Wall-crossing and free fermion

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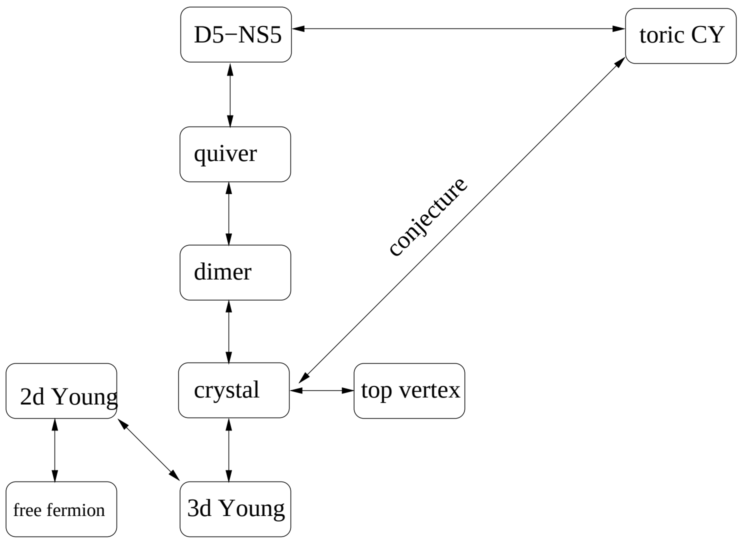
January 15, 2012

In this talk we describe some background of wall-crossing and the way we approach this problem via free fermion technique.



There is a chain of dualities from D5-NS5 brane system to a statistical crystal model. From D5-NS5 system to quiver diagram is based on the gauge theory on D5 brane where the NS5 brane is added to break the supersymmetry to $\mathcal{N} = 2$. D5 brane is chopped to different pieces. Each region of $(N, 0)$ brane corresponds to a quiver node. The open strings attached in between those regions represent the arrows from a node to another or to itself. From the quiver diagram we can build the universal quiver. Every face surrounded by clockwise or counterclockwise arrows forms the dimer node. From a dimer model we can build crystal model. The partition of a melting crystal model is the partition of the BPS bound states of the original D5-NS5 brane system.

There is a conjecture between the partition of the crystal and the bound states of the D-brane in the toric Calabi-Yau 3-fold or in the mathematical language Donaldson-Thomas invariants.

The two-dimensional complex free fermion theory has been well studied for long time. Recent ten years it has shown its power in string theory and algebraic geometry. In this talk we briefly review the vertex operators in the two-dimensional complex free fermion theory.

Since some simple crystal, such as cubic or colored cubic crystal, is related to 3D Young diagram. We talk about how to relate free fermion to 3D Young diagram. We know that a 3D Young diagram can be split into some 2D Young diagrams by slicing diagonally. The partitions corresponding to these 2D Young diagrams satisfy the interlacing condition. For each 2D Young diagram we can associate a fermion state. The transfer matrix is built up by the vertex operator. For cubic crystal the partition of melting crystal is called MacMahon function and there is already a developed way to calculate it via vertex operators.

What we want to do is using the free fermion technique to study the wall-crossing feature. In the unrefined case there has been some study. We consider in the refined case, i.e. motivic case the free fermion and wall-crossing feature for conifold and generalized conifold. We introduce the arrow diagram for a series of vertex operators and we define the shuffling mechanism to compute the motivic Donaldson-Thomas invariants in different chambers besides non-commutative Donaldson-Thomas (NCDT) chamber. We also developed a method to calculate the BPS partition in different chambers for the generalized conifold.

In the end we propose some open questions:

- Refined MacMahon

  We have seen that by using crystal melting model we can recover the refined BPS partition function except for the refined MacMahon function. Is it possible to adjust the counting weight to recover the exact refined BPS partition function?

- Crystal and motivic DT

  What is the intrinsic reason that counting crystal can reproduce the refined BPS partition function?

- Invariants and symmetries

  How is Heisenberg algebra categorification going to enter this story? Namely, how can we find operators which satisfy Kontsevich-Soibelman motivic wall-crossing formula?

