Levin-Wen model and tensor categories

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Topological order is an important subject in condensed matter physics every since the discovery of fractional quantum Hall effect. It also has applications in topological quantum computing. We will focus on a large class of non-chiral topological order in this talk and show that the representation theory of tensor category enters the study of topological order at its full strength.

Relations between the bulk and the boundary have proved important for the understanding of quantum Hall states. For example, the bulk electron wave function for the Moore-Read state [MR91] is constructed using conformal blocks of a certain conformal field theory, which also describes the edge modes (under suitable boundary conditions). Based on the success of this and similar theories, one might erroneously conclude that the bulk-boundary correspondence is one-to-one. It is, however, known that the boundary properties are generally richer than those of the bulk; in particular, the same bulk can have different boundaries. This phenomenon appears in its basic form when both the bulk and the boundary are gapped[1], but it should be relevant to some quantum Hall states as well.

A simple example of a topological phase that admits a gapped boundary is a $\mathbb{Z}_2$ gauge theory. Its Hamiltonian realizations include certain dimer models [MS00, MSP02]. Read and Chakraborty [RC89] studied the quasiparticle statistics and other topological properties of the $\mathbb{Z}_2$ phase. An exactly solvable Hamiltonian in this universality class (the "toric code" model) was proposed by the first author [K97]. Already in this simple example, as shown by Bravyi and Kitaev [BK98], the bulk "toric code" system has two topologically distinct boundary types.

An analogue of the toric code for an arbitrary finite group G was also proposed in [K97]. Levin and Wen [LW04] went even further, replacing the group (or, rather, its representation theory) by a *unitary tensor category*[2]. Both models may be viewed as Hamiltonian realizations of certain TQFTs (or state sums in the sense of Turaev and Viro [TV92]), which were originally introduced to define 3-manifold invariants. Thus, the Kitaev model corresponds to a special case of the Kuperberg invariant [Ku91] (the general case was considered in [BMCA10]), whereas the Levin-Wen model corresponds to the Barrett-Westbury invariant [BW93].

Boundaries for the Kitaev model have been studied recently [BSW10]. In this paper, we will outline our constructions of all possible boundaries and defects in Levin-Wen models. We cannot readily defend the word "all" in this claim since our method is limited to a particular class of models. As a parallel development, boundary conditions for Abelian Chern-Simons theories have also been characterized [KS10a, KS10b]. However, an alternative approach is possible, where one postulates some general properties (such as the fusion of quasiparticles) and studies algebraic structures defined by those axioms. This idea has long been implemented for bulk 2d systems [FRS89, FG90], with the conclusion that the quasiparticles are characterized by a *unitary modular category* (see Appendix E in Ref. [K05] for review). A similar theory of gapped boundaries has been contemplated by the first author and will appear in a separate paper.

[1] A Hamiltonian is called "gapped" if the smallest excitation energy, i.e. the difference between the two lowest eigenvalues, is bounded from below by a constant that is independent of the system size.

[2] More exactly, a unitary finite spherical fusion category.

In the Levin-Wen model associated with a unitary tensor category $\mathcal{C}$, the bulk excitations are objects of the unitary modular category $Z(\mathcal{C})$, the monoidal *center* of $\mathcal{C}$ (a generalization of Drinfeld's double). This result follows from the original analysis by Levin and Wen, but we will derive it from a theory of excitations on a domain wall between two phases. Indeed, bulk excitations may be viewed as excitations on a trivial domain wall between two regions of the same phase. In the simpler case of a standard boundary between the Levin-Wen model and vacuum, the excitations are objects of the category $\mathcal{C}$. Thus, the boundary theory uniquely determines the bulk theory by taking the monoidal center. On the other hand, the bulk can not completely determine the boundary because the same modular category may be realized as the center of different tensor categories, say, $\mathcal{C}$ and $\mathcal{D}$. Nevertheless, the bulk theory uniquely determines the boundary theory up to Morita equivalence. This is the full content of the bulk-boundary duality in the framework of Levin-Wen models. We will explicitly construct a $\mathcal{D}$ boundary for the $\mathcal{C}$ Levin-Wen model using the notion of a module over a tensor category.

Besides bulk-boundary duality, we also emphasize an interesting correspondence between the dualities among bulk theories (as braided monoidal equivalences) and "transparent", or "invertible" domain walls (or defect lines). In particular, for Morita equivalent $\mathcal{C}$ and $\mathcal{D}$, we will construct a transparent domain wall between the $\mathcal{C}$ and $\mathcal{D}$ models. One can see explicitly how excitations in one region tunnel through the wall into the other region, which is just another lattice realization of the same phase. In the mathematical language, this tunneling process gives a braided monoidal equivalence between $Z(\mathcal{C})$ and $Z(\mathcal{D})$. Moreover, the correspondence between transparent domain walls and equivalences of bulk theories is bijective. If $\mathcal{C}$ and $\mathcal{D}$ are themselves equivalent (as monoidal categories), the domain wall can terminate, and the transport of excitations around the endpoint defines an automorphism of $Z(\mathcal{C})$. The possibility of quasiparticles changing their type due to a transport around a point-like defect was mentioned in [K05]. Such defects were explicitly constructed and studied by Bombin [B10] under the name of "twists", though in his interpretation the associated domain wall is immaterial (like a Dirac string). General domain walls between phases are not transparent. A particle injected into a foreign phase leaves behind a trace (a superposition of domain walls). This process is also described using tensor category theory.

Another result of our work is a uniform treatment of different excitation types. As already mentioned, bulk quasiparticles are equivalent to excitations on the trivial domain wall. A wall between two models $\mathcal{C}$ and $\mathcal{D}$ can be regarded as a boundary of a single phase $\mathcal{C} \boxtimes \mathcal{D}^{\leftrightarrow}$ if we fold the plane. Thus, it is sufficient to consider boundary excitations. We characterize them as superselection sectors (or irreducible modules) of a local operator algebra. This construction provides a crucial link between the physically motivated notion of excitation and more abstract mathematical concepts. We will also show how the properties of boundary excitations can be translated into tensor-categorical language, which leads to the mathematical notion of a module functor. This view of excitations also works perfectly well for boundary points between different types of domain walls. They can be described by more general module functors. A domain wall (also called a defect line) has codimension 1; an excitation connecting multiple domain walls is a defect of codimension 2. One can go further to consider defects of codimension 3, which are given by natural transformations between module functors. The correspondence between physical notions and the tensor-categorical formalism is summarized in Table 1. A Levin-Wen model together with defects of codimension 1, 2, 3 provides the physical meaning behinds the so-called extended Turaev-Viro topological field theory [TV92, L08, KB10, Ka10].

