Introduction to differential topology

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Contents

Chapter 1

Some basic concepts

Differential topology studies basically the topology of Differential manifolds. As the prequisitive of this course, we assume the readers to have some basic knowledge of the subjects below which we will introduce briefly here in the beginning chapter: differential manifolds, curves and functions defined on manifolds, differential maps between manifolds, (co)tangent space, (co)tangent bundle, vector field, differential forms and the exterior differentiation and wedge product of them, Lie-Poisson bracket, Lie derivative, orientation of manifolds and integration of differential n-forms, Stokes theorem etc.

Definition 1.1 (differential manifolds)**.** *An $n-$dimensional differential manifold is a set M with injective maps $f_\alpha : U_\alpha(\subset \mathbb{R}^n) \to M$ s.t.*

(1) $M = \bigcup\limits_{\alpha \in A} f_\alpha(U_\alpha)$

(2) When $W = f_\alpha(U_\alpha) \bigcap f_\beta(U_\beta) \neq \emptyset$, where $f_\alpha^{-1}(W), f_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$, the mappings $f_\beta^{-1} \circ f_\alpha : f_\alpha^{-1}(W) \to \mathbb{R}^n$ and $f_\alpha^{-1} \circ f_\beta : f_\beta^{-1}(W) \to \mathbb{R}^n$ are differentiable.

(U_α, f_α) where α belongs to the index set, are called charts and the sets of all charts $\{(U_\alpha, f_\alpha)\}_{\alpha \in A}$ is called atlas.

the usual definition also includes the next condition

(3) $\{(U_\alpha, f_\alpha)\}$ is maximal, which means if there is a combination of (U, f) and f with any mapping in $\{(U_\alpha, f_\alpha)\}$ satisfies the above condition, then $(U, f) \in \{(U_\alpha, f_\alpha)\}$

There are several examples of differential manifolds, like $\mathbb{R}^n$, S^1(the unit circle), Σ_g(Riemann surface with genus g), torus T^n, sphere surface S^n, projective space $\mathbb{RP}^2 = \mathbb{R}^3 - \{0\}/\sim = S^2/\sim$, lie groups $SU(2) \cong S^3$ and $SO(3) \cong \mathbb{RP}^3$.

Remark 1.1. *M is not priorly a topological space but the images of the mapping f_α naturally give M a topological structure in that $\{f_\alpha(U_\alpha)\}_{\alpha \in A}$ serves as the topological base of M.*

Without the condition (3) we will have $V \in top(M)$(the sets of all open subsets of M) $\Leftrightarrow f_\alpha^{-1}(V) = f_\alpha^{-1}(V \bigcap f_\alpha(U\alpha))$ is open in $\mathbb{R}^n$

Remark 1.2. *There are usually two technical conditions*

(1) $top(M)$ is Hausdorff

(2) $top(M)$ is second countable.

But in our more general definition, there are also non-Hausdorff manifolds like "Y" where the intersection "point" is acturally two different points.

Remark 1.3. *if M and N are differential manifolds, then $M \times N$ is also a manifold with induced differential structure.*

Definition 1.2 (curves)**.** *let M be a manifold, a differential curve through $p \in M$ is a map $\gamma : (-\epsilon, \epsilon) \to M$ satisfying $\gamma(0) = p$ and $\forall (U_\alpha, f_\alpha)$ $p \in f_\alpha(U_\alpha) \Rightarrow f_\alpha^{-1} \circ \gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ is differentiable.*

Definition 1.3 (differentiable function). *A differentiable function on M is a function $h : M \to \mathbb{R}$ s.t. $h \circ f_\alpha : U_\alpha \to \mathbb{R}$ is differentiable $\forall \alpha \in A$*

We denote by $C^\infty(M)$ the set of all differentiable function on M and it is naturally an $\mathbb{R}$-algebra.

Definition 1.4 (differential map). *a differential map from a manifold M to another manifold N is a continuous map $F : M \to N$ and $\forall (U_\alpha, \varphi_\alpha)$ of M and (V_β, ϕ_β) of N, $\phi_\beta^{-1} \circ F \circ \varphi_\alpha$ is differentiable.*

Differential curves and differential fuctions are all simple examples of differential maps.

Definition 1.5 (Lie groups). *A lie group G is a group G with a manifold structure and the group product $m : G \times G \to G$ and inverse $i : G \to G$ are both differentiable maps between manifolds.*

Definition 1.6 (tangent space). *there are two different definition of tangent space on a point $p \in M$:*

1° *(Geometric) $T_pM = \Gamma(M,p)/\sim$, where $\Gamma(M,p)$ denotes all the curves γ on M satisfying $\gamma(0) = p$, $\gamma_1 \sim \gamma_2 \Leftrightarrow \frac{d}{dt}|_0 \varphi_\alpha \circ \gamma_1 = \frac{d}{dt}|_0 \varphi_\alpha \circ \gamma_2$, where φ_α is any map defined on a subset in $\mathbb{R}^n$ in the manifold structure defined above.*

2° *(Algebraic) T_pM is the set of derivation $v : C^\infty(M) \to \mathbb{R}$ of $\mathbb{R}$-algebra $C^\infty(M)$ which satisfies $v(fg) = v(f)g(p) + f(p)v(g)$.*

Remark 1.4. *the defintion 1° $\Leftrightarrow$ 2° and 2° $\Rightarrow T_pM$ is a vector space of basis $\{\frac{\partial}{\partial x^i}|_p\}$.*

Definition 1.7 (cotangent space). *there are also 2 different definition:*

1° *T_p^*M is the dual space of the tangent space*

2° *$T_p^*M = C^\infty(M)/\sim$, where $f_1 \sim f_2 \Leftrightarrow \frac{d}{dt}|_p f_1 \circ \gamma = \frac{d}{dt}|_p f_2 \circ \gamma, \forall \gamma \in \Gamma(M,p)$*

Definition 1.8 (the grassmann algebra). *we donote by $\Lambda^k(T_p^*M)$ the k-multilinear skew-symmetric functions on T_pM and $\Lambda^*(T_p^*M) = \oplus_{k=0}^n \Lambda^k(T_p^*M)$, where n is the dimension of the manifold. And we have a operation called wedge product $\wedge$ on $\Lambda^*(T_p^*M)$ which satisfies the graded commutativity:$\alpha \wedge \beta = (-1)^{deg\alpha \cdot deg\beta}$, the algebra together with the wedge product $(\Lambda^*(T_p^*M), \wedge)$ is a so-called graded commutative algebra.*

In the special cases we have: $\Lambda^0(T_p^*M) = \mathbb{R}$, $\Lambda^1(T_p^*M) = T_p^*M$, $\Lambda^1(T_p^*M) \cong \mathbb{R}$ if the manifold M is orientable.

Definition 1.9 (tangent and cotangent bundle). *$TM = \bigcup_{p \in M} T_pM$ is a manifold with charts $\tilde{\varphi}_\alpha : \tilde{U}_\alpha = U_\alpha \times \mathbb{R}^n \to TM, (a,b) \to \sum_{i=1}^n b_i \frac{\partial}{\partial x^i}|_{\varphi_\alpha(a)}$ we can define T^*M and $\Lambda^k T^*M$ similarly.*

Definition 1.10 (section). *$s : M \to TM$ is called a section if $\pi \circ s = id_M$ where π is the projection from TM to M.*

We denote $\Gamma(TM)$ the space of sections of TM, then $\mathfrak{X}(M) := \Gamma(TM)$ is the vector space of vector fields on M. furthermore, the differential k-form $\Omega^k(M) = \Gamma(\Lambda^k(T^*M))$

Theorem 1.1. $\mathfrak{X}(M) \cong \{Derivations C^\infty(M) \to C^\infty(M)\}$

we have the Leibniz rule $X(fg) = x(f)g + fX(g)$, so for $X, Y \in \mathfrak{X}(M)$ $XY : C^\infty(M) \to C^\infty(M)$ is not a derivation, but the lie-poisson braket of them $[X,Y] = XY - YX \in \mathfrak{X}$ is indeed a derivation.

Example 1.1.

The lie group G with the left transportation $lg_* : T_eG \overset{\cong}{\to} T_gG$ a isomorphism, so we know $TG \cong T_eG \times G$ is parallelizeble.