Free fermion and wall-crossing

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December 19, 2011

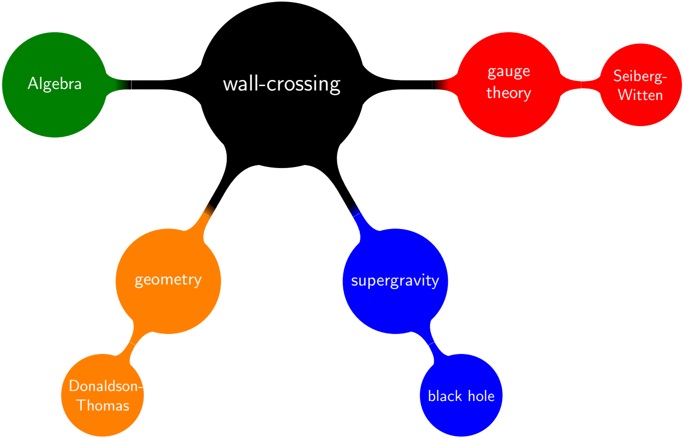
Motivation

- For string theorists
  It is called BPS states counting which means counting the D brane bound states.

Motivation

- For string theorists
  It is called BPS states counting which means counting the D brane bound states.
- For mathematician
  Invariants of moduli spaces of virtual dimension zero associate with Calabi-Yau 3-fold X
  1. holomorphic curves in X with fixed genus and degree, e.g. Gromov-Witten invariants
  2. coherent sheaves with a fixed Chern character, e.g. Donaldson-Thomas (DT) invariants

The world of wall



Kontsevich-Soibelman wall-crossing formula

There is a symplectomorphism transformation for functions over algebraic torus over lattice Γ

$$U_\gamma : X_{\gamma'} \longrightarrow X_{\gamma'}(1+\sigma(\gamma)X_\gamma)^{\langle\gamma',\gamma\rangle} \tag{1}$$

where U_γ can be expressed as the operator

$$U_\gamma = \exp\left(\sum_n \frac{\sigma(n\gamma)}{n^2}\{X_{n\gamma},\cdot\}\right) \tag{2}$$

Wall-crossing formula

$$\prod_\gamma^{\curvearrowright} U_\gamma^{\Omega(\gamma,u_+)} = \prod_\gamma^{\curvearrowright} U_\gamma^{\Omega(\gamma,u_-)} \tag{3}$$

Motivic wall-crossing formula

On the quantum algebraic torus the associative algebra is generated by $\hat{e}_\gamma$ s.t.

$$[\hat{e}_{\gamma_1}, \hat{e}_{\gamma_2}] = \left(q^{\frac{1}{2}\langle\gamma_1,\gamma_2\rangle} - q^{-\frac{1}{2}\langle\gamma_1,\gamma_2\rangle} \right) \hat{e}_{\gamma_1+\gamma_2} \tag{4}$$

The wall-crossing formula is

$$\prod_{\gamma}^{\curvearrowright} A_\gamma^{mot}(u_+) = \prod_{\gamma}^{\curvearrowright} A_\gamma^{mot}(u_-) \tag{5}$$

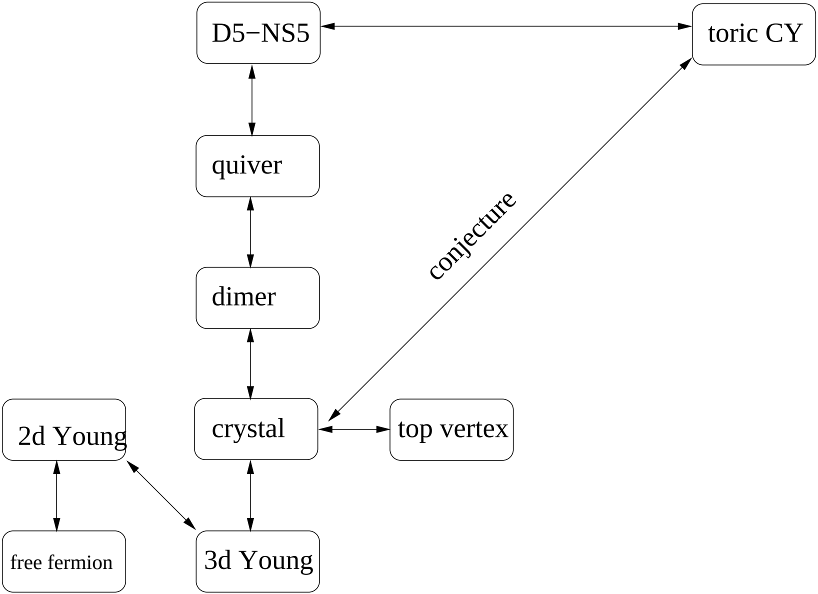
where A_γ^{mot} is a quantum analog of the classical symplectomorphism $U_\gamma^{\Omega(\gamma)}$.