Our constructions also provides a physics background of the so-called extended Turaev-Viro topological field theory [TV92][BW93][KB10][L08][Ka10], in which the unitary tensor category $\mathcal{C}$ (or even better, the bicategory of $\mathcal{C}$-modules) is assigned to a point, a bimodule is assigned to a framed interval, the category $Z(\mathcal{C})$ is assigned to a circle, etc. The last step is to assign a number,

Ingredients of Levin-Wen models	Tensor-categorical notions
bulk Levin-Wen model	unitary tensor category $\mathcal{C}$
edge labels in the bulk	simple objects in $\mathcal{C}$
excitations in the bulk	objects in $Z(\mathcal{C})$, the monoidal center of $\mathcal{C}$
boundary type	$\mathcal{C}$-module $\mathcal{M}$
edge labels on a ${}_{\mathcal{C}}\mathcal{M}$-boundary	simple objects in $\mathcal{M}$
excitations on a ${}_{\mathcal{C}}\mathcal{M}$-boundary	objects in the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ of $\mathcal{C}$-module functors
bulk excitations fusing into a ${}_{\mathcal{C}}\mathcal{M}$-boundary	$Z(\mathcal{C}) = \mathrm{Fun}_{\mathcal{C}\vert\mathcal{C}}(\mathcal{C},\mathcal{C}) \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ $(\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{C}) \mapsto (\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M} \xrightarrow{\mathcal{F}\boxtimes \mathrm{id}_{\mathcal{M}}} \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M})$.
domain wall	$\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{N}$
edge labels on a ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-wall	simple objects in $\mathcal{N}$
excitations on a ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-wall	objects in the category $\mathrm{Fun}_{\mathcal{C}\vert\mathcal{D}}(\mathcal{N},\mathcal{N})$ of $\mathcal{C}$-$\mathcal{D}$-bimodule functors
fusion of two walls	$\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$
invertible ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-wall	$\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent, i.e. $\mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{N}^{\mathrm{op}} \cong \mathcal{C}$, $\mathcal{N}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{D}$.
defects of codimension 2 ($\mathcal{M}$-$\mathcal{N}$-excitations)	objects $\mathcal{F}, \mathcal{G} \in \mathrm{Fun}_{\mathcal{C}\vert\mathcal{D}}(\mathcal{M},\mathcal{N})$
defect of codimension 3	$\mathcal{C}$-$\mathcal{D}$-bimodule natural transformation $\phi : \mathcal{F} \to \mathcal{G}$

Table 1: Dictionary between ingredients of Levin-Wen models and tensor-categorical notions.

called the Turaev-Viro invariant to a 3-manifold (possibly, with corners).

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Levin-Wen Models and Tensor Categories

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a joint work with Alexei Kitaev

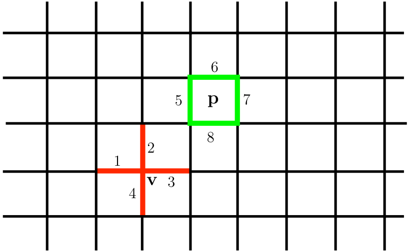
Goals:

- to provide a more rigorous and systematic study of Levin-Wen models;
- to enrich Levin-Wen models to include boundaries and defects of codimension 1,2,3;
- to show how the representation theory of tensor category enters the study of topological order at its full strength;
- to provide the physical meaning behind the so-called extended Turaev-Viro topological field theories;

Kitaev's Toric Code Model

- Kitaev's Toric Code Model is equivalent to Levin-Wen model associated to the category $\mathrm{Rep}_{\mathbb{Z}_2}$ of representations of $\mathbb{Z}_2$.
- It is the simplest example that can illustrate the general features of Levin-Wen models.

Kitaev's Toric Code Model



$$\mathcal{H} = \otimes_{e \in \text{all edges}} \mathcal{H}_e; \qquad \mathcal{H}_e = \mathbb{C}^2.$$

$$H = -\sum_v A_v - \sum_p B_p.$$

$$A_v = \sigma_x^1 \sigma_x^2 \sigma_x^3 \sigma_x^4; \qquad B_p = \sigma_z^5 \sigma_z^6 \sigma_z^7 \sigma_z^8.$$

Vacuum properties of toric code model:

A vacuum state $|0\rangle$ is a state satisfying $A_v|0\rangle = |0\rangle, B_p|0\rangle = |0\rangle$ for all v and p.

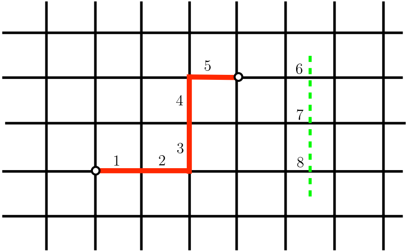
- If surface topology is trivial (a sphere, an infinite plane), the vacuum is unique.
- Vacuum is given by the condensation of closed strings, i.e.

$$|0\rangle = \sum_{c \in \text{all closed string configurations}} |c\rangle.$$

Excitations

- The "set" of excitations determines the topological phase.
- An excitation is defined to be super-selection sectors (irreducible modules) of a local operator algebra.
- There are four types of excitations: $1, e, m, \epsilon$. We denote the ground states of these sectors as $|0\rangle, |e\rangle, |m\rangle, |\epsilon\rangle$. We have

$$\begin{aligned} \exists v_0, \qquad & A_{v_0}|e\rangle = -|e\rangle, \\ \exists p_0, \qquad & B_{p_0}|m\rangle = -|m\rangle, \\ \exists v_1, p_1, \qquad & A_{v_1}|\epsilon\rangle = -|\epsilon\rangle, \ B_{p_1}|\epsilon\rangle = -|\epsilon\rangle. \end{aligned}$$