A vector field on G is called left invariant if $X_g = lg_* X_e$, we denote by $\mathfrak{g}$ the space of left invariant vector fields on G.

$T^*G \cong T_e^*G \times G$ is the left invariant 1-forms and the left invariant differential forms $\Lambda^*(\mathfrak{g}^*) \subset \Omega^*(G)$ is a graded commutative algebra.

We know $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ so $(\mathfrak{g}, [,])$ is a subalgebra of lie algebra $(\mathfrak{X}(G), [,])$.

Definition 1.11 (the category of manifolds)**.** *As all other category, the category of manifolds* $\mathbf{mfd}$ *involve two sets: the object set which contains all manifolds called* $\mathbf{mfd}^0$*, and all the morphism set* $\mathbf{mfd}^1$ *of differential maps between each two of the manifolds in the object set, we say* $F \in \mathbf{mfd}^1(M,N)$ *if* $F : M \to N$ *is a differential map. And the morphisms meet the following conditions:*

(1) Composition: if $F \in \mathbf{mfd}^1(M,N), G \in \mathbf{mfd}^1(N,L)$*, then* $G \circ F \in \mathbf{mfd}^1(M,L)$*,*

(2) Associativity: $(H \circ G) \circ F = H \circ (G \circ F)$

(3) Identity: $id_M \in \mathbf{mfd}^1(M,M)$

An Isomorphism is a morphism $F \in \mathbf{mfd}^1(M,N)$ satisfying $F \circ F^{-1} = id_N$ and $F^{-1} \circ F = id_M$, i.e. it is a diffeomorphism.

A question is what need to be idetified to show **mfd** is a category. The answer is merely that the composition of differentiable maps is still a differentiable map, all the others are trivial.

Definition 1.12 (some basic categories)**.**
1.$\mathbf{mfd}$*: The category of manifolds with the morphism of mapping between manifolds.*
2.$\mathbf{mfd}^\times$*: The subcategory of* **mfd** *with the morphism of isomorphism.*
3.$\mathbf{mfd}_*$*:The category of manifolds with a marked point.*
4.$\mathbf{Vect}_\mathbb{R}$*:The category of vector spaces with the morphism of linear transitions between vector spaces.*
5.$\mathbf{GCA}_\mathbb{R}$*:The category of graded commutative algebra.*

Definition 1.13 (some basic functors)**.**
1.Tagent space $T : \mathbf{mfd}_* \to \mathbf{Vect}_\mathbb{R}$ *s.t.* $(M,p) \to T_pM$ *and* $F \to F_*$ *which is covariant.*
2.Cotangent space $T^* : \mathbf{mfd}_* \to \mathbf{Vect}_\mathbb{R}$*:similar as above and can be easily extended to* $\wedge^* T^* : \mathbf{mfd}_* \to \mathbf{GCA}_\mathbb{R}$ *which is contravariant*
3.Function Algebra: $\Omega^0 : \mathbf{mfd} \to \mathbf{CA}_\mathbb{R}$ *and can be easily extended to* $\Omega^* : \mathbf{mfd}_* \to \mathbf{GCA}_\mathbb{R}$*.*
4.Vector field $\mathfrak{X} : \mathbf{mfd}^* \to \mathbf{LA}_\mathbb{R} \subset \mathbf{Vect}_\mathbb{R}$ *and we have the proposition that* $\mathfrak{X} \cong Der(C^\infty(M))$*.*

For a differential map$F : M \to N$, we have the induced fuctors $F_{*p} : T_pM \to T_{F(p)}N$ which is covariant:$(G \circ F)_{*p} = G_{*F(p)} \circ F_{*p}$, and $F^{*p} : \Lambda^k(T^*_{F(p)}N) \to \Lambda^k(T^*_pM)$ which is contravariant: $(G \circ F)^{*p} = F^{*p} \circ G^{*F(p)}$

And we also have functor $F^* : \Omega^k(N) \to \Omega^k(M)$, with $F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$ and $(G \circ F)^*\alpha = F^* \circ G^*\alpha$ and fuctors

$$\Omega^k : \mathbf{mfd} \to \mathbf{Vect}_\mathbb{R}, M \to \Omega^k(M),$$

$$\Omega^* : \mathbf{mfd} \to \mathbf{GCA}_\mathbb{R}, M \to \Omega^*(M),$$

$$\Omega^0 : \mathbf{mfd} \to \mathbf{CA}_\mathbb{R}$$

We have a question here: What about vector field? Is $\mathfrak{X}(M)$ a functor? The answer is no. Unlike pulling back differential forms, we cannot "push forward" vector fields in usual situation.

Actually we have a functor $* : \mathbf{mfd}^* \to \mathbf{Vect}_\mathbb{R}$ for $\mathbf{mfd}^*$ of which the morphisms are all diffeomorphism and which is a groupoid of the category **mfd**.

Chapter 2

The structure of $Der(\Omega^*(M))$

2.1 Graded Leibnitz Rule and Graded Lie Bracket

As we know, the exterior differentiation $d : \Omega^* \to \Omega^*$ is an element of $Der(\Omega^*(M))$ with degree **1** and satisfies the "Graded Leibnitz Rule" i.e $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\tilde{\alpha}}\alpha \wedge d\beta$ where $\tilde{\alpha}$ is defined as $deg(\alpha)$.
Here, we come up with a question:given $X, Y \in Der(\Omega^*(M))$, whether XY is in $Der(\Omega^*(M))$? If not, what combination will work? Initiate by the method dealing with $Der(C^\infty(M))$, we just calculate XY and YX and try to find what coefficient should be multiplied in order to satisfied the "Grade Leibnitz Rule". Consequently, we get the below definition:

$$[X, Y] = XY - (-1)^{\tilde{X}\tilde{Y}}YX$$

and its action is

$$[X, Y](\alpha \wedge \beta) = ([X, Y]\alpha) \wedge \beta + (-1)^{(\tilde{(Y)}+\tilde{X})\tilde{\alpha}}\alpha \wedge ([X, Y]\beta).$$

Besides, we achieve the **GradedJacobiIdentity**:

$$(-1)^{\tilde{X}\tilde{Z}}[[X, Y], Z] + (-1)^{\tilde{Y}\tilde{X}}[[Y, Z], X] + (-1)^{\tilde{Z}\tilde{Y}}[[Z, X], Y] = 0.$$

So $(Der(\Omega^*(M)), [,])$ is a graded Lie algebra($\mathbf{GLA}_{\mathbb{R}}$).

2.2 The existence of derivation of degree 0

We notice the d is of degree 1 and $d^2 = 0$. Then a natural question is: what about the existence of other degrees? Firstly, let's introduce a derivation of degree $-\mathbf{1}$. $\forall X \in \mathfrak{X}(M)$, define $i_X : \Omega^{k+1}(M) \to \Omega^k(M)$ to be

$$i_X\omega(X_1, \cdots, X_k) = \omega(X, X_1, \cdots, X_k).$$

And it is easily derived that

$$i_X(\alpha_0 \wedge \cdots \wedge \alpha_k) = \sum_{i=0}^{k}(-1)^i < X, \alpha_i > \alpha_0 \wedge \cdots \wedge \hat{\alpha_i} \wedge \cdots \wedge \alpha_k.$$

Proposition 2.1. $(1) i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^{\tilde{\alpha}}\alpha \wedge i_X\beta$
$(2) i_Xi_Y = -i_Yi_X \qquad (3) F^*iX\alpha = i_{F_*^{-1}(x)}F^*\alpha.$

Remark 2.1. *(1)and(2) can be got directly by the definition. (1) implies that $i_X \in Der(\Omega^*(M))$ and (2) shows that $[i_X, i_Y] = 0$. To prove (3), we just need to examine the validity for a pure tensor because of the linearity of F^* and i_X.*

Now, we can give an answer to our above question about the existence of derivations with degree **0**. Consider $[i_X, d] = i_Xd + di_X$. Restricted to $\Omega^0(M)$, it turns out that $[i_X, d](f) = i_Xdf = X(f)$.What's more, $X \mapsto [i_X, d]$ defines a **GLA**-homomorphism $\mathfrak{X}(M) \to Der(\Omega^*(M))$ with the proposition $[i_{[X,Y]}, d] = [[i_X, d], [i_Y, d]]$.