Dualities of BPS counting

Free fermion for $\mathbb{C}^3$

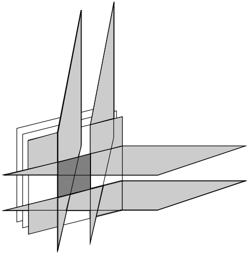
Refined wall-crossing for $\mathcal{O}(-1) \oplus_{\mathbb{P}^1} \mathcal{O}(-1)$

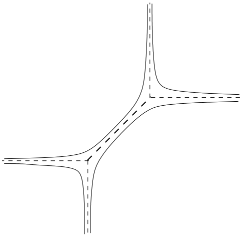
Open questions


D5−NS5
toric CY
quiver
conjecture
dimer
2d Young
crystal
top vertex
free fermion
3d Young

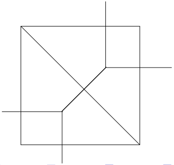
D5-NS5 brane and toric Calabi-Yau geometry

T-duality (mirror symmetry) between D5-NS5 brane configuration and toric diagram

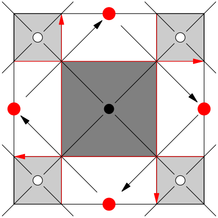
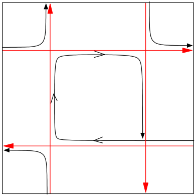
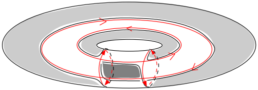




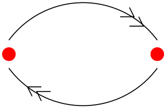
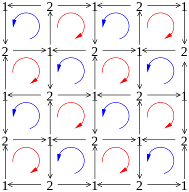
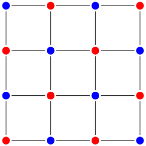
	0	1	2	3	4	5	6	7	8	9
D5	*	*	*	*		–		–		
NS5	*	*	*	*	–	–				
NS5	*	*	*	*			–	–		



Quiver and dimer model

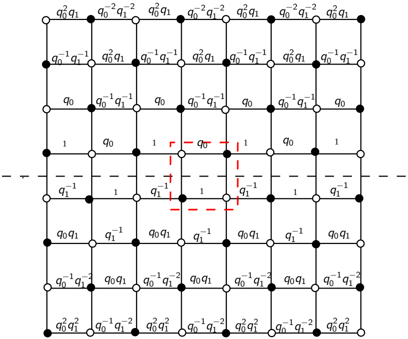




The quiver, universal quiver, and dimer diagram for conifold are

Crystal

(e) a screenshot of paper by Young and Bryan arXiv:0802.3948



(f) dimer assigned with weight

Conjecture:
The partition function of statiscal model of crystal melting is related to the BPS partition function.

Dualities of BPS counting

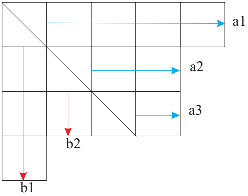
Free fermion for $\mathbb{C}^3$

Refined wall-crossing for $\mathcal{O}(-1) \oplus_{\mathbb{P}^1} \mathcal{O}(-1)$

Open questions

Free fermion and 2D Young diagram correspondence

- 2D Young diagram λ



- The correspondence

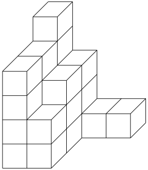
$$\lambda \Leftrightarrow |\lambda\rangle = \prod_{i=1}^{d(\lambda)} \psi^{*}_{-(a_i+\frac{1}{2})} \psi_{-(b_i+\frac{1}{2})} |0\rangle \tag{6}$$

where

$$a_i = \lambda_i - i, \quad b_i = \lambda_i^t - i \tag{7}$$

3D Young diagram Y_3

- Plane partitions
  Definition: 2D Young diagram with weakly decreasing number filling in rows and columns



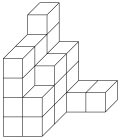
5	3	1	1
4	3		
4	2		

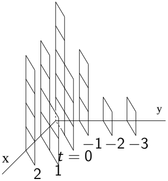
- MacMahon function is the generating function of Y_3

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = \sum_{\pi \in \text{all } Y_3} q^{|\pi|} \qquad (8)$$

2D and 3D Young diagram

- The diagonal diagram





- Interlacing condition of the diagonal slices

$$\begin{cases} \lambda(t) \succ \lambda(t+1) & t > 0 \\ \lambda(t) \succ \lambda(t-1) & t < 0 \end{cases} \tag{9}$$

where

$$\lambda \succ \mu \quad \Longleftrightarrow \quad \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \cdots \tag{10}$$

- Vertex operators

$$\Gamma_{\pm}(x) = \exp\left\{\sum_{n=1}^{\infty} \frac{x^n}{n}\alpha_{\pm n}\right\} \tag{11}$$

$\alpha_{\pm n}$ are the creation and annihilation operator for the bosonization of free fermion.

