WHAT IS A GLOBAL FIELD?

JEAN-BENOIT BOST

A ***global field*** K is either

• a finite degree extension field of $\mathbb{Q}$, i.e.,

$$K = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(P(x))$$

where $P(x) = x^d + a_1 x^{d-1} + \cdots + a_d \in \mathbb{Q}[x] \setminus \mathbb{Q}$ is an irreducible polynomial of degree d such that $P(\alpha) = 0$;

or

• a finite degree extension field of $\mathbb{F}_p(T)$, where p is a prime number, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and

$$\mathbb{F}_p(T) := \left\{ \frac{P}{Q} \,\middle|\, P\,,\, Q \in \mathbb{F}_p[T]\,,\, Q \neq 0 \right\}.$$

As $[K : \mathbb{F}_p(T)] < +\infty$,

$$\mathbb{F}_p \subseteq \{\alpha \in K \,|\, \alpha \text{ algebraic over } \mathbb{F}_p\} =: k$$

is a finite field extension $\mathbb{F}_p \subseteq \mathbb{F}_q$. We will call k the ***field of constants*** of K. There is an element $t \in K \setminus k$ such that K is a finite separable extension of $k(t)$. Since t is transcendental over k, the field $k(t)$ is isomorphic to $k(T)$, the field of rational functions over k. Thus, we have an isomorphism

$$K = k(t)[\alpha] \cong k(T)[X]/(P(T\,,\, X))$$

where $P(T\,,\, X) \in k[T\,,\, X]$ is an absolutely irreducible polynomial such that $\deg_X P \geq 1$ and $\frac{\partial P}{\partial X} \neq 0$.

Two natural questions:

Why is the above definition relevant?

Why is it meaningful to consider number fields and function fields on the same footing?

There must be some analogies between number fields and function fields!

In the following, I will give a historical survey on the subject. Here is an outline:

(1) The work of Dedekind, Weber, ... , Grothendieck;

(2) The work of Hensel, ... , Tate, Langlands;

(3) Algebraic varieties over global fields.

However, nothing concerning Galois cohomology will be touched in this very incomplete survey.

1. Dedekind, Weber, ... , Grothendieck: Algebra

A global field K is a field isomorphic to the field of rational functions $\kappa(C)$ of some integral scheme C of finite type over $\mathbb{Z}$ which has Krull dimension 1. The nonempty open subsets of C are characterized as subsets of the form $C \setminus F$, where F is a finite set of closed points of C. Choose a finite open cover $C = \bigcup_{1 \leq i \leq n} U_i$,

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where each $U_i = \operatorname{Spec} A_i$ is affine. Since the U_i's are of finite type $\mathbb{Z}$ as C is, for each i,

$$A_i \cong \mathbb{Z}[x_1, \cdots, x_v]/I_i$$

for some prime ideal I_i of $\mathbb{Z}[x_1, \ldots, x_v]$. For each i, the field of fractions of A_i is K and since $\dim A_i = 1$,

$$\{\text{ maximal ideals of } A_i\} = \{\text{ non-zero prime ideals of } A_i\}\,.$$

The canonical map $C \to \operatorname{Spec}\mathbb{Z}$ gives natural homomorphisms $\mathbb{Z} \to A_i\,,\ \forall\ i$. Consider the fiber square

$$\begin{array}{ccc} C_{\mathbb{Q}} & \longrightarrow & C \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec}\mathbb{Q} & \longrightarrow & \operatorname{Spec}\mathbb{Z} \end{array}$$

$C \to \operatorname{Spec}\mathbb{Z}$ being a morphism between integral schemes of dimension 1, either the generic fiber $C_{\mathbb{Q}}$ of $C \to \operatorname{Spec}\mathbb{Z}$ is nonempty, which means the structural morphism $C \to \operatorname{Spec}\mathbb{Z}$ is dominant; or the the image of $C \to \operatorname{Spec}\mathbb{Z}$ consists of a single closed point of $\operatorname{Spec}\mathbb{Z}$. The generic point of C lies in each nonempty U_i. So the first case happens if and only if the natural map $\mathbb{Z} \to A_i$ is injective for every i, or equivalently, $A_i \otimes \mathbb{Q} \neq 0$ for every i. In this case, one proves that $A_i \otimes \mathbb{Q} \xrightarrow{\sim} K$. The field K is finitely generated as a $\mathbb{Q}$-algebra, so K is a number field. In the second case, the morphism $C \to \operatorname{Spec}\mathbb{Z}$ factors as

$$C \longrightarrow \operatorname{Spec}\mathbb{F}_p \longrightarrow \operatorname{Spec}\mathbb{Z}$$

for some prime number p, or equivalently, $\ker(\mathbb{Z} \to A_i) = p\mathbb{Z}$ for each i. In this case, $p = 0$ in K so that K has characteristic $p > 0$ and is a function field.