2.3 View of Geometry and Lie Derivation

Proposition 2.2. $\mathfrak{X}(M) \cong \{$ *smooth families* $(F_t)_{t\in(-\epsilon,\epsilon)}$ *of diffeomorphism* F_t:$M \to M$, $F_0 = id_M\}/\sim$.

Given $(F_t)_{t\in(-\epsilon,\epsilon)}$, $F_t^* : C^\infty(M) \to C^\infty(M)$ is an algebra homomorphism $\forall t \in (-\epsilon, \epsilon)$ and $F_t^*(fg) = F_t^*(f) \cdot F_t^*(g)$. Define $X(f) = \frac{d}{dt}|_0 F_t^*(f)$, then it is to verify that $X \in \mathfrak{X}(M)$.
Conversely, we claim that $\forall x \in \mathfrak{X}(M)$, $\exists (F_t)_{t\in(-\epsilon,\epsilon)}$such that $\frac{d}{dt}|_0 F_t^*(f) = X(f)$.
Proof: For fixed $p \in M$, $X_p = \frac{d}{dt}|_0 F_t^*(p) \Rightarrow X_p$ is tangent vector of the curves. $F_t(p)$ can be taken to be the solution curve of the ODE:

$$\begin{cases} dF_t(p) &= X_{F_t(p)} \\ F_0(p) &= p \end{cases}$$

$\exists$ unique solution $F_t(p)$ smoothly depending on t and p. #

Proposition 2.3. $[X, Y] = \frac{d}{dt}|_{t=0} F_{t*}^{-1}(Y) = -\frac{d}{dt}|_0 F_{t*} Y$, *where* $X = \frac{d}{dt}|_{t=0} F_t^*$

Proof:

$$\begin{aligned} [X, Y]_p f &= X_p Y(f) - Y_p X(f) \\ &= \frac{d}{dt}|_0 F_t^*[Y(f)] - Y \frac{d}{dt}|_0 F_t^* f \\ &= \frac{d}{dt}|_0 (Y(f) \circ F_t) - Y_p \frac{d}{dt}|_0 (f \circ F_t) \\ &= \frac{d}{dt}|_0 Y_{F_t(p)} (f \circ F_{-t}) \\ &= (\frac{d}{dt}|_{t=0} F_{t*}^{-1})_p (Y) \end{aligned}$$

Since we have $F^* X(f) = i_{F_*^{-1}} X dF^* f = F^{-1} X(F^* f)$ and $Y \frac{d}{dt}|_0 F_t^* f = \frac{d}{dt}|_0 (F_{t*} Y) f$.

Remark 2.2. *For different proof, we may refer to "Lecture Notes on Differential Geometry" by Chern.*

Now, we give the definition of **Lie Derivation**:

$$L_X \alpha = \frac{d}{dt}|_0 F_t^* \alpha.$$

And we can get the "Cartan's Formula":$L_X = [i_X, d]$. To prove the formula, we just need to show that $L_X \in Der(\Omega^*(M))$ and verify it's valid for f and df which is easy to do since we've known the proposition of i_x and F^*.
Recall that $\mathfrak{X}(M) \cong Der(\Omega^*(M))$. Then what about $\Omega^*(M)$? The Answer is that the graded Lie algebra of graded derivation of $\Omega^*(M)$ is generated by i_X, $X \in \mathfrak{X}(M)$ and d module the relation

$$[d, d] = 0, [i_u, i_v] = 0, [L_v, i_w] = i_{[v,w]}.$$

Let's summarize what we have done with the language of category.
1)$\bar{\mathfrak{X}}$: $\mathbf{mfd}^\times \to \mathbf{GLA}_\mathbb{R}, M \mapsto Der(\Omega^*(M)) := \bar{\mathfrak{X}}, F \mapsto F_*$with the propositions:

$$F_* i_X = i_{F_*} X, F_* d = dF_*, F_*[i_X, d] = [i_{F_* X}, d].$$

2)$(\Omega^*(M), d)$ is a differential graded commutative algebra ($\mathbf{DGCA}_\mathbb{R}$).
Ω^* : $\mathbf{mfd} \to \mathbf{DGCA}_\mathbb{R}$,$M \to (\Omega^k(M), d)$. For $F : M \to N$, $F^*(\Omega^*(N), d) \to (\Omega^*(M), d)$,

$$F^*(\alpha \wedge \beta) = F^* \alpha \wedge F^* \beta, F^* d\alpha = dF^* \alpha, (G \circ F)^* = F^* \circ G^*.$$

Example 2.1. (M, ω) *is a symplectic manifold.*$\forall f \in \Omega^0(M), \exists! X_f \in \mathfrak{X}(M), i_{Xf} \omega = df, L_{Xf} = 0 \Rightarrow$ *Flow of* X_f *preserve* ω.
$i_{[X_f, X_g]} \omega = [L_{X_f}, i_{X_g}] \omega = d i_{X_f} i_{X_g} \omega + i_{X_f} i_{X_g} \omega = d(i_{X_f} i_{X_g} \omega)$.
$g, f = \omega(X_g, X_f)$ *is called Poisson bracket. And we have* $[X_f, X_g] = X_{g,f}$. *So we get an map from* $\Omega^0(M) \to \mathfrak{X}(M)$, *and the image is a Lie algebra and the pull-back gives a Poisson Algebra structure on* $\Omega^0(M)$.

Chapter 3

de Rham cohomology

3.1 Introduction

Definition 3.1 (de Rham cohomology)**.**
For the chain complex

$$\cdots \to A^{k-1} \stackrel{d^{k-1}}{\to} A^k \stackrel{d^k}{\to} A^{k+1} \stackrel{d^{k+1}}{\to} A^{k+2} \to \cdots$$

we define

$$H^k(M) = \frac{ker(d_k)}{Im(d_{k-1})}, k = 0, 1, ...n$$

Q : when is a closed form exact?
1. Poincaré Lemma: closed 1-form is locally exact.
2. Closed 1-form on simply connected region is exact.

Example 3.1.

$$H^k(\mathbb{R}) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

1° $H^k(\mathbb{R}) = 0\ \forall k > 1$ *because* $\Omega^k(\mathbb{R}) = 0, \forall k > 1$

2° $H^0(\mathbb{R}) = \{f \in \Omega^0(\mathbb{R}) : df = 0\} \cong \mathbb{R})$

3° $H^1(\mathbb{R})$
$\forall \alpha \in \Omega^1(\mathbb{R})$ *s.t.* $d\alpha = 0$, $\alpha = fdx$, *let* $g(x) = \int_0^x f(t)dt$, *then* $dg = \alpha$, *so* $H^1(\mathbb{R}) = 0$

Example 3.2.

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

1° *similarly* $H^k(\mathbb{R}) = 0\ \forall k > n$

2° $H^0(\mathbb{R}) = \mathbb{R}$

3° *for* $1 \le k \le n-1$, $\forall \alpha \in \Omega^k(\mathbb{R}^n)$ *s.t* $d\alpha = 0$ $\exists \beta \in \Omega^{k-1}(\mathbb{R}^n)$ *s.t.* $\alpha = d\beta$ *by Poincare Lemma which we will stated below.*

Example 3.3. $H^k(S^1)$

1° $H^k(S^1) = 0 \forall k > 1$

2° $H^0(S^1) = \mathbb{R}$

3° $H^1(S^1)$

$\forall\alpha \in \Omega^1(\mathbb{R})$ s.t. $d\alpha = 0$, we may assume $\alpha = f(\theta)d\theta$, let $g(\theta) = \int_0^\theta f(t)dt$, we have problem because θ is not sigle-valued. We consider the map $\Omega^1(S^1) \to \mathbb{R} : \alpha \to \int_0^{2\pi} \alpha$, then

$$\alpha \to \int_0^{2\pi} \alpha = 0 \Leftrightarrow \exists g \in \Omega^0(S^1) \quad s.t. \quad \alpha = dg,$$

so we have the exact form is the kernel of this map, so we know that $H^1(S^1)$ is equal to the image, i.e. $\mathbb{R}$.