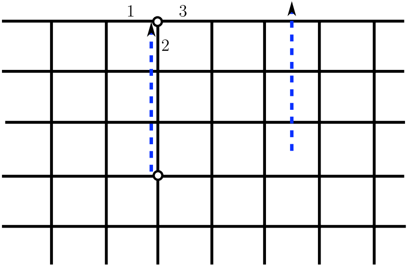
$$1 = e \otimes e \sim \sigma_z^1 \sigma_z^2 \sigma_z^3 \sigma_z^4 \sigma_z^5 |0\rangle,$$

$$1 = m \otimes m \sim \sigma_x^6 \sigma_x^7 \sigma_x^8 |0\rangle,$$

$$e \otimes m = \epsilon.$$

☞ $1, e, m, \epsilon$ are simple objects of a braided tensor category $Z(\mathrm{Rep}_{\mathbb{Z}_2})$ which is the monoidal center of $\mathrm{Rep}_{\mathbb{Z}_2}$.

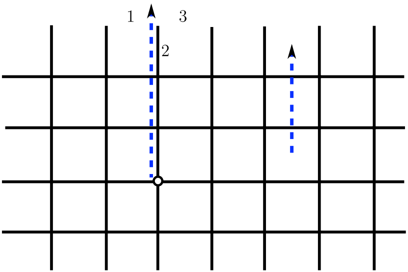
A smooth edge



$$\begin{array}{ccccc} 1 & \longrightarrow & 1 & e \longrightarrow e \\ m & \longrightarrow & 1 & \epsilon \longrightarrow e \end{array}$$

☞ This assignment actually gives a monoidal functor $Z(\mathrm{Rep}_{\mathbb{Z}_2}) \to \mathrm{Rep}_{\mathbb{Z}_2} = \mathrm{Fun}_{\mathrm{Rep}_{\mathbb{Z}_2}}(\mathrm{Rep}_{\mathbb{Z}_2}, \mathrm{Rep}_{\mathbb{Z}_2})$.

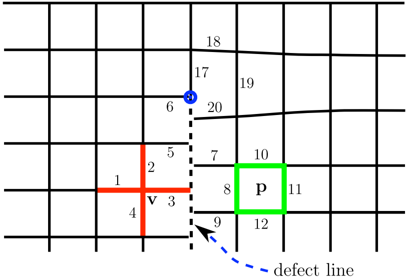
A rough edge



$$
\begin{array}{ccccc}
1 & \longrightarrow & 1 & m \longrightarrow m \\
e & \longrightarrow & 1 & \epsilon \longrightarrow m
\end{array}
$$

☞ This assignment gives another monoidal functor $Z(\mathrm{Rep}_{\mathbb{Z}_2}) \to \mathrm{Rep}_{\mathbb{Z}_2} = \mathrm{Fun}_{\mathrm{Rep}_{\mathbb{Z}_2}}(\mathrm{Hilb}, \mathrm{Hilb})$.

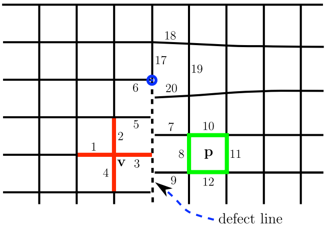
defects of codimension 1, 2



$$B_{p_1} = \sigma_x^7 \sigma_x^3 \sigma_x^2 \sigma_x^5; \quad B_{p_2} = \sigma_x^3 \sigma_x^7 \sigma_x^8 \sigma_x^9;$$

$$B_Q = \sigma_x^6 \sigma_y^{17} \sigma_z^{18} \sigma_z^{19} \sigma_z^{20}.$$

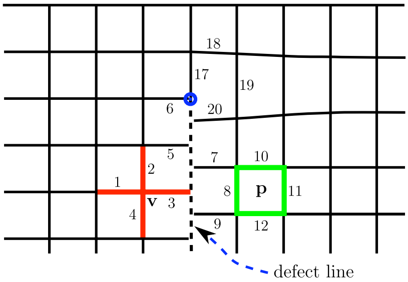
defects of codimension 1



$$1 \mapsto 1 \mapsto 1, \qquad e \xrightarrow{\sigma_z^3} \mathrm{Ext}_{3|7,8,9}^{defect} \xrightarrow{\sigma_x^8} m,$$
$$m \mapsto \mathrm{Ext}_{7|3,2,5}^{defect} \mapsto e, \qquad \epsilon \mapsto \mathrm{Ext}_{2,5,7,8,9,3}^{defect} \mapsto \epsilon.$$

☞ This assignment gives an invertible monoidal functor $Z(\mathrm{Rep}_{\mathbb{Z}_2}) \to \mathrm{Fun}_{\mathrm{Rep}_{\mathbb{Z}_2}|\mathrm{Rep}_{\mathbb{Z}_2}}(\mathrm{Hilb}, \mathrm{Hilb}) \to Z(\mathrm{Rep}_{\mathbb{Z}_2})$.

defects of codimension 2



$$B_Q = \sigma_x^6 \sigma_y^{17} \sigma_z^{18} \sigma_z^{19} \sigma_z^{20}$$

☞ Two eigenstates of B_Q correspond to two simple $\mathrm{Rep}_{\mathbb{Z}_2}$-$\mathrm{Rep}_{\mathbb{Z}_2}$-bimodule functors $\mathrm{Hilb} \to \mathrm{Rep}_{\mathbb{Z}_2}$.