Actually, the "curve" C may assumed to be regular. Then in the decomposition

$$C = \bigcup_{1 \leq i \leq n} U_i\,, \quad U_i = \operatorname{Spec} A_i$$

each A_i is Dedekind domain which is finitely generated as a $\mathbb{Z}$-algebra. Further, there is a canonical "maximal" regular C: in the number field case, call

$$\mathcal{O}_K := \{\alpha \in K \mid \exists\ P \in \mathbb{Z}[X] \setminus \mathbb{Z} \text{ a monic polynomial such that }\ , P(\alpha) = 0\}$$

the ***ring of integers*** of K. Then the canonical regular model for K is $C = \operatorname{Spec}\mathcal{O}_K$. In the function field case, C is a smooth projective, geometrically irreducible curve over $k \cong \mathbb{F}_q$. For example, if $K = \mathbb{F}_q(T)$, $C = \mathbb{P}^1_{\mathbb{F}_q}$; if $K = \mathbb{F}_q(x)[Y]/(Y^2 - P(x))$, where $P \in \mathbb{F}_q[x]$ has degree 3, $\gcd(P, P') = 1$ and the characteristic p is $\neq 2$, then C is an elliptic curve (with affine equation $y^2 = f(x)$).

Historical comments. (1) Dedekind and Weber are the first to apply "arithmetic" approaches to the theory of algebraic functions in one variable. By a "function field in one variable over $\mathbb{C}$", we mean a finite degree extension field K of $\mathbb{C}(T)$. Such a field has the form

$$K \cong \mathbb{C}(T)[X]/(P(T, X))\,,$$

where $P \in \mathbb{C}[T\,, X]$ is irreducible of degree $\deg P \geq 1$. Elements $R(t, x)$ of such a field K are rational functions on the algebraic curve

$$C := \{(t\,, x) \in \mathbb{C}^2 \mid P(t, x) = 0\} \subseteq \mathbb{A}^2(\mathbb{C})$$

After removing singularities of C, there exists a bianalytic isomorphism onto the complement of a finite subset in a compact Riemann surface $\widetilde{C}^{\mathrm{an}}$.

$$\begin{array}{ccc} C \setminus \{\text{singular points}\} & \xrightarrow{\text{bianalytic iso.}} & \widetilde{C}^{\mathrm{an}} \setminus \text{a finite subset} \\ \downarrow & & \downarrow \\ \mathbb{A}^1(\mathbb{C}) & & \mathbb{P}^1(\mathbb{C}) \end{array}$$

The Riemann surface $\widetilde{C}^{\mathrm{an}}$ is deduced from C by deleting a finite subset of singular points, and then by "adding" a finite set of points.

By Riemann's work, the function field K is isomorphic to the field $\mathcal{M}(\widetilde{C}^{\mathrm{an}})$ of meromorphic functions on the compact Riemann surface $\widetilde{C}^{\mathrm{an}}$:

$$K \cong \mathcal{M}(\widetilde{C}^{\mathrm{an}})\,.$$

Thus the field may be investigated by analytic methods, e.g. by using $\overline{\partial}$, $\partial\overline{\partial} = i\Delta$ and Dirichlet principle, etc. The lack of rigorous analytic foundations was a motivation for Dedekind and Weber to develop a purely algebraic approach.

According to Weil's Rosetta Stone, there should be an enormous amount of mathematics naturally divided into three parts:

• number fields, i.e., finite extension fields of $\mathbb{Q}$;

• function fields in one variable over a finite field (ever studied by Artin and Schmidt), i.e., $K = \mathbb{F}_q(t)[x]$ with an algebraic relation $P(t,\,x) = 0$;

• function fields in one variable over $\mathbb{C}$ (ever studied by Riemann), i.e., $K = \mathbb{C}(t)[x]$ with an algebraic relation $P(t,\,x) = 0$;

each with its own framework and techniques and each written in its own language in similar texts.