- Γ_{-} can be treated as the creation operator while Γ_{+} the annihilation one of nearest neighbor slices, i.e.

$$\Gamma_{\substack{- \\ +}}|\mu\rangle = \sum_{\substack{\lambda \succ \mu \\ \lambda \prec \mu}} |\lambda\rangle, \tag{12}$$

Commutation relation

$$\Gamma_{+}(x)\Gamma_{-}(y) = \frac{1}{1-xy}\Gamma_{-}(y)\Gamma_{+}(x) \tag{13}$$

Crystal melting for $\mathbb{C}^3$

- The statistics of a melting cubic crystal is identified with the partition function of D0-D6 bound states of $\mathbb{C}^3$
- MacMahon partition function

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = \quad (14)$$

$$\langle \underbrace{q^{L_0}\Gamma_+(1)\cdots q^{L_0}\Gamma_+(1)}_{\infty} q^{L_0} \underbrace{\Gamma_-(1)q^{L_0}\cdots\Gamma_-(1)q^{L_0}}_{\infty} \rangle$$

where we perform commutation relation of vertex operators.

$$\prec\lambda(-2)\prec\lambda(-1)\prec\lambda(0)\succ\lambda(1)\succ\lambda(2)\succ$$

Dualities of BPS counting

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Refined wall-crossing for $\mathcal{O}(-1) \oplus_{\mathbb{P}^1} \mathcal{O}(-1)$

Open questions

Marginal stability wall

- SUSY condition for mass and central charge of a state

$$M \geq |Z(\gamma)| \tag{15}$$

when "$=$", it is called a BPS state.

- Marginal stability wall

$$\gamma = \gamma_1 + \gamma_2, \quad Z(\gamma) = Z(\gamma_1) + Z(\gamma_2) \tag{16}$$

$$M_{2-\text{particle}} = (p_1 + p_2)^2 \geq M_1 + M_2 \tag{17}$$

$$M_{2-\text{particle}} \geq |Z_1| + |Z_2| \geq |Z_1 + Z_2| \tag{18}$$

Therefore 2-particle BPS states $M_{2-\text{particle}} = |Z_1 + Z_2|$ are separated from 1-particle BPS states $M_1 = |Z_1|, M_2 = |Z_2|$ unless $|Z_1 + Z_2| = |Z_1| + |Z_2|$, i.e., at the wall.

D0-D2-D6 bound states

- Charge lattice

D0	D2	D4	D6
H_0	H_2	H_4	H_6
q_0	Q	P	p^0
n	$-m$	0	1

$\gamma = ndV - m\beta + 1$

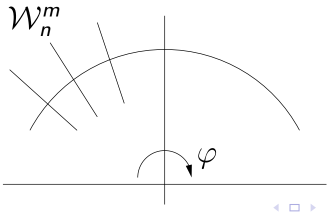
- Central charge of BPS states

$$Z(\gamma) = -\int_X \gamma \wedge e^{-t} \qquad (19)$$

where the complexified Kähler moduli is $t = z\mathcal{P} + \Lambda e^{i\varphi}\mathcal{P}'$

- Marginal stability walls in the moduli space of BPS states

$$ArgZ(\gamma) = ArgZ(\gamma_1) \qquad (20)$$



Index of BPS states

Define an index by

$$\Omega(\gamma) = \mathrm{Tr}_{\mathcal{H}^{\gamma}_{BPS,u}}(-1)^{2J}, \tag{21}$$

where $\gamma \in \Gamma$ the charge lattice and $J \in \mathfrak{so}(3)$ is the generator of rotations around any axis.

$$\Omega(\gamma) \sim \text{Euler characteristic of the stable sheaf moduli space}$$
$$\sim \text{Donaldson-Thomas Invariants}$$

