What is the geometric interpretation of TMF? Can we get some intuition from physics?
- For ordinary cohomology and K-theory, the geometric interpretations are known:
  - Singular cohomology : differential forms
  - K-theory: vector bundles
- Segal: 2-dimensional conformal field theory over M could be used as cocycles for some elliptic cohomology theory. It would associate Hilbert spaces to loops in M, and Hilbert-Schmidt operators to conformal surfaces (with boundary) in M.

Definition

The d-dim bordism category has
Objects: $(d-1)$-dim closed manifolds
Morphism from X to Y: d-dim bordisms from X to Y



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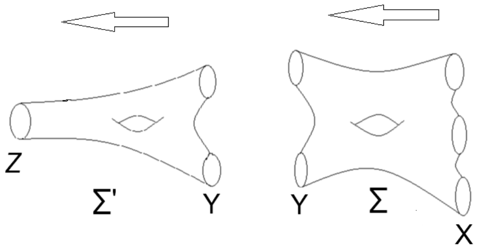
Definition(Preliminary)

A d-dim field theory is a smooth symmetric monoidal functor $d\text{-Bord} \to TV$



Definition of field theory

A d-dimensional field theory E assigns to any closed smooth $(d-1)$-manifold Y a topological vector space $E(Y)$, and to a d-dimensional bordism Σ from Y_0 to Y_1 a continuous linear map $E(\Sigma) : E(Y_0) \to E(Y_1)$, such that the following four requirements are satisfied:

1. If Σ_1 and Σ_2 are bordisms from X_0 to X_1 which are isomorphic relative to boundaries, then $E(\Sigma_1) = E(\Sigma_2)$

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2. Let Σ be a bordism from X to Y, and Σ' a bordism from Y to Z, then
$$E(\Sigma' \cup_Y \Sigma) = E(\Sigma') \circ E(\Sigma)$$

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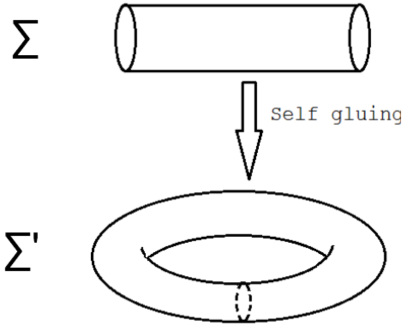
4. $E(X \coprod X') = E(X) \otimes E(X'), E(\Sigma \coprod \Sigma') = E(\Sigma) \otimes E(\Sigma')$

Partition function

- Partition function
  - Requirement (4) implies that $E(\emptyset)$ is a 1-dimensional vector space
  - Let Σ be a closed d-manifold, we can consider Σ as a bordism from $\emptyset$ to itself, and hence $E(\Sigma)$ is a complex number.
  - By the smoothness, we get a smooth function on the moduli space of closed d-manifolds, which is called the *partition function* of E.



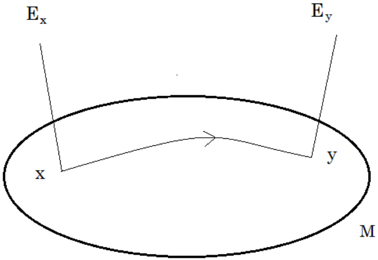
Theorem

Let $E(\Sigma) \in Hom(H, H)$ be of trace class , then $E(\Sigma') \in \mathbb{C}$ is the trace of $E(\Sigma)$

Stolz-Teichner Program

To realize cohomology theories by field theories.

- K-theory: vector bundles with connections over M can be thought of as a 1-dim field theory over M



- (Stolz-Teichner) de Rham cohomology of M can be represented by concordance classes of $0|1$-dim SUSY field theories over M

Stolz-Teichner's theorem

Theorem

(Stolz-Teichner)Let E be a Euclidean field theory of dimension $2|1$ and degree n, then the partition function $Z_E : \mathbb{H} \to \mathbb{C}$ is a holomorphic modular form of weight n with integral coefficients.