3.2 Poincaré lemma

Definition 3.2 (Homotopy). *$f_0, f_1 : M \to N$ are homotopic if $\exists F : M \times I \to N$ such that $F(0, x) = f_0(x), F(1, x) = f_1(x)$.*

Example 3.4. *M is contractible if Id_M is homotopic to a constant map.*

$\forall\alpha \in \Omega^k(N), F^*(\alpha) \in \Omega^k(M \times I), K_\alpha = \int_0^1 < \frac{\partial}{\partial t}, F^*\alpha > dt \in \Omega^{k-1}M$. We have the next lemma:

Lemma 3.1. *if f_0 and f_1 are homotopic then $f_1^*\alpha - f_0^*\alpha = Kd\alpha + dK\alpha$.*

Proof. Assume $F^*\alpha = \omega + dt \wedge \eta$, $i_t : M \to M \times t \subset M \times I$, $F_t = F \circ i_t : M \to N$

$$\alpha := F_t^*\alpha = i_t^* F^*\alpha = i_t^*\omega =: \omega_t$$

$$\eta =< \frac{\partial}{\partial t}, F^*\alpha >= i_{\frac{\partial}{\partial t}} F^*\alpha d\omega - dt \wedge d_x\eta = dF^*\alpha = F^*d\alpha$$

so

$$< \frac{\partial}{\partial t}, d\omega >= d_x < \frac{\partial}{\partial t}, F^*\alpha > + < \frac{\partial}{\partial t}, F^*d\alpha >$$

$$\begin{aligned} \alpha_1 - \alpha_0 &= \omega_1 - \omega_0 = \int_0^1 dt \frac{\partial \omega_t}{\partial t} = \int_0^1 dt < \frac{\partial}{\partial t}, d\omega > \\ &= \int_0^1 d_x < \frac{\partial}{\partial t}, F^*\alpha > dt + \int_0^1 < \frac{\partial}{\partial t}, F^*d\alpha > dt \\ &= dK\alpha + Kd\alpha \end{aligned}$$

□

Theorem 3.2 (Poincaré Lemma). *if M is contractible, $\alpha \in \Omega^k(M)(k > 1)$ then $d\alpha = 0 \Leftrightarrow \alpha$ is exact i.e. $\exists\beta \in \Omega^{k-1}(M), \alpha = d\beta$*

Proof. Use the above lemma. Let $f_1 = id_M$, $f_0 = p \in M$

$$\alpha_0 = f_0^*\alpha = 0, \alpha_1 = \alpha$$

so

$$\alpha = \alpha_1 - \alpha_0 = dK\alpha + Kd\alpha = d\beta, \quad \beta = K\alpha$$

□

Remark 3.1. *The homotopy lemma can also be proved by Cartan's fomula. Define $v_t \in \Gamma(F_t^*TN)$ by $\frac{d}{dt}F_t(p) = v_t(F_t(p))$, we have Cartan's fomula:*

$$\frac{d}{dt}F_t^*(\alpha) = L_{v_t}F_t^*\alpha = di_{v_t}F_t^*\alpha + i_{v_t}dF_t^*\alpha$$

so we have

$$F_1^*\alpha - F_0^*\alpha = \int_0^1 L_{v_t}F_t^*\alpha dt = dK\alpha + Kd\alpha$$

where $K\alpha = \int_0^1 dt i_{v_t} F_t^*\alpha$

Cartan's fomula is the infinitesimal version of the homotopy fomula.

Corollary 3.3. *$H^k(M) = 0 (k \geq 0)$ if M is contractible.*

Example 3.5 (Darboux's theorem)**.**
Consider a symplectic manifold (M, ω), for p with local coordinate $x(p) = 0$, we have $\omega_p = \sum_{i=1}^{n} dx^i \wedge dx^{n+i}$

we give a proof of it as an application of Cartan's fomula:
Let $\omega_t = t\omega + (1-t)\omega_p$, $t \in [0,1]$, we have $\omega_0 = \omega_p$ and $\omega_1 = \omega$
Take $\beta = \omega_1 - \omega_0$, $d\beta = 0$ so $\exists \alpha$ s.t. $d\alpha = \beta$(use homotopy $\varphi_s(x) = sx, s \in [0,1]$)
$\exists$ a family of vector field $[v_t]$ s.t. $i_{v_t}\omega_t = -\alpha$
let ϕ_t be the flow satisfying $\frac{d\phi_t}{dt} = v_t \circ \phi_t$

$$\frac{d\phi_t^*\omega_t}{dt} = \phi_t^*(L_{v_t}\omega + \frac{d\omega_t}{dt}) = \phi_t^*(di_{v_t}\omega_t + d\alpha) = 0$$

$\Rightarrow \phi_t^*\omega_t = \omega_0$, $\forall t \in [0,1]$
In particular $\phi_1^*\omega = \omega_p = \sum_{i=1}^{n} dx^i \wedge dx^{i+n}$

3.3 generalization

Definition 3.3. *The graded vector space $C^* = \oplus_{q \in \mathbb{Z}} C^q$ is called a differential complex if there are homology*

$$\cdots \xrightarrow{d} C^{q-1} \xrightarrow{d} C^q \xrightarrow{d} C^{q+1} \xrightarrow{d} \cdots$$

where $d : C^* \to C^*$ is of degree 1 s.t. $d^2 = 0$. This differential complex is denoted by $C = (C^*, d_C)$ Then we define the cohomology group

$$H^q(C) = H^q(C^*, d) = \frac{kerd \cap C^q}{Imd \cap C^q}$$

Example 3.6. *A vector space is a differential complex:*

$$C : 0 \to 0 \to v \to 0 \to 0$$

$$H^k(C) = \begin{cases} v & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Example 3.7. *A linear map $f : A \to B$ induces a differential complex:*

$$0 \to 0 \to A \xrightarrow{f} B \to 0 \to 0$$

with $H^0(C) = kerf$, $H^1(C) = cokerf = B/Imf$

Example 3.8. *A short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where $kerf = 0$, $\quad Img = C$, $\quad kerg = Imf$ is a differential complex.

Example 3.9 (de Rham complex)**.** *de Rham complex of a manifold is a differential complex:*

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^\infty(M) \to 0$$

de Rham cohomology $H^k_{dR}(M) = H^k(\Omega^*(M), d)$
In particular $\Omega^*(\mathbb{R}^n), d)$ is exact at degree $k \neq 0$

$\mathbf{DGV}_{\mathbb{R}}$ denotes the category of differential graded vector space(also differential complexes) over $\mathbb{R}$

Definition 3.4. *A map $f : A \to B$ between differential complexes consists of $f_k : A^k \to B^k$ s.t. $fd_a = d_B f$, i.e. the next diagram commutes:*

$$\begin{array}{ccccccc} \to & B^{k-1} & \overset{d_B}{\to} & B^k & \overset{d_B}{\to} & B^{k+1} & \to \\ & \uparrow f_{k-1} & & \uparrow f_k & & \uparrow f_{k+1} & \\ \to & A^{k-1} & \overset{d_A}{\to} & A_k & \overset{d_A}{\to} & A^{k+1} & \to \end{array}$$

Remark 3.2. *Let $H^* : \mathbf{DGV}_{\mathbb{R}} \to \mathbf{GV}_{\mathbb{R}}$, mapping the differential complex C to $H^*(C) = \oplus_{k\in\mathbb{Z}} H^k(C)$, then H^* is a functor in the following sense: let $f : A \to B$ be a map between differential complexes, then $H^*(f) : H^*(A) \to H^*(B)$ is defined this way:*

Let $[\alpha]$ be a cohomology class in $H^k(A)$, where $d_A\alpha = 0$, then $H^*(f)[\alpha] = [f\alpha]$, we can easily check this is well-defined because $d_B(f\alpha) = fd_A\alpha = 0$

Remark 3.3. *Let $H^*_{DR}(M) = \oplus_{k=0}^n H^k_{DR}(M) \in \mathbf{GV}_{\mathbb{R}}$ be the de Rham cohomology groups on M, then $H^*_{DR} : \mathbf{mfd} \to \mathbf{GV}_{\mathbb{R}}$ is a functor composed by $\Omega^* : \mathbf{mfd} \to \mathbf{DGCA}_{\mathbb{R}}$ and $H^* : \mathbf{DGCA}_{\mathbb{R}} \to \mathbf{DGV}_{\mathbb{R}}$ maps differential forms to cohomology groups: $H^*_{DR} = H^* \circ \Omega^*$*

3.4 Relations between complexes

Q:What would be differential complexes of differential complexes?