Refined index is defined by [Dimofte & Gukov arXiv:0904.1420]

$$\Omega^{\mathrm{ref}}(\gamma; y) = \mathrm{Tr}_{\mathcal{H}^{\gamma}_{BPS,u}}(-y)^{2J} \tag{22}$$

$$\Omega^{\mathrm{ref}}(\gamma; y \to 1) = \Omega(\gamma) \tag{23}$$

Some description of "refined"

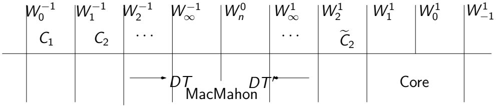
refined		quantum		motivic
y	$\leftrightarrow$	$-q^{\frac{1}{2}}$	$\leftrightarrow$	$\mathbb{L}^{\frac{1}{2}}$

- Physically we distinguish left and right spin, turing on Ω background
- Mathematically

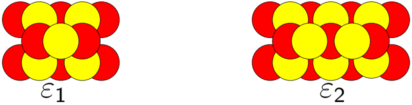
$$\begin{aligned}\Omega^{\text{ref}}(\gamma; y) &\sim \text{Poincare polynomial of the stable sheaf moduli spa} \\ &\sim \text{Refined/Motivic Donaldson-Thomas Invariants}\end{aligned}$$

Wall-crossing for the resolved conifold

- Wall



- 2-colored stone diagram for chamber 0 and 1



- The relation between 2D partition and free fermion is preserved. But for different slices the vertex operators have different arguments.

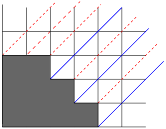
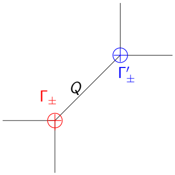


Figure: The red (dotted) lines denote the left or right moving of the slices, and the blue (solid) lines denote the up or down moving

Crystal melting for resolved conifold

Vertex operators

Toric diagram



$$\Gamma_\pm(x) \;=\; \exp\left\{\sum_n \frac{x^n}{n}\alpha_{\pm n}\right\}$$

$$\Gamma'_\pm(x) \;=\; \exp\left\{\sum_n \frac{(-1)^{n-1}x^n}{n}\alpha_{\pm n}\right\}$$

$$\Gamma'_{\substack{-\\+}}|\mu\rangle = \sum_{\substack{\lambda^t \succ \mu^t \\ \lambda^t \prec \mu^t}} |\lambda\rangle,$$

How do we count refined in the crystal melting?

arXiv: 1010.0348 collaborated with Haitao Liu

- Arrow diagrams for chamber n are

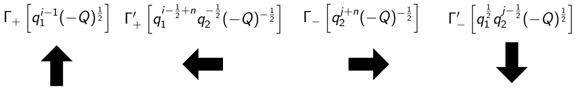
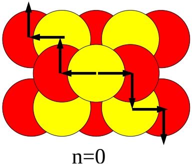
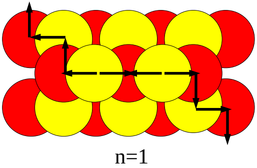


Figure: Arrow diagrams for chamber n of the conifold

- Stone diagrams with arrows are





Vertex Operators

If we define $\overline{A}_{\pm}(x)$ by

$$\overline{A}_{+}(x) := \widehat{Q}_{01}^{\frac{1}{2}}\widehat{Q}_{1}^{\frac{1}{2}}(\widehat{Q}_{01}^{\frac{1}{2}}\Gamma_{+}(x)\widehat{Q}_{01}^{-\frac{1}{2}})\widehat{Q}_{1}(\widehat{Q}_{02}^{\frac{1}{2}}\Gamma'_{+}(x)\widehat{Q}_{02}^{-\frac{1}{2}})\widehat{Q}_{1}^{-\frac{1}{2}}\widehat{Q}_{01}^{\frac{1}{2}},$$

$$\overline{A}_{-}(x) := \widehat{Q}_{02}^{\frac{1}{2}}\widehat{Q}_{1}^{\frac{1}{2}}(\widehat{Q}_{02}^{\frac{1}{2}}\Gamma_{-}(x)\widehat{Q}_{02}^{-\frac{1}{2}})\widehat{Q}_{1}(\widehat{Q}_{01}^{\frac{1}{2}}\Gamma'_{-}(x)\widehat{Q}_{01}^{-\frac{1}{2}})\widehat{Q}_{1}^{-\frac{1}{2}}\widehat{Q}_{02}^{\frac{1}{2}}$$