Outline

Basics of unitary tensor category

unitary tensor category $\mathcal{C}$ = unitary spherical fusion category

- semisimple: every object is a direct sum of simple objects;
- finite: there are only finite number of inequivalent simple objects, $i, j, k, l \in \mathcal{I}$, $|\mathcal{I}| < \infty$; $\dim \mathrm{Hom}(A, B) < \infty$.
- monoidal: $(i \otimes j) \otimes k \cong i \otimes (j \otimes k)$; $\mathbf{1} \in \mathcal{I}$, $\mathbf{1} \otimes i \cong i \cong i \otimes \mathbf{1}$;
- the fusion rule: $\dim \mathrm{Hom}(i \otimes j, k) = N_{ij}^k < \infty$;
- $\mathcal{C}$ is not assumed to be braided.

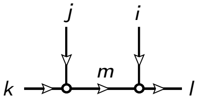
Theorem (Müger): The monoidal center $Z(\mathcal{C})$ of $\mathcal{C}$ is a modular tensor category.

Fusion matrices

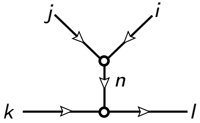
The associator $(i \otimes j) \otimes k \xrightarrow{\alpha} i \otimes (j \otimes k)$ induces an isomorphism:

$$\mathrm{Hom}((i \otimes j) \otimes k, l) \xrightarrow{\cong} \mathrm{Hom}(i \otimes (j \otimes k), l)$$

Writing in basis, we obtain the fusion matrices:



$$= \sum_n F^{ijk;l}_{mn}$$



(1)

Levin-Wen models

We fix a unitary tensor category $\mathcal{C}$ with simple objects $i, j, k, l, m, n \in \mathcal{I}$.

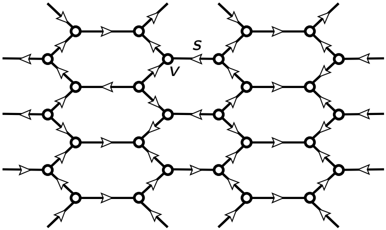


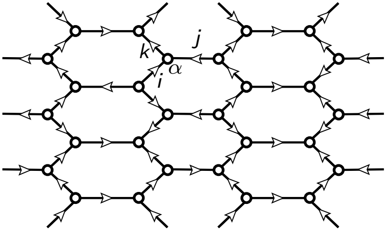
Figure: Levin-Wen model defined on a honeycomb lattice.

$$\mathcal{H}_s = \mathbb{C}^{\mathcal{I}}, \quad \mathcal{H}_v = \oplus_{i,j,k} \mathrm{Hom}_{\mathcal{C}}(i \otimes j, k).$$

$$\mathcal{H} = \otimes_s \mathcal{H}_s \otimes_v \mathcal{H}_v.$$

Hamiltonian

Chose a basis of $\mathcal{H}$, $i, j, k \in \mathcal{I}$ and $\alpha^{i'j';k'} \in \mathrm{Hom}_{\mathcal{C}}(i' \otimes j', k')$,



$$H = -\sum_v A_v - \sum_p B_p.$$

$$A_v|(i,j;k|\alpha^{i',j';k'})\rangle = \delta_{i,i'}\delta_{j,j'}\delta_{k,k'}|(i,j;k|\alpha^{i',j';k'})\rangle.$$

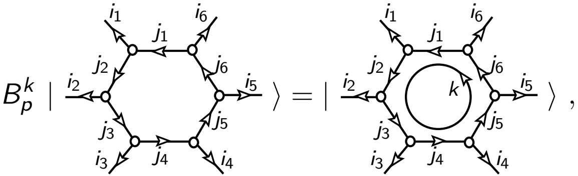
If the spin on v is such that A_v acts as 1, then it is called stable.

The definition of B_p operator

$$B_p := \sum_{i \in \mathcal{I}} \frac{d_i}{\sum_k d_k^2} B_p^i$$

- If there are unstable spins around the plaquette p, B_p^i act on the plaquette as zero.

- If all the spins at the corners are stable, then $B_{\mathbf{p}}^k$ is defined as follow: suppressing all the spin labels,



the right hand side of which is a sum of hexagons (without the k-loop) obtained by first fusing the k-loop with each j-edge then evaluating 6 triangles.

- B_p is a projector. A_v and B_p commute.

Ground states

$$A_v|0\rangle = |0\rangle, \qquad B_p|0\rangle = |0\rangle.$$

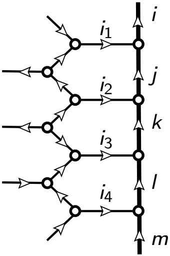
If the model is defined on a surface Σ, then the space of ground states is exactly given by the $TV(\Sigma)$. It has been known for a long time. But only rigorously proved recently by Kirillov Jr. (2011)

Remark:

- Given a unitary tensor category $\mathcal{C}$, we obtain a lattice model.
- Conversely, Levin-Wen showed how the axioms of the unitary tensor category can be derived from the requirement to have a fix-point wave function of a string-net condensation state.

Edge theories

If we cut the lattice, we automatically obtain a lattice with a boundary with all boundary strings labeled by simple objects in $\mathcal{C}$.

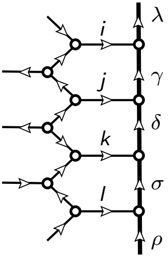


We will call such boundary as a $\mathcal{C}$-boundary or $\mathcal{C}$-edge.

Question: Are there any other possibilities?

$\mathcal{M}$-edge

It is possible to label the boundary strings by a different finite set $\{\lambda, \sigma, \dots\}$ which can be viewed as the set of inequivalent simples objects of another finite unitary semisimple category $\mathcal{M}$.

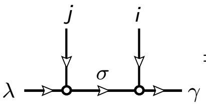


The requirement of giving a fix-point wave function of string-net condensation state is equivalent to require that $\mathcal{M}$ has a structure of $\mathcal{C}$-module. We call such boundary an ${}_{\mathcal{C}}\mathcal{M}$-boundary or ${}_{\mathcal{C}}\mathcal{M}$-edge.