Example 3.10. *short exact sequence of differential complexes: $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ where $A, B, C \in (\mathbf{DGV}_{\mathbb{R}})^0$, $f, g \in (\mathbf{DGV}_{\mathbb{R}})^1$, and the sequences $0 \to A_k \overset{f_k}{\to} B_k \overset{g_k}{\to} C_k \to 0$ are exact $\forall k \in \mathbb{Z}$*

Example 3.11. *$U, V \subset M$ are open sets and $M = U \cup V$, then the following sequence is exact:*

$$0 \to \Omega^*(M) \overset{I}{\to} \Omega^*(U) \oplus \Omega^*(V) \overset{J}{\to} \Omega^*(u \cap V) \to 0$$

where $I_k : \alpha \to (\alpha|_U, \alpha|_V)$ and $J_k : (\beta_U, \beta_V) \to \beta_U|_{U\cap V} - \beta_V|_{U\cap V}$

Definition 3.5. *For a short exact sequence of chain complexes $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ we define*

$$\delta_* : H^k(C) \to H^{p+1}(A)$$

to be the linear map given by

$$\delta_*([c]) = \left[(f^{p+1})^{-1}\left(d_B^p((g^p)^{-1}(c))\right)\right]$$

We leave to the reader to check that this is well-defined. You can read [1] for reference.

Theorem 3.4 (long exact homology sequence)**.** *Let $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ be a short exact sequence of chain complexes. Then the sequence*

$$\cdots \to H^k(A) \overset{f_*}{\to} H^k(B) \overset{g_*}{\to} H^k(C) \overset{\delta_*}{\to} H^{k+1}(A) \overset{f_*}{\to} H^{k+1}(B) \to \cdots$$

is exact.

Corollary 3.5 (Mayer-Vietoris)**.** *$U, V \subset M$ are open sets and $M = U \cup V$, then the following sequence is exact:*

$$\cdots \to H^k(M) \overset{I_*}{\to} H^k(U) \oplus H^k(V) \overset{J_*}{\to} H^k(U \cap V) \overset{\delta_*}{\to} H^{k+1}(M) \to \cdots$$

Here the maps I_* and J_* are induced maps of I and J in the last example.

This is the direct result of the above theorem and the last example.

Here we give a simple but concrete example as an application of the Mayer-Vietoris sequence.

Example 3.12.

$$H^k(S^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ 0 & otherwise \end{cases}$$

let $U = \{x \in S^2 : x^3 \geq -1/4\}$ and V the set symmetric to U with respect to the $x^1 - x^2$ plane. Then we have the following long exact sequence:

$$\begin{array}{ccccccccc}
0 & \to & H^0(S^2) & \to & H^0(U) & \oplus & H^0(V) & \to & H^0(U\cap V) \\
 & & \| & & \| & & \| & & \| \\
 & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\
\to & H^1(S^2) & \to & H^1(U) & \oplus & H^1(V) & \to & H^1(U\cap V) & \\
 & & & \| & & \| & & \| & \\
 & & & 0 & & 0 & & \mathbb{R} & \\
\to & H^2(S^2) & \to & H^2(U) & \oplus & H^2(V) & & & \\
 & & & \| & & \| & & & \\
 & & & 0 & & 0 & & &
\end{array}$$

The group isomorphisms listed above are based on obeservations: $U \cong V \cong \mathbb{R}^2$ and $U \cap V \cong S^1 \times I$. Thus we obtain the results by exactness of the sequence.

Next we explore the relations between homology groups of homotopic manifolds.

Definition 3.6 (Homotopy of chain complexes)**.** *f_0 and $f_1 : A \to B$(maps of complexes) are homotopic if $\exists K : B^i \to A^{i-1}$ such that $f_0 - f_1 = dK + Kd$.*

Proposition 3.6. *f_0 and $f_1 : A \to B$(maps of complexes) are homotopic $\Longrightarrow f_{0*} = f_{1*} : H^*(A) \to H^*(B)$.*

Proof. $\forall[\alpha] \in H^k(A), d\alpha = 0,\ f_{0*}[\alpha] = [f_0\alpha] = [f_1\alpha - dK\alpha - Kd\alpha] = [f_1\alpha] = f_{1*}[\alpha]$ □

Corollary 3.7. *f_0 and $f_1 : M \to N$ are homotopic $\Longrightarrow f_0^* = f_1^* : H^*(N) \to H^*(M)$*

This is direct from the above proposition and Lemman 3.1.

Corollary 3.8. *$H^*(M \times I) \cong H^*(M)$.*

Proof. Define $\pi : M \times I \to M, (x, t) \mapsto x$ and $i : M \to M \times I, x \mapsto (x, 0)$. Notice that $\pi \circ i = id_M$ and $i \circ \pi$ is homotopic to $i_{M\times I}$. □

Chapter 4

Compact Supported de Rham Cohomology

Definition 4.1. $H_c^k(M) := H^k(\Omega_c^*(M), d)$,*where* $\Omega_c^*(M) = \{\alpha \in \Omega^*(M) | supp\ \alpha\ is\ compact\}$.

Example 4.1.

$$H_c^k(point) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & otherwise. \end{cases}$$

Example 4.2.

$$H_c^k(\mathbb{R}) = \begin{cases} 0 & k = 0 \\ \mathbb{R} & k = 1. \end{cases}$$

Remark 4.1. *The calculation of the first example is just by definition. The second one needs the Poincare lamma for dimension-1,i.e if* $\alpha \in \Omega_c^1(\mathbb{R})$ *and* $d\alpha = 0$ *then we have* $\alpha = d\beta$.

Example 4.3.

$$H_c^k(\mathbb{R}^n) = \begin{cases} 0 & k \neq n \\ \mathbb{R} & k = n. \end{cases}$$

To achieve this, we need the proposition below which can be found in the book "Differential Forms in Algebraic Topology " by R.Bott and L.W.Tu.

Proposition 4.1. $H_c^*(M) = H_c^{*+1}(M \times \mathbb{R})$.

Remark 4.2. *To prove the proposition, we need to construct* $I\alpha = \int_{-\infty}^{\infty} < \frac{\partial}{\partial t}, \alpha > dt$ *and* $J_\alpha = \int_{-\infty}^{t} < \frac{\partial}{\partial t}, GI\alpha - \alpha >$ *for* $G : \Omega_c^*(M \times I) \to \Omega_c^{*+1}(M \times \mathbb{R})$ *such that we have the equation* $GI\alpha - \alpha = dJ\alpha + Jd\alpha$ *which implies the homotopy.*

Next, let's talk about the functorial property of Ω_c^*.Notice that for $f : M \to N$, we may not have $f^* : \Omega_c^*(N) \to \Omega_c^*(M)$ since the inverse image of a compact set needn't be a compact set. However, if f is an open embedding, $f^* : \Omega_c^*(N) \to \Omega_c^*(M)$ is well defined. Just as what we have done for Ω^*, we can also consider the Mayer-Vietoris sequence with respect to Ω^*. The Mayer-Vietoris sequence is mainly used to calculate some homology groups under finite conditions. So a nature question is when $H^*(M)$ and $H_c^*(M)$ are finitely dimensional. First, let's introduce the definition of "good open cover".

Definition 4.2. *A open cover* $\mathcal{U} = \{U_\alpha\}$ *is called a good cover if all-non-empty finite intersection are diffeomorphism to* $\mathbb{R}^n$.

Example 4.4 (Good cover for $\mathbb{R}^n$). *For fixed* $r > 0$, *inductively define* x_i *such that* $\{B_r(x_i)\}$ *are disjoint:* $x_0 = 0, x_l$ *satisfies that* $|x_l - x_i| \geq 2r$ *for each* $i \leq l$ *and* $|x_l| = min$. *Then* $\{B_{2r}(x_i)\}_{i=1}^{\infty}$ *is a good cover of* $\mathbb{R}^n$.

Theorem 4.2. *If* M *has a finite good open cover, then* $H^*(M)$ *and* $H_c^*(M)$ *are finitely dimensional.*

Theorem 4.3 (Poicaré Duality). *When* M *is orientable and has a finite good cover, then* $H^q \cong (H_c^{n-q})^*$.