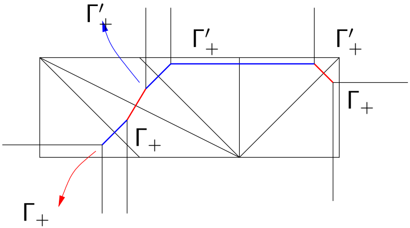
Then we can show that

$$\begin{aligned} Z_{NCDT}^{cystal} &:= \langle 0|\overline{A}_{+}(1)\cdots\overline{A}_{+}(1)\overline{A}_{-}(1)\cdots\overline{A}_{-}(1)|0\rangle \\ &= Z_{BPS}^{ref}(q_1, q_2, Q)|_{NCDT} \text{ (up to refined MacMahon).} \end{aligned}$$

For chamber n we construct

$$\begin{aligned}
\mathcal{Z}_{crystal} &= \langle 0| \prod_{i=1}^{\infty} \Gamma_{+}\left[q_1^{i-1}(-Q)^{\frac{1}{2}}\right] \Gamma'_{+}\left[q_1^{i-\frac{1}{2}+n} q_2^{\frac{1}{2}}(-Q)^{-\frac{1}{2}}\right] \\
&\times \Gamma_{-}\left[q_2(-Q)^{-\frac{1}{2}}\right] \Gamma'_{+}\left[q_1^{n-\frac{1}{2}} q_2^{-\frac{1}{2}}(-Q)^{-\frac{1}{2}}\right] \\
&\times \Gamma_{-}\left[q_2^{2}(-Q)^{-\frac{1}{2}}\right] \Gamma'_{+}\left[q_1^{n-\frac{3}{2}} q_2^{-\frac{1}{2}}(-Q)^{-\frac{1}{2}}\right] \\
&\cdots \times \Gamma_{-}\left[q_2^{n}(-Q)^{-\frac{1}{2}}\right] \Gamma'_{+}\left[q_1^{\frac{1}{2}} q_2^{-\frac{1}{2}}(-Q)^{-\frac{1}{2}}\right] \\
&\times \prod_{j=1}^{\infty} \Gamma_{-}\left[q_2^{j+n}(-Q)^{-\frac{1}{2}}\right] \Gamma'_{-}\left[q_1^{\frac{1}{2}} q_2^{j-\frac{1}{2}}(-Q)^{\frac{1}{2}}\right] |0\rangle \\
&= M_{\delta=1}(q_1, q_2) M_{\delta=-1}(q_1, q_2) \prod_{i,j=1}^{\infty} (1 - q_1^{i-\frac{1}{2}} q_2^{j-\frac{1}{2}} Q) \\
&\quad \prod_{\substack{i+j>n+1 \\ i,j\geq 1}}^{\infty} (1 - q_1^{i-\frac{1}{2}} q_2^{j-\frac{1}{2}} Q^{-1})
\end{aligned}$$

Generalized conifold



We have some progress on constracting the free fermion formalism of crystal melting for NCDT chamber and DT chamber and computing the partition function of D0-D2-D6 bound states.

The refined MacMahon function $\mathcal{M}(q_1, q_2)$ is defined by [Behrend, Bryan, Szendrői arXiv:0909.5088]

$$\mathcal{M}(q_1, q_2) = \prod_{d=0}^{6} \left(M_{\frac{d-3}{2}}(q_1, q_2) \right)^{(-1)^d b_d},$$

where b_d is the Betti number of the Calabi-Yau threefold X of degree d and $M_\delta(q_1, q_2)$ is the refined MacMahon function defined by

$$M_\delta(q_1, q_2) = \prod_{i,j=1}^{\infty} (1 - q_1^{i-\frac{1}{2}+\frac{\delta}{2}} q_2^{j-\frac{1}{2}-\frac{\delta}{2}})^{-1}.$$

Dualities of BPS counting

Free fermion for $\mathbb{C}^3$

Refined wall-crossing for $\mathcal{O}(-1) \oplus_{\mathbb{P}^1} \mathcal{O}(-1)$

Open questions

- Refined MacMahon
  We have seen that by using crystal melting model we can recover the refined BPS partition function except for the refined MacMahon function (defined by Behrend et. al.). Is it possible to adjust the counting weight to recover the exact refined BPS partition function?

- Crystal and motivic DT
  What is the intrinsic reason that counting crystal can reproduce the refined BPS partition function?

- Invariants and symmetries
  How is Heisenberg algebra categorification going to enter this story? Namely, how can we find operators which satisfy KS motivic wall-crossing formula?

Thank you!