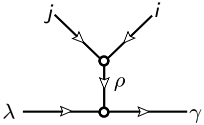
$\mathcal{C}$-module $\mathcal{M}$:

For $i \in \mathcal{C}$, $\gamma, \lambda \in \mathcal{M}$,

- $i \otimes \gamma$ is an object in $\mathcal{M}$ ($\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$)
- $\dim \mathrm{Hom}_{\mathcal{M}}(i \otimes \gamma, \lambda) = N_{i,\gamma}^{\lambda} < \infty$;
- $\mathbf{1} \otimes \gamma \cong \gamma$;
- associator $(i \otimes j) \otimes \lambda \xrightarrow{\alpha} i \otimes (j \otimes \lambda)$;
- fusion matrices:

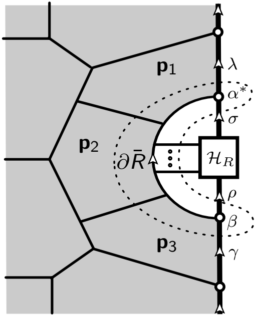


$$= \sum_n F^{ijk;l}_{mn}$$



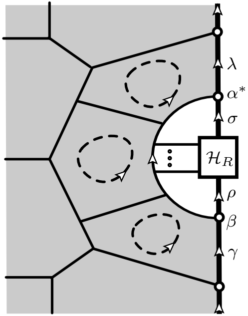
(2)

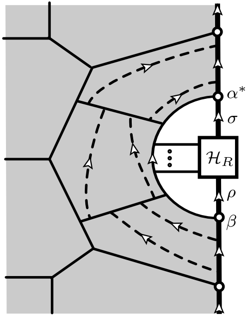
Boundary excitations



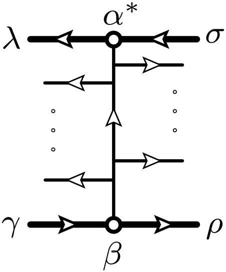
For a given region $\bar{R}$ (with $n = 2$-external $\mathcal{C}$-legs), $\bar{R} = \partial\bar{R} \cup R$, an excitation is given by a Hilbert subspace $\mathrm{Im}P_{\bar{R}} \subset \mathcal{H}_{\bar{R}} = \mathcal{H}_{\partial\bar{R}} \otimes \mathcal{H}_R$ such that the projector $P_{\bar{R}}$ commutes with the action of $\otimes_{i=1}^3 B_{\mathbf{p}_i}$ on the plaquettes immediately outside $\partial\bar{R}$.

The action of $\otimes_{i=1}^{3} B_{\mathbf{p}_i}$ on the plaquettes immediately outside ∂R can be written as $\sum_r Q_r^{ext} \otimes Q_r^{\partial \bar{R}}$ where Q_{ext}^r acts on $\mathcal{H}_{ext}$ and $Q_{\partial R}^r$ on $\mathcal{H}_{\partial \bar{R}}$. The linear independence of Q_r^{ext} implies that $Q_{\partial \bar{R}}^r$ commute with $P_{\bar{R}}$.





$\{Q^r_{\partial R}\}$ generate an algebra $A^{(n)}_{\mathcal{M}\mathcal{M}}$ (n=2) spanned by:



Theorem: A boundary excitation = a module over $A^{(n)}_{\mathcal{M}\mathcal{M}}$.

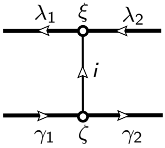
Theorem: $A^{(m)}_{\mathcal{M}\mathcal{M}}$ and $A^{(n)}_{\mathcal{M}\mathcal{M}}$ are Morita equivalent.

The case $n = 0$

Local operator algebra: $A_{\mathcal{M}\mathcal{M}} := A^{(0)}_{\mathcal{M}\mathcal{M}}$.

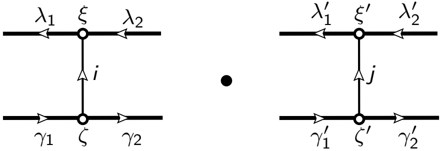
$$A_{\mathcal{M}\mathcal{M}} := \oplus_{i,\lambda_1,\lambda_2,\gamma_1,\gamma_2} \mathrm{Hom}_{\mathcal{M}}(i \otimes \lambda_2, \lambda_1) \otimes \mathrm{Hom}_{\mathcal{M}}(\gamma_1, i \otimes \gamma_2).$$

For $\xi \in \mathrm{Hom}_{\mathcal{M}}(i \otimes \lambda_2, \lambda_1)$ and $\zeta \in \mathrm{Hom}_{\mathcal{M}}(\gamma_1, i \otimes \gamma_2)$, the element $\xi \otimes \zeta \in A_{\mathcal{M}\mathcal{M}}$ can be expressed by the following graph:

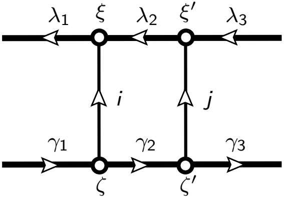
$\xi \otimes \zeta =$ 

for $i \in \mathcal{C}$ and $\lambda_1, \lambda_2, \gamma_1, \gamma_2 \in \mathcal{M}$.

The multiplication $A_{\mathcal{M}\mathcal{M}} \otimes A_{\mathcal{M}\mathcal{M}} \xrightarrow{\bullet} A_{\mathcal{M}\mathcal{M}}$ is defined by



$= \delta_{\lambda_2 \lambda'_2} \delta_{\gamma_2 \gamma'_2}$



where the last graph is a linear span of graphs in $A_{\mathcal{M}\mathcal{M}}$ by applying F-moves twice and removing bubbles.