Remark 4.3. *If M is compact and orientable, then $H^q(M) \cong (H^{n-q}(M))^*$ and we can define inner product by $< [\alpha], [\beta] >= \int_M \alpha \wedge \beta$ thus induce the linear map $PD : H^q(M) \to (H^{n-q})^*$. If the map PD is isomorphism for $U, V, U \cap V$, then it is also isomorphism for $U \cap V$. This proposition can be proved by the M-V sequence and the $5 - lemma$.*

Example 4.5. *Let G be a compact Lie group, $\mathfrak{g}$ is the space of left-invariant vector and $\mathfrak{g}^*$ is the space of left-invariant 1-form. The Grassman Algebra $\bigwedge^*(\mathfrak{g}^*) \subset \Omega^*(G)$ is a graded subalgebra. Then we have $\bigwedge^*(\mathfrak{g}^*)$ and $\Omega^*(G)$ are homotopic equivalent.*
Proof: Define i to be the inclusion map and pi by $\pi(\alpha) = \int L_g^* \alpha dv_g$, then it is clear that $\pi \circ i = id, i \circ d = d \circ i, \pi \circ d = d \circ \pi$. We need to show $i \circ \pi$ is homotopic to $id_{\Omega^*(G)}$. $\forall g \in G, \exists$ 1-parameter subgroup g_t generated by $v \in \mathfrak{g}, g_0 = e, g_1 = g. \forall \alpha \in \Omega^*(G), i \circ \pi\alpha - \alpha = dH\alpha + Hd\alpha$, where $H\alpha = \int_G H_g \alpha dv_g$ and $H_g \alpha = \int_0^1 i_v L_{g_t}^* v dt$.

Example 4.6. *$G = T^n$, g is trivial, $d_g = 0 \Longrightarrow H^*(T^n) \cong \bigwedge^*(\mathbb{R}^n)$.*

Notice that $S^n = \mathbb{R}^n \cup \{\infty\}$, so we have an another view to the compact supported de Rham complex on $\mathbb{R}^n$.

The next is a short exact sequence:

$$0 \to \Omega_c^*(\mathbb{R}^n) \to \Omega^*(S^n) \to \Omega_\infty^*(S^n) \to 0$$

where $\Omega_\infty^*(S^n)$ stands for the germs of differential forms at infinity point on the sphere surface.

We notice that $\Omega_\infty^*(S^n) \cong \Omega_0^*(\mathbb{R}^n)$ and by Poincaré lemma on $\mathbb{R}^n$ $H^*(\Omega_0^*(\mathbb{R}^n, d) = \mathbb{R}$ then we have

$$H^k(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & k \neq 0 \end{cases} \Rightarrow H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases}$$

by Mayer-Vietoris sequence. And this is similar to the famous Thom Isomorphism.

Theorem 4.4 (Thom Isomorphism). *For M a compact manifold with an orientable n-rank vector bundle E there is an isomorphism $H_c^*(E) \cong H^{*-n}(M)$*

Chapter 5

Sard theorem; Transversality

5.1 Sard theorem

Sard theorem is about the critical value of a transformation. As a simple example, for a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\mathbb{R}^n) \subset \mathbb{R}^m$ is of measure zero if and only if f is not surjective(in fact $f(\mathbb{R}^n)$ is of codimension at least 1.)

Definition 5.1. *(1)For a smooth map $f : M \to N$ $p \in M$ is called regular point of F if $dF_p : T_pM \to T_{f(p)}N$ is surjective. A regular value of f is a point $q \in N$ s.t. $f^{-1}(q)$ consists only of regular points of f.*

(2)$p \in M$(resp. $q \in N$) is called an irregular point(resp. value) if it's not a regular one.

(3)$p \in M$ is called a critical point of f if $dF_p = 0$. Image of critical points are critical values.

Proposition 5.1. *q is a regular value of $f \Rightarrow F^{-1}(q) \subset M$ is a submanifold.*

Definition 5.2. *Let D be the set of irregular points, and we define $D_k = \{p \in M | D^l f = 0 for 1 \leq l \leq k\}$.*

Then we have D_1 the set of critical points and $D \supset D_1 \supset \ldots \supset D_k$ all closed.

Theorem 5.2 (Sard)**.** *The irregular value set $f(D)$ is of measure 0.*

Proof. Since M has countable topological base, it's sufficient to just consider $f : U(\subset \mathbb{R}^n) \to \mathbb{R}^m$

We prove by induction on $n = dimM$:

step 1: For k large, $F(D_k)$ has measure 0.

Assume U is in a unit cube w, subdivide w into N^n subcubes. For such a subcube w', $w' \cap D_k \neq \emptyset \Rightarrow f(w')$ is in a ball of radius of $bN^{-(k+1)}(f(x) - f(p) \leq br^{k+1})$. Then the measure of $f(D_k)$ is bounded by $N^n b^m N^{-(k+1)m} = b^m N^{(n-(k+1)m)} \Rightarrow f(D_k)$ has measure zero where $k > \frac{n}{m} - 1$

step 2: $f(D_1)$ has measure 0.

For any $p \in D_k \setminus D_{k+1}$ assume $\partial^{k+1}/\partial w \partial \tilde{z}_1 \cdots \partial \tilde{z}_k \neq 0$, in a small ball $B_r(p)$, $B_r(p) \cap (D_k \setminus D_{k+1}) \subset \{w = \partial^k f/\partial \tilde{z}_1 \cdots \partial \tilde{z}_k = 0\}$, where $w, \tilde{z}_1, \cdots, \tilde{z}_n$ is a coordinate.

Thus by induction on dimension we finish this step.

step 3: $f(D \setminus D_1)$ has measure 0.

For any $p \in D \setminus D_1$, assume $\frac{\partial f_1}{x_1} \neq 0$. There is a small ball $B_r(p)$ with coordinate $h(x) = (f(x_1), x_2, \cdots, x_n)$, s.t. $g(t, z) = f \circ h^{-1}(t, z) = (f, g_2(t, z), \cdots, g_n(t, z)) = (t, g^t(z))$. $(t, z) \in B_r(p) \cap (D \setminus D_1) \Leftrightarrow z$ is an irregular point of g^t, namely $f(D \setminus D_1) = \cup_t g^t(D(g^t))$

By induction $g^t(D(g^t))$ has $(m-1)$-measure 0 $\forall t$, then by Fubini theorem $f(D \setminus D_1)$ has m-measure 0. □

Remark 5.1. *Actually the theorem above has a refinement:*

The irregular value set $f(D)$ is of Hausdorff codimension≥ 1.

We can prove this in the following steps:

step 1: $f(D_k)$ has Hausdorff dimension$\leq n/(k+1)$.

step 2: $f(D_1)$ has Hausdorff dimension$= 0$.

step 3: $f(D)$ has Hausdorff dimension$\leq m-1$.

And we need the Fubini-type theorem below:

Theorem 5.3. *Let* $\mathbb{R}_t^{n-1} := \{x \in \mathbb{R}^n | x_n = t\}$*, let* $C \subset \mathbb{R}_{[0,1]}^{n-1} \subset \mathbb{R}^n$ *be compact and* $C_t := C \cap \mathbb{R}_t^{n-1}$ *has Hausdorff codimension*$\geq l, \forall t$*, then* C *has Hausdorff codimension*$\geq l$*.*

5.2 Sard Smale Theorem

In this section we focus on Banach manifolds. Instead of Euclidean spaces, Banach spaces are locally homeomorphic to Banach spaces. A Banach manifold should be separable and para-compact. Banach manifolds are one possibility of extending manifolds to infinite dimensions.

Definition 5.3. B_1, B_2 *are Banach Spaces, continuous linear transform* $F : B_1 \to B_2$ *is called Fredholm if* $F(B_1) \subset B_2$ *is closed and* $kerF, cokerF$ *are finite dimensional. A smooth map* $F : B_1 \to B_2$ *is called Fredholm if each tangent map is Fredholm.*

Then we have a problem that our definition of measure seems to have no natural extension here. Thus we introduce the Baire category theorem.