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The proof of the theorem consists of the four ingredients:

1. Modular transformation property
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2. Holomorphicity
3. Integral Fourier coefficients
4. Independence of length scale

Complex elliptic genus

Let M be an (almost) complex manifold of dimension d, then we can define a two-variable elliptic genus in a manner similar to the Witten genus.

$$Elliptic\ genus(M;\tau,z) = y^{\frac{d}{2}}\langle ch(E_{q,y})td(TM),[M]\rangle$$

where $q = e^{2\pi i\tau}, y = e^{2\pi iz}, E_{q,y}$ is given by

$$E_{q,y} = \bigotimes_{n=0}^{\infty} \bigwedge_{-y^{-1}q^n} TM^* \otimes \bigotimes_{n=1}^{\infty} \bigwedge_{-yq^n} TM \otimes \bigotimes_{n=1}^{\infty} S_{q^n} TM^* \otimes \bigotimes_{n=1}^{\infty} S_{q^n} TM$$

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Proposition

If M has trivial first Chern class, i.e., $c_1(M) = 0$, then *Elliptic genus*$(M;\tau,z)$ is a Jacobi form.

Jacobi forms

Definition

A Jacobi form of weight k and index t on $SL_2(\mathbb{Z})$ is a holomorphic function
$\Phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$, satisfying the two transformation properties:

- $\Phi(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}) = (c\tau + d)^k e^{\frac{2\pi itcz}{c\tau+d}} \Phi(\tau, z)$, for $(a, b; c, d) \in SL_2(\mathbb{Z})$
- $\Phi(\tau, z + \lambda\tau + \mu) = e^{-2\pi it(\lambda^2\tau + 2\lambda z)} \Phi(\tau, z)$ for $(\lambda, \mu) \in \mathbb{Z}^2$
- $\Phi(\tau, z)$ has a Fourier expansion of the form $\Phi(\tau, z) = \sum a_{m,n} e^{2\pi im\tau} e^{2\pi inz}$

Geometric understanding of Jacobi forms: Sections of certain line bundles on the universal tori $\mathbb{C} \times \mathbb{H}/\mathbb{Z}^2 \times SL_2(\mathbb{Z})$

Super Hermitian structure

- Physical interpretation of elliptic genus: the partition function of the $N = 2$ super-conformal σ-model with target M.
- Our goal is to show rigorously that the partition function of a $2|2$-field theory is a Jacobi form in the framework of Stolz-Teichner program.

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- Super Hermitian structure
  1. Local model: $H = \mathbb{C} \times \Pi\mathbb{C}^2$
  2. "Isometry" group of the local model:

$$Iso(H) = Rotation \ltimes Translation$$

Super Hermitian structure

- Translation: $H \times H \to H$

$$(z, \bar{z}, \theta_1, \theta_2) \times (z', \bar{z}', \theta_1', \theta_2')$$

$$\mapsto (z + z', \bar{z} + \bar{z}' + \theta_1\theta_2' + \theta_2\theta_1', \theta_1 + \theta_1', \theta_2 + \theta_2').$$

- Rotation:

$$U(1) \times U(1) \times H \to H$$

$$(b, c) \times (z, \bar{z}, \theta_1, \theta_2) \mapsto (bcz, \bar{b}\bar{c}\bar{z}, \bar{b}\theta_1, \bar{c}\theta_2)$$

Moduli of closed super Hermitian manifolds

Definition

A super Hermitian structure on a supermanifold M is a maximal atlas of M such that, all charts are open subsets of H and all transition functions belong to the super Hermitian group $\mathrm{Isom}(H)$.

Example

Given a flat $U(1)$ bundle L on a pointed tori M, the super space $\Pi(L \oplus L^{-1})$ has a canonical structure of super Hermitian manifolds.

Partition function

- On a closed, pointed Riemann surface M of genus 1, we have the following identifications:
  - *flat* $U(1)$ *bundles*
  - *holomorphic line bundles with vanishing* c_1
  - M

Proposition

The moduli space of closed super Hermitian manifolds is isomorphic to the moduli space of line bundles on pointed tori with vanishing c_1.

Theorem

The partition function of a super Hermitian field theory E $Z_E : \mathbb{R}_+ \times \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ *have the following properties:*

1. Z_E *doesn't depend on the first variable* l

Hence $Z|_{\{1\}\times\mathbb{H}\times\mathbb{C}}$ *is an integral Jacobi form*

Theorem

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2. Z_E *is holomorphic with respect to the two variables* τ *and* z
3. Z_E *has integral Fourier coefficients*
4. Z_E *satisfies the transformation properties of Jacobi forms*

Hence $Z|_{\{1\}\times\mathbb{H}\times\mathbb{C}}$ *is an integral Jacobi form*

Thank you!