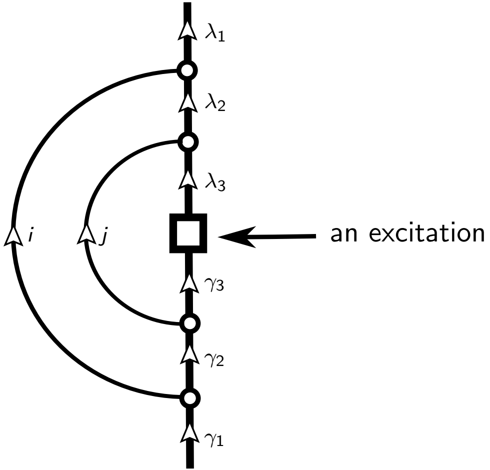
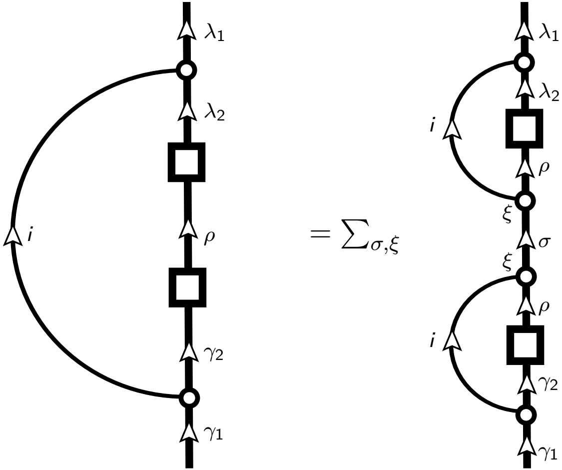


Figure: Two elements of local operator algebra $A_{\mathcal{M}\mathcal{M}}$ act on an edge excitation (up to an ambiguity of the excited region).



$A_{\mathcal{M}\mathcal{M}}$ is bialgebra with above comultiplication.

With some small modifications, one can turn $A_{\mathcal{M}\mathcal{M}}$ into a weak C^*-Hopf algebra so that the boundary excitations form a finite unitary fusion category (Hayashi99, Szlachanyi00; Ostrik01, etc.).

Theorem [Ostrik, Kitaev-K.]:

The category of $A_{\mathcal{M}\mathcal{M}}$-modules $\cong \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$.

A physical proof: use the set-up to show that excitations are classified by closed string operators that commute with the Hamiltonian (Levin-Wen). It is fairly straightforward to show that the latter objects are equivalent to $\mathcal{C}$-module functors.

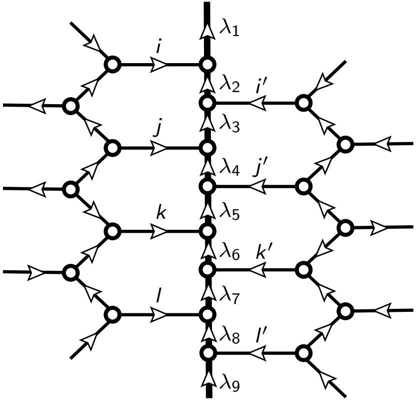
Close the boundary to a circle, a closed string operator on it is nothing but a systematic reassignment of boundary string labels and spin labels:

$$\begin{aligned} \gamma &\mapsto F(\gamma) \in \mathcal{M}, \\ \mathrm{Hom}_{\mathcal{M}}(i \otimes \gamma, \lambda) &\mapsto \mathrm{Hom}_{\mathcal{M}}(i \otimes F(\gamma), F(\lambda)) \end{aligned}$$

This assignment is essentially the same data forming a functor from $\mathcal{M}$ to $\mathcal{M}$. Physical requirements (Levin-Wen) add certain consistency conditions which turn it into a $\mathcal{C}$-module functor.

Theorem: Excitations on a ${}_{\mathcal{C}}\mathcal{M}$-edge are given by simple objects in the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ of $\mathcal{C}$-module functors.

a defect line or a domain wall



$i, j, k, l \in \mathcal{C}$, $\lambda_1, \dots, \lambda_9 \in \mathcal{M}$, $i', j', k', l' \in \mathcal{D}$. $\mathcal{C}$ and $\mathcal{D}$ are unitary tensor categories and $\mathcal{M}$ is a $\mathcal{C}$-$\mathcal{D}$-bimodule. We call such defect ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$-defect line or ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$-wall.

- A $\mathcal{M}$-edge can be viewed as ${}_{\mathcal{C}}\mathcal{M}_{\mathsf{Hilb}}$-wall.
- Conversely, if we fold the system along the ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$-wall, we obtain a doubled bulk system determined by $\mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}}$ with a single boundary determined by $\mathcal{M}$ which is viewed as a $\mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}}$-module.

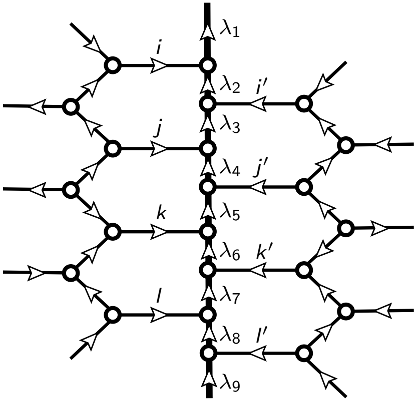
$$\text{a } {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}\text{-wall} = \text{a } {}_{\mathcal{C}\boxtimes\mathcal{D}^{\mathrm{op}}}\mathcal{M}\text{-edge}$$

Therefore, we have:

$$\begin{array}{rcl} {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}\text{-wall excitations} & = & {}_{\mathcal{C}\boxtimes\mathcal{D}^{\mathrm{op}}}\mathcal{M}\text{-edge excitations} \\ & = & \mathrm{Fun}_{\mathcal{C}\boxtimes\mathcal{D}^{\mathrm{op}}}(\mathcal{M},\mathcal{M}) \\ & = & \mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M},\mathcal{M}) \end{array}$$

$\mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M},\mathcal{M}) :=$ the category of $\mathcal{C}$-$\mathcal{D}$-bimodule functors.

As a special case, $i, j, k, l, \lambda_1, \dots, \lambda_9, i', j', k', l' \in \mathcal{C} = \mathcal{M} = \mathcal{D}$.



a line in $\mathcal{C}$-bulk $=$ a ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$-wall

$\mathcal{C}$-bulk excitations $=$ ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$-wall excitations

$= \mathrm{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C}, \mathcal{C}) = Z(\mathcal{C})$

- A ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$-wall can fuse with a ${}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$-wall into a ${}_{\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})_{\mathcal{E}}$-wall.

- ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$-wall (or ${}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$-wall) excitations can fuse into ${}_{\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})_{\mathcal{E}}$-wall as follow:

$$(\mathcal{M} \xrightarrow{F} \mathcal{M}) \mapsto (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{F \boxtimes_{\mathcal{D}} \mathrm{id}_{\mathcal{N}}} \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})$$

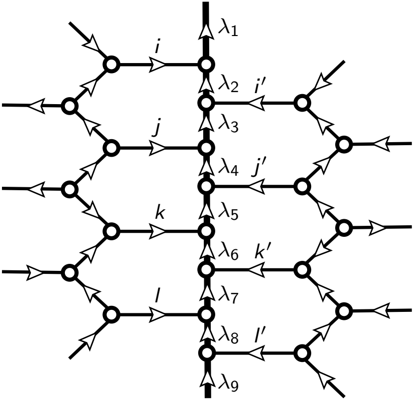
$$(\mathcal{N} \xrightarrow{G} \mathcal{N}) \mapsto (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{\mathrm{id}_{\mathcal{M}} \boxtimes_{\mathcal{D}} G} \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})$$

As a special case $\mathcal{M} = \mathcal{D}$: we obtain

the fusion of bulk excitations into wall excitations

as a monoidal functor:

$$(\mathcal{D} \xrightarrow{F} \mathcal{D}) \mapsto (\mathcal{D} \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{F \boxtimes_{\mathcal{D}} \mathrm{id}_{\mathcal{N}}} \mathcal{D} \boxtimes_{\mathcal{D}} \mathcal{N})$$



$$\text{a cospan}: Z(\mathcal{C}) \xrightarrow{L_{\mathcal{M}}} \mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{M}) \xleftarrow{R_{\mathcal{M}}} Z(\mathcal{D})$$

$$L_{\mathcal{M}}: \quad (\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{C}) \quad \longmapsto \quad (\mathcal{M} \cong \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M} \xrightarrow{\mathcal{F} \boxtimes_{\mathcal{C}} \mathrm{id}_{\mathcal{M}}} \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M} \cong \mathcal{M})$$

$$R_{\mathcal{M}}: \quad (\mathcal{D} \xrightarrow{\mathcal{G}} \mathcal{D}) \quad \longmapsto \quad (\mathcal{M} \cong \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{D} \xrightarrow{\mathrm{id}_{\mathcal{M}} \boxtimes_{\mathcal{D}} \mathcal{G}} \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{D} \cong \mathcal{M})$$

Definition: If $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{C}$ and $\mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M} \cong \mathcal{D}$, then $\mathcal{M}$ and $\mathcal{N}$ are called invertible; $\mathcal{C}$ and $\mathcal{D}$ are called Morita equivalent.

Theorem (Müger, Etingof-Nikshych-Ostrik, Kitaev)
$\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent iff $Z(\mathcal{C})$ is equivalent to $Z(\mathcal{D})$ as braided tensor categories.

- Invertible $\mathcal{C}$-$\mathcal{C}$-defects form a group called Picard group $\mathrm{Pic}(\mathcal{C})$.
- We denote the auto-equivalence of $Z(\mathcal{C})$ as $\mathrm{Aut}(Z(\mathcal{C}))$.

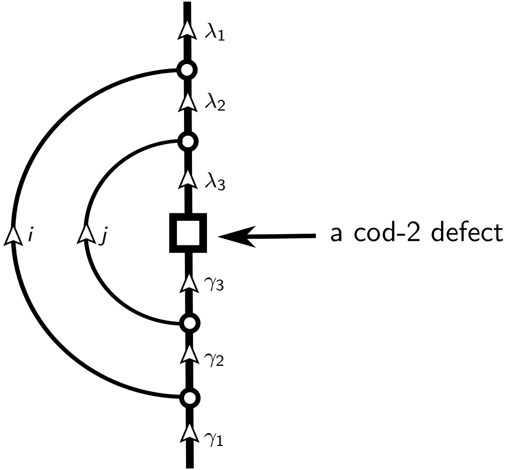
Theorem (Etingof-Nikshych-Ostrik09, Kitaev-K.09):

$$\mathrm{Aut}(Z(\mathcal{C})) \cong \mathrm{Pic}(\mathcal{C}).$$

Defects of codimension 2

- A defect of codimension 2 is a junction between a ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$-wall and a ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-wall. It corresponds to a module functor $\mathcal{F} \in \mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{N})$.
- Any excitation can be viewed as a defect of codimension 2.
- Any defect of codimension 2 is an excitation in the sense that it can be realized as a super-selection sector of a local operator algebra $A_{\mathcal{M}\mathcal{N}}$.

Action of $A_{\mathcal{M}\mathcal{N}}$ on defects of codimension 2



$$\lambda_1, \lambda_2, \lambda_3 \in \mathcal{M}, \gamma_1, \gamma_2, \gamma_3 \in \mathcal{N}$$

1. The category of $A_{\mathcal{M}\mathcal{N}}$-modules $= \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$.

2. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{C}$-$\mathcal{D}$-walls, then we have the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{M}) \arrow[d, "\mathcal{F}^*"] & \\
Z(\mathcal{C}) \arrow[ur, "L_{\mathcal{M}}"] \arrow[dr, "L_{\mathcal{N}}"'] & \mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{N}) & Z(\mathcal{D}) \arrow[ul, "R_{\mathcal{M}}"'] \arrow[dl, "R_{\mathcal{N}}"] \\
 & \mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{N}, \mathcal{N}) \arrow[u, "\mathcal{F}_*"'] &
\end{tikzcd}$$

Defects of codimension 3 (instantons)

If one takes into account the time direction, one can define a defect of codimension 3 by a natural transformation ϕ between module functors.

The Hamiltonian:

$$H \to H + H_t.$$

where H_t is a local operator defined using ϕ (an instanton).