Definition 5.4. *In a complete metric space* X*, countable intersection of open dense subset is called Baire set of* X

Lemma 5.4. *Baire set of* X *is dense in* X*.*

Proof. Assume $S = \cap_{i=0}^{\infty} U_i$ where U_i are open dense. $\forall p \in X, r > 0, \exists B_{r_1}(p_1) \subset B_{r/2}(p_2) \cap U_1 (U_1$ is open dense). By induction, $\exists B_{r_k}(p_n) \subset B_{r_{k-1}/2}(p_{k-1}) \cap U_k \subset \cap_{i=1}^{k} U_i$.

$\{p_n\}$ is Cauchy sequence $\Rightarrow p_k \to p_0 \in S \cap B_{r/2}(p) \Rightarrow$ S is dense in X □

Theorem 5.5 (Sard-Smale)**.** $F : B_1 \to B_2$ *is a Fredholm map, then the regular value set is a Baire set of* B_2

Remark 5.2. *Refinement:*

$F(D) \subset B_2$ *is of Hausdorff codimension*≥ 1

5.3 Transversality

Definition 5.5. *Given a differential map* $F : M \to N$ *a submanifold* $A \subset N$ *is called transverse to* F *if* $\forall p \in F^{-1}(A)$*,* $F_{*p}T_p(M) + T_{F(p)}A = T_{F(p)}N$*.*

Proposition 5.6. $A \pitchfork F \Rightarrow F^{-1}(A) \subset M$ *is a submanifold of codimension* $n - r$*.*

Theorem 5.7. *1. There is a small pertubation of* A *that is transverse to* F*. 2. There is a small pertubation of* F *that is transverse to* A*.*

We have the observation that F is transverse to $A \Rightarrow$ Graph$(F) \subset M \times N$ is transverse to $\pi_N^{-1}(A) \subset M \times N$. Then 1$\Rightarrow$2. Then we can just prove 1.

Proof. For simplicity assume A is compact. The non-transverse point set $D \subset A$ is a closed subset. $\forall p \in D$, take a tubular neighbourhood $U \cong (U \cap A) \times B^{n-r}$ with projection $\pi : U \to B^{n-r}$. There is the map $\pi \circ F : M \to B^{n-r}$. Let $A_y = \pi^{-1}(y)$ for $y \in B^{n-r}(A \cap U = A_0)$.

F is transverse to $A_y \Leftrightarrow y$ is a regular value of $\pi \circ F$. Sard theorem $\Rightarrow F$ is transverse to A_y for suitable small y.

Glue such A_y to A in $U \setminus U'$, the new D would be in a small neighbourhood of $D \setminus U'$, where $p \in U' \subset U$. By compactness of A finite many such U' would cover D. So after finitely many such pertubation A is transverse to F. □

Remark 5.3. *(1) Generally, with paracompact replacing compact, the proof still works.*

(2) If A *is transverse to* F *on a closed subset* $A' \subset A$*, then pertubation can be made to fix* A'*.*

(3) The transversality theorem is still true when F *is a Fredholm map between Banach Manifolds and* A *is finite-dimensional.*

Chapter 6

Morse Theory

Definition 6.1 (Morse function). *A function f on a manifold is called a Morse function if all its critical points are non-degenerate(A critical point is non-degenerate if the Hessian $H_f(p)$ is non-degenerate).*

Definition 6.2 (Morse Index). *The number of negative eigenvalues of $H_f(p)$ is called the Morse Index.*

Lemma 6.1 (Morse). *Assume $0 \in \mathbb{R}$ is a non-degenerate critical point of $f : \mathbb{R}^n \to \mathbb{R}$. Then by suitable change of coordinate preserving $0 \in \mathbb{R}$, $f(x) = f(0) + q(x)$, when $q(x)$ is the quadratic term of f.*

Proof. By assumption, $y = Df(x)$ is a local coordinate near 0. $Df(0) = 0, Df(x) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$. For a function g, $g(y) = \int_0^1 \frac{\partial g}{\partial y^i}(ty)dt$, then $v = v^i \frac{\partial}{\partial x^i}$ is a vector field such that $v(f) = -g(y)$. Similar for $f_g = (1-s)(f(0)+q) + sf$, where $g = \frac{d}{ds} f_s = f - f(0) - q$. Get v_s such that$v_s(f_s) = -g(y)$, then v_s generate flow ϕ_s such that $\frac{d}{ds}\phi_s(p) = v_s(\phi_s(p))$. So $\frac{d}{ds} f_s \circ \phi_s(p) = \frac{df_s}{ds}|_{\phi_s(p)} + v_s(f_s)|_{\phi_s(p)} = 0$, thus $f \circ \phi_1 = f_1 \circ \phi_1 = f_0 \circ \phi_0 = f(0) + q$. □

Q : What about other critical points?

Theorem 6.2 (Reeb). *If M is compact without boundary, and possesses a Morse function with only two critical points, then M is homeomorphic to $\mathbb{S}^n$.*

Proof. Assume $f(M) = [-1, 1]$ and p_{-1}, p_1 are the two critical points. By Morse Lemma, $\exists$ coordinate $\{x\}$ near $p_{\pm 1}$ such that $f(x) = \pm(1 - |x|^2)$. Pick a Riemann metric g on M such that $g = dx^2$ near $p_{\pm 1}$. Let $v = \frac{\nabla f}{|\nabla f|^2}$, then $v(f) = 1$. Let $D_r \subset \mathbb{R}^n$ be the disk of radius r. For $\epsilon > 0$ and small enough, consider ODE

$$\begin{cases} \frac{dx}{dr} & = v. \\ x(0) & = p \in S_\epsilon = \partial D_\epsilon \end{cases}$$

The solution is $x(p, r)$ for $(p, r) \in S_\epsilon \times [0, 1]$. Define a diffeomorphism $\phi_- : D_1^* \to f^{-1}[-1, 0]$. Similarly, we can get $\phi_+ : D_1 \to M_+ = f^{-1}[0, 1]$. Thus, $M_0 = M_+ \bigcap M_- \cong S^{n-1}$. □

Remark 6.1. *Let's review how we deal with Reeb theorem above. First, we get two discs near the two critical points. Then, we construct an ODE or a flow from one boundary to the other where the existence is ensured by the ODE theory about the existence of solution in large area. Many more details can be found in "New Lectures on Differential Topology" by "Zhang Zhusheng".*

Definition 6.3 (Compatible Vector Field). *v is a compatible vector field with respect to f if $v(f) = 1$ and $v = \frac{\nabla f}{|\nabla f|^2}$ under flat metric near critical point.*

Assume $[a, b]$ contains no critical points, then the flow of v determines $M[a, b] \cong M_a \times [a, b]$. Let $\{w_t\}_{t \in [a,b]}$ be a family of vector field on M_a such that $w_t = 0$ for t near a, b(vanishing near boundary). It induce a horizontal vector field w on $M[a, b]$. $v + w$ is also a compatible vector field. The change has the effect of twisting M_a by ϕ_t at level for $t \in [a.b]$, where $\{\phi_t\}_{t \in [a,b]}$

is the flow of $\{w_t\}_{t\in[a,b]}$. Near a critical point p, $f(x,y) = f(p) + |x|^2 - |y|^2$. There are the unstable submanifold $w_p^+ = \{y = 0\}$ and the stable submanifold $w_p^- = \{x = 0\}$ that intersect transversely at point at p. v is tangent to $w_p^{+/-}$, which can be extended by the flow of v to form a submanifold of M. $M = \bigsqcup_{p\in crit(f)} w_p^+ = \bigsqcup_{p\in crit(f)} w_p^-$.