Historical comments. (2) From Kronecher's viewpoint, basic objects of study of arithmetic or algebraic geometry are (in modern languages) schemes of finite type over $\mathbb{Z}$. The objects of interest consist of an ideal $I = (P_1, \dots, P_f)$ of a polynomial ring $\mathbb{Z}[X_1, \dots, X_N]$ and the quotient ring $A = \mathbb{Z}[X_1, \dots, X_N]/I$. The ring A is an integral domain if and only if I is a prime ideal. In this case, we write K for the fraction field of A.

In the hierarchy by Krull (=Krnoecher=absolute) dimension, $\dim A = 0$ if and only if $A = K$. In this case, K is a finite field. $\dim A = 1$ if and only if $A \neq K$ and any non-zero prime ideal of A is maximal. In this case, K is a global field.

Another discovery of Kronecker is that if we put

$$N_p := \#\{(x_1, \dots, x_N) \in \mathbb{F}_p^N \mid P_1(x_1, \dots, x_N) = \cdots = P_f(x_1, \dots, x_N) = 0\}\,,$$

then the series

$$\sum_{p\,:\,\text{prime}} \frac{N_p}{p^s}$$

converges for $s \in \mathbb{C}$, $\mathrm{Re}(s) > d := \dim A$ and

$$\sum_{p\,:\,\text{prime}} \frac{N_p}{p^s} = n \cdot \log(s-d)^{-1} + O(1) \quad \text{as} \;\; s \to d_+\,,$$

where n is the number of d-dimensional irreducible components of $\operatorname{Spec} A$.

More generally, we may consider solutions of the system $P_1 = \cdots = P_f = 0$ in any finite field $\mathbb{F}_q$, which corresponds to maximal ideals $\mathfrak{m}$ of A such that $q = N_\mathfrak{m} := \#(A/\mathfrak{m})$.

After Euler, Riemann, Dedekind, Artin, Hasse and Weil, one defines

$$\zeta_\mathcal{X}(s) = \prod_{\mathfrak{m}\in\mathcal{X}_0} (1 - N_\mathfrak{m})^{-1}$$

for $\mathcal{X} = \operatorname{Spec} A$. One has

$$\log \zeta_\mathcal{X}(s) = \sum_p \frac{N_p}{p^s} \ + \ \text{error terms}\,.$$

The function $\zeta_{\operatorname{Spec}\mathbb{Z}}$ is the usual zeta function in analytic number theory.

Some most important problems concerning the zeta functions involves the study of analytic continuation of $\zeta_\mathcal{X}$, of zeros and poles of $\zeta_\mathcal{X}$, etc.

When $\mathcal{X}$ is defined over a finite field $\mathbb{F}_p$, great contributions have be made by Dwork, Grothendieck and Deligne. In the case $\dim\mathcal{X} = 1$, there are remarkable works by Hecke and Schmidt. When $\dim\mathcal{X} = 2$ and $\mathcal{X}_\mathbb{Q}$ is an elliptic curve, we have the famous theorem of Wiles. In the case where $\mathcal{X}_\mathbb{Q}$ is a Shimura variety, a lot of problems remain to be interesting topics for further study.

2. Hensel, ... , Tate, Langlands: Analysis and Representation Theory

An ***absolute value*** on a field K is a map

$$|\cdot| : K \to \mathbb{R}_+$$

such that the following properties hold for all $x,\, y \in K$

(1) $|x+y| \le |x| + |y|$;
(2) $|xy| = |x|\cdot|y|$;
(3) $|x| = 0 \iff x = 0$.

An absolute value $|\cdot|$ on K is called ***ultimetric*** if $|x+y| \le \max\{|x|\,,\,|y|\}$ for all $x,\, y \in K$.

Given an absolute value $|\cdot|$, $d_{|\cdot|}(x,y) := |x-y|$ defines a distance on K. The absolute value $|\cdot|$ is called ***nontrivial*** if there exits some $x \in K^*$ such that $|x| \ne 1$. Two absolute values $|\cdot|$ and $|\cdot|'$ are said to be ***equivalent*** if $d_{|\cdot|}$ and $d_{|\cdot|'}$ define the same topology on K. This condition is equivalent to saying that there is a positive real number $\alpha \in \mathbb{R}_+^*$ such that $|\cdot|' = |\cdot|^\alpha$.

