

# Slopes and Harder-Narasimhan Filtrations in Arithmetic and Geometry

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## Reference

# Preface

These notes are the faithful record of a course “Slopes and Harder-Narasimhan Filtrations in Arithmetic and Geometry”, which is given by Prof. Jean-Benoît Bost during July and August in 2010, in the Mathematical Sciences Center, Tsinghua University. All the manuscript is taken by Chunhui Liu (E-mail: [ichws@163.com](mailto:ichws@163.com)) from Department of Mathematical Sciences, Tsinghua University.

The theory of slopes and of Harder-Narasimhan filtrations—initially devised for the study of vector bundles on projective algebraic curves and then on projective varieties—makes sense in a general abstract framework, which turns out to have applications in very diverse contexts, including the theory of  $p$ -adic representations (Fontaine), analytic differential equations in the complex and  $p$ -adic domains, the theory of shtukas (Drinfeld-Lafforgue), Diophantine geometry and diophantine approximation on algebraic groups, Hermitian vector bundles in Arakelov geometry, ... In these various domains, the general formalism of slopes and Harder-Narasimhan filtrations provides a simple and unified view of constructions and proofs, and turns out to be very helpful both conceptually and technically.

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