Dictionary 1:

Ingredients in LW-model	Tensor-categorical notions
a bulk lattice	a unitary tensor category $\mathcal{C}$
string labels in a bulk	simple objects in a unitary tensor category $\mathcal{C}$
excitations in a bulk	simple objects in $Z(\mathcal{C})$ the monoidal center of $\mathcal{C}$
an edge	a $\mathcal{C}$-module $\mathcal{M}$
string labels on an edge	simple objects in a $\mathcal{C}$-module $\mathcal{M}$
excitations on a $\mathcal{M}$-edge	$\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$: the category of $\mathcal{C}$-module functors
bulk-excitations fuse into an $\mathcal{M}$-edge	$Z(\mathcal{C}) = \mathrm{Fun}_{\mathcal{C}\mid\mathcal{C}}(\mathcal{C},\mathcal{C}) \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ $(\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{C}) \mapsto (\mathcal{C}\boxtimes_{\mathcal{C}}\mathcal{M} \xrightarrow{\mathcal{F}\boxtimes \mathrm{id}_{\mathcal{M}}} \mathcal{C}\boxtimes_{\mathcal{C}}\mathcal{M})$.

Dictionary 2:

Ingredients in LW-model	Tensor-categorical notions
a domain wall	a $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{N}$
string labels on a $\mathcal{N}$-wall	simple objects in a $\mathcal{C}$-$\mathcal{D}$-bimodule ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$
excitations on a $\mathcal{N}$-wall	$\mathrm{Fun}_{\mathcal{C}\vert\mathcal{D}}(\mathcal{N},\mathcal{N})$: the category of $\mathcal{C}$-$\mathcal{D}$-bimodule functors
fusion of two walls	$\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$
an invertible ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-wall	$\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent, i.e. $\mathcal{N} \otimes_{\mathcal{D}} \mathcal{N}^{\mathrm{op}} \cong \mathcal{C}$, $\mathcal{N}^{\mathrm{op}} \otimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{D}$.
bulk-excitation fuse into a ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-wall	$Z(\mathcal{C}) = \mathrm{Fun}_{\mathcal{C}\vert\mathcal{C}}(\mathcal{C},\mathcal{C}) \to \mathrm{Fun}_{\mathcal{C}\vert\mathcal{D}}(\mathcal{N},\mathcal{N})$ $(\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{C}) \mapsto (\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N} \xrightarrow{\mathcal{F}\boxtimes \mathrm{id}_{\mathcal{N}}} \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N})$.
defects of codimension 2: a $\mathcal{M}$-$\mathcal{N}$-excitation	simple objects $\mathcal{F}, \mathcal{G} \in \mathrm{Fun}_{\mathcal{C}\vert\mathcal{D}}(\mathcal{M},\mathcal{N})$
a defect of codimesion 3 or an instanton	a natural transformation $\phi : \mathcal{F} \to \mathcal{G}$

Outline

- Levin-Wen models enriched by defects of codimension 1,2,3 can be viewed as a categorified theory of Fröhlich-Fuchs-Runkel-Schweigert's theory for rational CFTs with defects of codimension 1, 2.

- It provides a physical meaning behind the so-called extended Turaev-Viro topological field theories.

- Algebraic structures appeared in extended Turaev-Viro TQFT can be summarized as a conjectured boundary-to-bulk (or holography) functor between two tri-categories as we will discuss.

The building blocks of the lattice models:

$$\begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=70, "\mathcal{M}"] \arrow[r, bend right=70, "\mathcal{N}"'] \arrow[r, phantom, "\mathcal{F}\Downarrow \overset{\phi}{\Rrightarrow} \Downarrow\mathcal{G}"] & \mathcal{D} \end{tikzcd}$$

in which 0-1-2-3 cells form a tri-category, or "equivalently",

$$\begin{tikzcd}[column sep=huge] \mathcal{C}\text{-Mod} \arrow[r, bend left=70, "\mathcal{F}_{\mathcal{M}}"] \arrow[r, bend right=70, "\mathcal{F}_{\mathcal{N}}"'] \arrow[r, phantom, "\varphi\Downarrow \overset{m}{\Rrightarrow} \Downarrow\varphi'"] & \mathcal{D}\text{-Mod} \end{tikzcd}$$

A tri-category of excitations (?):

$$\begin{tikzcd}
 & & Z(\mathcal{M}) \arrow[dl, "\mathcal{F}_*"'] \arrow[dr, "\mathcal{G}_*"] & & \\
Z(\mathcal{C}) \arrow[urr, "L_{\mathcal{M}}"] \arrow[drr, "L_{\mathcal{N}}"'] & Z(\mathcal{M},\mathcal{N})_{\mathcal{F}} \arrow[rr, "Z(\phi)"] & & Z(\mathcal{M},\mathcal{N})_{\mathcal{G}} & Z(\mathcal{D}) \arrow[ull, "R_{\mathcal{M}}"'] \arrow[dll, "R_{\mathcal{N}}"] \\
 & & Z(\mathcal{N}) \arrow[ul, "\mathcal{F}^*"] \arrow[ur, "\mathcal{G}^*"'] & &
\end{tikzcd}$$

$Z(\mathcal{M}) := \mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M},\mathcal{M})$, $Z(\mathcal{N}) := \mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{N},\mathcal{N})$,
$\mathcal{F},\mathcal{G} \in Z(\mathcal{M},\mathcal{N}) := \mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M},\mathcal{N})$.

Conjecture (Functoriality of Holography): The assignment Z is a functor between two tricategories.

Remark: It also says that the notion of monoidal center is functorial.

General philosophy: for $n+1$-dim extended TQFT,

$$\mathrm{pt} \mapsto n\text{-category of boundary conditions}.$$

Extended Turaev-Viro (2+1) TQFT: the bicategory of boundary conditions of LW-models $= \mathcal{C}$-Mod,

$$\mathrm{pt}_{+,-} \quad \mapsto \quad \mathcal{C}, \mathcal{D} \text{ or } (\mathcal{C}\text{-Mod} \cong \mathcal{D}\text{-Mod}),$$

$$\text{an interval} \quad \mapsto \quad {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}, {}_{\mathcal{D}}\mathcal{N}_{\mathcal{C}} \ \text{(invertible)}$$

$$S^1 \quad \mapsto \quad Tr(\mathcal{C}) = Z(\mathcal{C}),$$

Thank you!