The set of ***places*** of K, denoted by $V(K)$, is the set of all non-trivial absolute values on K modulo the above equivalence relation. By convention, when K is a Riemannian field (i.e., function field of an algebraic curve over $\mathbb{C}$) or a global function field (i.e., a finite extension of $\mathbb{F}_p(T)$), we restrict ourselves to places represented by absolute values that are trivial on the field of constants $\mathbb{C}$ or $k = \mathbb{F}_q$.

Given $v = [|\cdot|] \in V(K)$, define

$$K_v := \text{the completion of } \ K \text{ with respect to } \ d_{|\cdot|}\,.$$

Then K_v is an extension field of K, and $|\cdot|$ extends to an absolute value on K_v.

For a Riemannian field $K \cong \mathcal{M}(\widetilde{C}^{\mathrm{an}})$, there is a bijection

$$\widetilde{C}^{\mathrm{an}} \xrightarrow{\sim} V(K)\;; \quad P \longmapsto [|\cdot|_P]$$

where for $f \in \mathcal{M}(\widetilde{C}^{\rm an})$, the value $|f|_P$ is defined as

$$|f|_P := \exp(-v_P(f))\,, \quad \text{with } \ v_P(f) := \text{the valuation of } f \text{ at } P\,.$$

For a global function field K, say the function field of a curve C over a finite field $k = \mathbb{F}_q$. Let C_0 be the set of closed points of C and let

$$C(\overline{\mathbb{F}}_q) := \bigcup_{n\geq 1} C(\mathbb{F}_{q^n})\,.$$

We may identify C_0 with the set of orbits of the Galois action of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$ on the set $C(\overline{\mathbb{F}}_q)$. Give $P \in C_0$ corresponding to the orbit of a point $x \in C(\overline{\mathbb{F}}_q)$, we say the ***field of definition*** of x is $\mathbb{F}_{q^n}$ if $\kappa(P) = \mathbb{F}_{q^n}$. Set $N_P := |\kappa(P)|$ and defined an absolute valuation $|\cdot|_P$ on K by

$$|f|_P := N_P^{-v_P(f)}\,,$$

where v_P denotes the valuation at P (or at x). Then there is a bijection

$$C_0 \xrightarrow{\sim} V(K)\,; \quad P \longmapsto [|\cdot|_P]$$

For a number field K, the ring of integers $\mathcal{O}_K$ is a Dedekind domain. Every non-zero ideal I of $\mathcal{O}_K$ has a unique factorization into products of non-zero prime ideals:

$$I = \prod_{\mathfrak{p}\neq 0} \mathfrak{p}^{v_{\mathfrak{p}}(I)}$$

where every $v_{\mathfrak{p}}(I) \geq 0$ and $v_{\mathfrak{p}}(I) = 0$ for almost all $\mathfrak{p}$. For each $\mathfrak{p} \in (\mathrm{Spec}\,\mathcal{O}_K)_0$, the residue field $\mathbb{F}_{\mathfrak{p}} := \mathcal{O}_K/\mathfrak{p}$ is a finite field with prime field $\mathbb{F}_p$ if p is the prime number such that $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$. Set $N_{\mathfrak{p}} := \#\mathbb{F}_{\mathfrak{p}} = p^{[\mathbb{F}_{\mathfrak{p}}:\mathbb{F}_p]}$ and define a function $v_{\mathfrak{p}} : K \to \mathbb{Z} \cup \{+\infty\}$ by

$$v_{\mathfrak{p}}(x) := \begin{cases} +\infty\,, & \text{if } \ x = 0 \\ v_{\mathfrak{p}}(x\mathcal{O}_K)\,, & \text{if } x \in \mathcal{O}_K \setminus \{0\} \\ v_{\mathfrak{p}}(a) - v_{\mathfrak{p}}(b)\,, & \text{if } \ x = a/b \text{ with } a,\, b \in \mathcal{O}_K\,,\, b \neq 0\,. \end{cases}$$