Definition 6.4 (Ordered Morse Function). *A Morse function if called ordered if $f(p_1) > f(p_2)$ when $I(p_1) > I(p_2), p_1, p_2 \in crit(f)$.*

Proposition 6.3. *A proper Morse function $f : M_{[a,b]} \to [a,b]$ can be modified in the interior to become ordered.*

Proof. Assume $p_1, p_2 \in crit(f), f(p_1) < f(p_2)$ and $I(p_1) > I(p_2)$ and there are no other critical points with value in $[f(p_1), f(p_2)]$.Pick a, b such that $f(p_1) < a < b < f(p_2)$. Then $S_1 = M_a \cap W_p^+ \cong S^{n-I(p_1)-1}$ and $S_2 = M_a \cap W_p^- \cong S^{I(p_2)-1}$.So

$$dim\ S_1 + dim\ S_2 = (n-1) + (I(p_2) - I(p_1)) - 1 < n-2$$

. By transversal theorem, $\exists$ flow $\{\phi_t\}_{t\in[0,1]}$ on M_a determined by $\{w_t\}_{t\in[a,b]}$ such that $\phi_1(S_1)S_2$. If $I(p_1) \geq I(p_2)$, replace v by $v + w$, we have $S_1 \cap S_2 = \emptyset$. Modify in a tubular neighborhood $U(w_{p_2}^-)$ to move $f(p_2)$ below $f(p_1)$. □

Theorem 6.4 (Poincaré Conjecture for $n \geq 5$). *If M is a simply connected homological sphere, then M is homeomorphism to S^n.*

Proposition 6.5. *Let $f : M[a,b] \to [a,b]$ be an ordered proper Morse function with only two critical points x_1, x_2 such that $a < f(x_1) < f(x_2) < b$ and $I(x_2) = I(x_1) + 1$. There is a unique flow line from x_1 to x_2. Then f can be modified in a manifold of $w_{p_2}^-$ such that there are no more critical points.*

Now, let's talk about the relation among Morse function, handle decomposition and CW-complex. We give some definition first.

Definition 6.5 (Handle). *Let M be a n-manifold with boundary. For a smooth embedding $h^\lambda : S^{\lambda-1} \times D^{n-\lambda} \to \partial M$, we may define $\tilde{M} := M \cup (D^\lambda \times D^{n-\lambda})/h^\lambda =: M + \tilde{h}^\lambda$ where $\tilde{h}^\lambda : D^\lambda \times D^{n-\lambda} \to \tilde{M}$ is called a handle.*

Definition 6.6 (Ordered Handle Decomposition). *An ordered handle decomposition of M is a filtration $M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that $M_\lambda = M_\lambda + \tilde{h_1}^\lambda + \cdots + \tilde{h_k}^\lambda$, where $h_i^\lambda \in \partial M_{\lambda-1}$ are disjoint.*

According to Smale, there is a equivalence order Morse function on M$\Longleftrightarrow$ Ordered Handle decomposition of M. And the critical points correspond to the handles.

Let $g^\lambda : S^{\lambda-1} \times \{0\} \to \partial M$,we may define $\hat{M} := M + \hat{g}^\lambda = M \cup (D^\lambda \times \{0\})/g^\lambda$, where $\hat{g^\lambda} : D^\lambda \times \{0\} \to \tilde{M}$ is a $\lambda - cell$. So we get $\hat{M}_0 \subset \hat{M}_1 \subset \cdots \subset \hat{M}_n = M$ such that $\hat{M}_\lambda = \hat{M}_{\lambda-1} + \hat{g_1}^\lambda + \cdots + \hat{g_k}^\lambda$. This gives a CW-complex string on M:

$$C_\lambda^{cw} = \oplus_{i=1}^k \mathbb{Z}[\hat{g}_i^\lambda], C_*^{cw} = \oplus_{\lambda\geq 0} C_\lambda^{cw}, (C_*^{cw}, \partial)\partial : C_{\lambda+1}^{cw} \to C_\lambda^{cw}.$$

For any cell $\hat{g}_i^\lambda$, there is a map:$f_i^\lambda : \hat{M} \to \hat{M}_\lambda / \hat{M}_{\lambda-1} + del\hat{g}_i^\lambda \cong S^\lambda$.

For any two cells, $\hat{g}_i^\lambda, \hat{g}_j^{\lambda+1}$, $\partial[g_j^\lambda + 1] = \sum_i deg(f_i^\lambda \circ g_j^{\lambda+1})[\hat{g}_i^\lambda]$, $f_i^\lambda \circ g_j^{\lambda+1} : S^\lambda \to S^\lambda$. $C_\lambda^M = \oplus_{p\in crit_\lambda(f)\mathbb{Z}[p]_-}$, where $[p]_-$ in the index λ critical point with orientation of w_p^-. And we have:

$$(C_*^{cw}, \partial_w) \cong (C_*^M, \partial_M)$$

.

Proposition 6.6. $degf_i^\lambda \circ g_j^{\lambda+1} = \sharp\{$*flow line from* $p_j^{\lambda+1}$ *to* $p_i^\lambda\}$.

Proposition 6.7. $H_*(C_*, \partial_M) = H_*(M, \mathbb{Z})$.

Theorem 6.8 (Poincaré duality). *M is oriented, so we have $(C_*^M(f), \partial) \cong (C_M^{n-*}(-f), d)$, thus $H_*(M,\mathbb{Z}) \cong H^{n-*}(M,\mathbb{Z})$, where $\partial : C_{\lambda+1}^M \to C_\lambda^M$ is matrix over $\mathbb{Z}$.*

Proposition 6.9. *A matrix $A = a_{ij}$, $a_{ij} \in \mathbb{Z}$ can be reduced to*

$$\begin{pmatrix} n_1 & & & & & \\ & \ddots & & & & \\ & & n_l & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

where $n_i | n_j$ if $i < j$ under following 3 types of row(column) transformation:
1. exchange rows(columns);
2. times -1 to a row(column);
3. add a row(column) to another.

Proposition 6.10. *If M is compact closed manifold.$\exists$ ordered Morse function with unique maximum and minimum.*

Proof. Let $q_1, q_2, \cdots, q_n$ be the maximum critical points. Since the function is ordered, so the level set will not cross any critical points that are not maximum until passing through all the maximum critical points. Before meeting a non-maximum critical points the level set near critical point s are actually disjoint sphere. Let p be the first joint point and assume it to be the intersected point of the spheres of q_1 and q_2. Then the index of p is $n-1$ and $\exists$ unique flow line from q_1 to p. By cancelation lemma (q,p) can be moved. So, step by step, we have only one maximum point. □

Corollary 6.11 (Heegaard Splitting)**.** *For orientable connected closed 3-manifold, $M = M_+ \cap M_-$ and $\partial M_+ = \partial M_- = \sigma_g$.*

Proof. $\exists$ ordered Morse function with unique maximum and minimum. Let g be index 1 critical points. $f : M \to [-1, 1]$, $M_- = f^{-1}[-1, 0]$. By Smale critical points and handle lemma, $M_- = D^3 + \tilde{h_1}^1 + \cdots + \tilde{g_1}^1, \partial M_- = \sigma_{g_1} = \sigma_{g_2} = \partial M_+$. □

Proposition 6.12. *If dim $M = 5$ and M is simply connected, then index 1 critical points can be eliminated by creating index 2,3 critical points(canceling pair). Then cancel 1,2 critical points and leaving new index 3 critical points.*

Corollary 6.13. *If $M(n \geq 5)$ is a simply connected homology sphere, $\exists$ Morse function with unique maximum/minimum and no critical points of index 1 and n-1.*

$\partial : C_3^M \to C_2^M$ is surjective($H_2 = 0$).So the standard form is $\begin{pmatrix} I \\ 0 \end{pmatrix}$ By cancelation theorem, the only critical points are the unique maximum and minimum. Thus by Reeb's Theorem, M is homeomorphic to $S^n (n \geq 5)$. What we need to finish the proof the Poincaré Conjecture for $n \geq 5$ is the geometrization of transformation (3):
1) connected sum of $\lambda - cells$ for $\lambda \geq 2$, $\tilde{g_3}^\lambda := \tilde{g_1}^\lambda \sharp\ \tilde{g_2}^\lambda$.
2) $\tilde{g_3}^\lambda \leftrightarrow \tilde{g_3}^\lambda$(cell/handle corresponding).
3) $M + \tilde{h_1}^\lambda + \tilde{h_2}^\lambda = (M + \tilde{h_1}^\lambda) + \tilde{h_2}^\lambda = (M + \tilde{h_1}^\lambda) + \tilde{h_3}^\lambda$.
4) $\tilde{g_3}^\lambda$ and $\tilde{g_1}^\lambda$ can be made disjoint in ∂M. Then $(M + \tilde{h_1}^\lambda) + \tilde{h_3}^\lambda = M + \tilde{h_1}^\lambda + \tilde{h_3}^\lambda$.

Definition 6.7 (Cobordism)**.** *Two compact manifolds M_0 and M_1 are called corbordant ($M_0 \sim M_1$) if $\exists W$ such that $\partial W = M_0 \bigcup M_1$.*

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