Then $x \mapsto |x|_{\mathfrak{p}} := N_{\mathfrak{p}}^{-v_{\mathfrak{p}}(x)}$ defines an absolute value on K. There is a bijection

$$(\mathrm{Spec}\,\mathcal{O}_K) \coprod (\{\sigma : K \hookrightarrow \mathbb{C}\}/\text{complex conjugation}) \xrightarrow{\sim} V(K)$$

which sends a $\mathfrak{p} \in (\mathrm{Spec}\,\mathcal{O}_K)$ to the place $[|\cdot|_{\mathfrak{p}}]$ and sends a complex embedding $\sigma : K \to \mathbb{C}$ to the place $[|\sigma(\cdot)|]$. If $K = \mathbb{Q}$, for each prime number p, one has

$$|p^n a/b|_p = p^{-n}\,; \quad \forall\, a,\, b\,,\, n \in \mathbb{Z}\,,\; b \neq 0 \text{ such that } p \nmid ab\,.$$

The absolute value corresponding to the unique embedding $\sigma : \mathbb{Q} \to \mathbb{C}$ is the usual archimedean absolute value on $\mathbb{Q}$.

From the above, we see that there is a uniform way of recovering the points of C from the field K. Actually, the analogy between number fields and global function fields become more satisfactory when one takes the archimedean places $[|\sigma(\sigma)|]$ into account.

As a basic example for analogies among the three kinds of fields mentioned before, we remark that there is a product formula for each of the three cases. For a Riemannian field, we have for each $f \in \mathcal{M}(\widetilde{C}^{\mathrm{an}})$,

$$\sum_{P \in \widetilde{C}^{an}} v_P(f) = 0\,, \quad \text{or equivalently} \;, \quad \prod_{P \in \widetilde{C}^{\mathrm{an}}} |f|_P = 1\,.$$

For a global function field, we have for each $f \in K^*$,

$$\sum_{P \in C_0} v_P(f) \log N_P = 0\,, \quad \text{or equivalently}\,, \quad \prod_{P \in C_0} |f|_P = 1\,.$$

For a number field, we have for each $x \in K^*$,

$$\sum_{\mathfrak{p}} v_{\mathfrak{p}}(x) \log N_{\mathfrak{p}} + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log |\sigma(x)| = 0\,,$$

or equivalently,

$$\prod_{P \in V(K)} |f|_v = 1\,,$$

where $|\cdot|_v = |\cdot|_{\mathfrak{p}}$ if $v = [|\cdot|_{\mathfrak{p}}]$; $|\cdot|_v = |\sigma(\cdot)|$ if $v = [\sigma(\cdot)]$ for some σ such that $\sigma(K) \subseteq \mathbb{R}$; and $|\cdot|_v = |\sigma(\cdot)|^2$ if $v = [\sigma(\cdot)]$ for some σ such that $\sigma(K) \nsubseteq \mathbb{R}$.

Another example is the analogous description of completions. For a Riemannian field, if v is a place corresponding to a point $P \in \widetilde{C}^{\mathrm{an}}$ and if z is a local analytic coordinate at P, then

$$K_v \xrightarrow{\sim} \mathbb{C}[[T]][T^{-1}]\,; \quad z \longmapsto T$$

and the ring of integers $\mathcal{O}_v := \{x \in K \,|\, |\cdot|_P \leq 1\}$ is

$$\mathcal{O}_v \xrightarrow{\sim} \mathbb{C}[[T]]\,.$$

For a global function field, if v is a place corresponding to a $P \in C_0$, $\mathbb{F}_P = \kappa(P)$ and z is a local parameter at P, then

$$K_v \xrightarrow{\sim} \mathbb{F}_P[[T]][T^{-1}]\,; \quad z \longmapsto T$$

and

$$\mathcal{O}_v \xrightarrow{\sim} \mathbb{F}_P[[T]]\,.$$

For a number field, if v is an archimedean place, associated to a complex embedding $\sigma : K \to \mathbb{C}$, then $K_v \xrightarrow{\sim} \mathbb{R}$ if $\sigma(K) \subseteq \mathbb{R}$ and $K_v \xrightarrow{\sim} \mathbb{C}$ if $\sigma(K) \nsubseteq \mathbb{R}$. If v is a non-archimedean place, associated to a non-zero prime ideal $\mathcal{O}_K$ with $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$, then K_v is a finite field extension of $\mathbb{Q}_P$ and $\mathcal{O}_v$ is the integral closure of $\mathbb{Z}_p$ in K_v.

Note that if K is a global field, then every completion K_v is a locally compact field and $\mathcal{O}_v$ is an open compact subring of K_v. A locally compact field with non-discrete topology is called a ***local field***. In fact, all local fields are obtained in this way, i.e., completions of global fields.

To study global fields, we may use some analytic tools. For example, there exists a Haar measure λ_v on each $(K_v\,, +)$ such that

$$\lambda_v(xE) = |x|_v \lambda_v(E)\,,$$

for all $x \in K_v$ and $E \subseteq K_v$. The ring of ***adèles*** of the global field K is defined as

$$\mathcal{A}_K := \{(x_v) \in \prod_{v \in V(K)} K_v \,|\, x_v \in \mathcal{O}_v \ \text{ for almost all } \ v\}$$

with coordinate-wise addition and multiplication. This is a locally compact ring and K is a discrete cocompact subring of $\mathcal{A}_K$ via the diagonal embedding $x \mapsto (v \mapsto x)$.

N.B.: There is an Haar measure $\lambda = \otimes_v \lambda_v$ on $\mathcal{A}_K$ such that $\lambda_v(\mathcal{O}_v) = 1$ for almost all v. One recovers the product formula by observing that multiplication by any $x \in K^*$ preserves the Haar measure λ. As a fact, any field K equipped with absolute values having the above property is actually a global field.

The group of ***idèles*** of the global field K is the group $\mathcal{J}_K := \mathcal{A}_K^*$ is the group of invertible elements in the ring $\mathcal{A}_K$ equipped with the subspace topology induced from the product topology on $\mathcal{A}_K \times \mathcal{A}_K$ via the injection

$$\mathcal{J}_K = \mathcal{A}_K^* \longrightarrow \mathcal{A}_K \times \mathcal{A}_K \; ; \quad x \mapsto (x, x^{-1}) \, .$$

The multiplicative group $\mathcal{J}_K$ is a locally compact group. Define the subgroup $\mathcal{J}_K^1$ by the exact sequence

$$1 \longrightarrow \mathcal{J}_K^1 \longrightarrow \mathcal{J}_K \longrightarrow \mathbb{R}_+^* \longrightarrow 1$$

where the map $\mathcal{J}_K \to \mathbb{R}_+^*$ is given by

$$(x_v) \longmapsto \prod_v |x_v|_v \, .$$

By the product formula, K^* is a subgroup of $\mathcal{J}_K^1$. In fact, K^* is a cocompact subgroup of $\mathcal{J}_K^1$. This is a fancy reformulation of Dirichlet's theorems: the ***class group***

$$\mathrm{Cl}(K) := \{\text{non-zero fractional ideals of } \mathcal{O}_K\}/\sim \, ,$$

where $\sim$ is the equivalence relation defined by

$$I \sim J \iff I = \lambda J \, , \quad \text{for some } \lambda \in K^* \, ,$$

is finite and $\mathcal{O}_K^*$ is a finitely generated abelian group of rank $\#\{\text{archimedean places}\} - 1$.

Actually, $\mathcal{J}_K/K^*$ contains a lot of deep non-trivial information. The group of characters

$$(\mathcal{J}_K/K^*)^\wedge := \{\chi \, : \, \mathcal{J}_K \to U(1) \text{ a continous group homomorphism} \,|\, \chi|_{K^*} = 1\}$$

may be identified with the group of Hecke's *Größencharaktere*. Analysis on $\mathcal{J}_K/K^*$ can give information about the L-function.

More generally, by the work of Gelfand and Langlands, for $G = \mathrm{GL}_N$, the group

$$G(\mathcal{A}_K)^1 := \{g \in \mathrm{GL}_N(\mathcal{A}_K) \,|\, |\det(g)| = 1\}$$

contains $G(K) = \mathrm{GL}_N(K)$ as a discrete subgroup of finite covolume. The group $G(\mathcal{A}_K)^1$ acts unitarily on $L^1(G(\mathcal{A}_K)^1/G(K))$. This representation is supposed to encode deep informations concerning the global field K, e.g, the Galois group $\mathrm{Gal}(\overline{K}/K)$, motives over K and their Hasse-Weil zeta functions.

3. Varieties over global fields

Let $K = \kappa(C)$ be a field as in Weil's Rosetta Stone and let X be an algebraic variety over K. One may find a model $\mathscr{X}$ of X over C relevant for studying X over K, i.e., a flat morphism $\mathscr{X} \to C$ with generic fiber X/K.

Examples. (1) The number field case. If X is a subvariety of $\mathbb{A}^N_{\mathbb{Q}}$ with ideal $I_X = \{P \in \mathbb{Q}[x_1, \ldots, x_N] \,|\, P|_X = 0\}$. Then $\mathscr{I} := I_X \cap \mathbb{Z}[x_1, \ldots, x_N]$ defines a model $\mathscr{X} \subseteq \mathbb{A}^N_{\mathbb{Z}}$.

(2) The Riemannian field case. If $X$is a smooth projective curve over $\mathbb{C}(C)$, one may find a smooth projective complex surface fibered over C.

In the geometric cases (i.e, the cases of a global function field or a Riemannian field), C is a smooth projective curve over $k = \mathbb{F}_q$ or $\mathbb{C}$, one may use some extra tools available when doing geometry over k.

Example 1. The Riemanniann field case. Let $\pi : \mathscr{X} \to C$ be a surjective morphism from a smooth projective surface $\mathscr{X}/\mathbb{C}$ to a smooth projective curve $C/\mathbb{C}$ with generic fiber X/K. Let $X(K)$ be the set of K-rational points of X and let $\mathscr{X}(C)$ be the set of regular (holomorphic) sections of π. We may identify $X(K)$ with the set of meromorphic sections of π so that $\mathscr{X}(C)$ is a subset of $X(K)$. Since $\mathscr{X}$ is projective, we have $\mathscr{X}(C) = X(K)$. Thus, $X(K)$ may be investigated using the geometry of complex surfaces.

Mordell conjecture for function fields over $\mathbb{C}$ (Manin): if the genus of X is ≥ 2, then $X(K)$ is finite.

An analogous but simpler situation is an abelian scheme:

$$\begin{array}{ccc} A & \longrightarrow & \mathscr{A} \\ \big\downarrow & & \big\downarrow \pi \\ \operatorname{Spec} K & \longrightarrow & C \end{array}$$

Now there is a natural bijection between $A(K)$ with the set $\mathscr{A}(C)$ of analytic sections of $\pi : \mathscr{A} \to C$. In this situation, there is an analytic proof of Mordell-Weil conjecture. We sketch the ideas of proof as follows:

There is an sequence of abelian sheaves over C^{an}:

$$0 \longrightarrow \Gamma \longrightarrow \mathrm{Lie}\mathscr{A} \longrightarrow \mathscr{A} \longrightarrow 0$$

which gives on fibers an exact sequence

$$0 \longrightarrow \mathbb{Z}^{2g} \longrightarrow \mathbb{C}^{2g} \longrightarrow \mathbb{C}^{2g}/(\mathbb{Z}^g + \Omega\mathbb{Z}^g) \longrightarrow 0$$

where g is the genus of the curve C. Taking cohomology gives an exact sequence

$$H^0(C\,,\, \mathrm{Lie}\mathscr{A}) \longrightarrow \mathscr{A}(C) \longrightarrow H^1(C\,,\, \Gamma)\,.$$

By a "curvature" argument, one can prove that "$\mathscr{A}$ has no fixed points". Together with "some positivity of the Hodge bundle", this yields $H^0(C\,,\, \mathrm{Lie}\mathscr{A}) = 0$. It follows that $\mathscr{A}(C) \to H^1(C\,,\, \Gamma)$ is an injection. The topoligcal Euler-Poincaré characteristic $\chi_{\mathrm{top}}(C\,,\, \Gamma)$ is

$$\chi_{\mathrm{top}}(C\,,\, \Gamma) = 2g\chi_{\mathrm{top}}(C) = 2g(2g-2) = 4g(g-1)\,.$$

So $H^1(C\,,\, \Gamma)$ is a finitely generated $\mathbb{Z}$-module of rank $4g(g-1)$. Hence, $A(K) \cong \mathscr{A}(C)$ is finite generated as a $\mathbb{Z}$-module, of rank at most $4g(g-1)$.

N.B.: A typical "Hodge theoretic" argument usually consists of a "sign of curvature" argument and a relation between coherent cohomology groups and Betti/topological cohomology groups.

Example. 2. The global function field case. Let $K = \mathbb{F}_q(C)$, i.e., C is a model of the function field K. Put

$$\zeta_K(s) := \zeta_C(s) = \prod_{P \in C_0} (1 - N_P^{-s})^{-1}\,.$$

Artin and Schmidt proved that $\zeta_K(s)$ is a rational function of q^{-s}. In fact, one has

$$\zeta_K(s) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

where $P(X) = \prod_{1 \le i \le 2g}(1 - a_i X) \in \mathbb{Z}[X]$. Moreover,

$$\#C(\mathbb{F}_{q^n}) = q^n + 1 - \sum_{i=1}^{2g} a_i^n\,.$$

The Riemann Hypothesis for ζ_K is equivalently to each of the following statements: (1) $|a_i| = \sqrt{q}$ for each i; (2) $|\#C(F_{q^n}) - q^n - 1| \le 2g\sqrt{g}^n$ for all $n \ge 1$.

We will give a sketch of the proof of the Hasse-Weil estimate:

$$|\#C(F_q) - q - 1| \le 2g\sqrt{q}\,.$$

Consider the projective surface $S := C \times_{F_q} C$ and use

• intersection theory on a projective surface: There is an intersection symmetric bilinear form $\langle \cdot\,,\, \cdot \rangle$ defined for curves on S, which has the following properties: if C_1 and C_2 meet properly, then

$$\langle C_1\,,\, C_2 \rangle = \#(C_1 \cap C_2)$$

and $\langle C_1\,,\, C_2 \rangle$ is invariant when C_1 or C_2 moves in an algebraic family. The Hodge index theorem (Castelnuovo-Severi conjecture) states that for curve classes $C_1, \ldots, C_n$ on S the matrix of intersection numbers $(\langle C_i\,,\, C_j \rangle)$ has signature

$$(-\,,\,-\,,\,\ldots\,,-\,,\,0,\,\ldots\,,0) \quad \text{or} \quad (+\,,\,-\,,\ldots\,,-\,,\,0\,,\ldots\,,0)\,.$$

Consequently, if there are $n_i \in \mathbb{R}$ such that $\sum_{i,\,j} n_i n_j \langle C_i\,,\, C_j \rangle > 0$, then

$$(-1)^{n+1} \det(\langle C_i\,,\, C_j \rangle) \ge 0\,.$$

• Frobenius morphisms: Over a field of characteristic $p > 0$, one has $(a+b)^p = a^p + b^p$. So there is a Frobenius morphism

$$F = F_q \;:\; C/\mathbb{F}_q \longrightarrow C/\mathbb{F}_q$$

given by the coordinate expression

$$(x_0\,,\ldots,x_N) \longmapsto (x_0^q\,,\ldots,x_N^q)$$

when C is embedded in a projective space $\mathbb{P}^N_{\mathbb{F}_q}$. We may then identify

$$C(\mathbb{F}_q) = C(\overline{\mathbb{F}}_q)^F\,.$$

Note that $F = F_q$ is a morphism of degree q from C to itself.

Sketch of proof of Hasse-Weil estimate. We consider the curve classes

$C_1 = V := \{P\} \times C\,;\;\; C_2 = H := C \times \{P\}\,;$

$C_3 = \Delta :=$ the diagonal of $C_{\overline{\mathbb{F}}_q} \times_{\overline{\mathbb{F}}_q} C_{\overline{\mathbb{F}}_q}$

$C_4 = G :=$ the graph of the Frobenius morphism F

over $S_{\overline{\mathbb{F}}_q} = C_{\overline{\mathbb{F}}_q} \times_{\overline{\mathbb{F}}_q} C_{\overline{\mathbb{F}}_q}$. Write $N := \#C(\mathbb{F}_q)$. The intersection matrix $\langle C_i \, , \, C_j \rangle$ is

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & q \\ 1 & 1 & 2-2g & N \\ 1 & q & N & q(2-2g) \end{pmatrix} .$$

Since $\langle C_1 + C_2 \, , \, C_1 + C_2 \rangle = \langle V + H \, , \, V + H \rangle = 2 > 0$, by Hodge index theorem,

$$\det(\langle C_i \, , \, C_j \rangle) = (N - (q+1))^2 - 4g^2 q \leq 0 \, ,$$

whence the Hasse-Weil estimate. □

A natural question is how to transfer these arguments to the number field case. Tools we have now in hand include:

• Coherent cohomology coming from heights, Diophantine geometry and Arakelov theory;

• (p-adic) Hodge theory coming from heights, Arakelov theory, étale cohomology, etc.

But what else?