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- All together, these actions define a 2-dimensional ℓ-adic representation $\rho_E : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_\ell)$.

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- The number of $\mathbb{F}_p$-points on E_p can be calculated by the formula $1 + p - \mathrm{tr}(\rho_E(\mathrm{Fr}_p))$.

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- such that $a_p = \mathrm{tr}(\rho_E(\mathrm{Fr}_p))$ for the prime p as above.
- This is a theorem of Wiles, Taylor-Wiles, Breuil-Conrad-Diamond-Taylor.

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- The number field case is more difficult even to state because there are automorphic forms that do not correspond to Galois representations.
- Since, classically, automorphic forms live on hermitian symmetric domain, it is more natural to include automorphic forms on arbitrary reductive group in the picture.

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- Unramified representations of $G(F_v)$, F_v being a nonarchimedean local field are classified by semisimple conjugacy classes of $\hat{G}(\mathbb{C})$

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- It is important to understand in which circumstance, the tensor product of local representation π_v is automorphic. Langlands' prediction is based on an elaborated form of Galois group.

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- There should be an extension L_F of the global Weil group W_F with homomorphism $L_{F_v} \to L_F$ well defined up to conjugacy.
- Tempered automorphic representations π are parametrized by homomorphisms $\phi : L_F \to \hat{G}$. Local parameters $\phi_v : L_{F_v} \to \hat{G}$ are obtained by restricting ϕ to L_{F_v}.

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- if at an unramified place v, $\pi_{H,v}$ is parametrized by a semisimple conjugacy class $s_v \in \hat{H}$, then π_v is also unramified and parametrized by the conjugacy class $\rho(s_v)$.

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- Many deep conjectures are also consequences of functoriality : general form of the Ramanujan conjecture, the Sato-Tate conjecture, the Artin conjecture ...

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- Endoscopy theory via the stabilization of the trace formula : Arthur, ...

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- They belong to the same **packet** of representations.
- Rotations of angle θ and $-\theta$ are not conjugate in $\mathrm{SL}_2(\mathbb{R})$ but become conjugate in either $\mathrm{GL}(2, \mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$. They are **stably conjugate**.

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- There should be a natural bijection between the packet Π_{ϕ_v} and the set of irreducible representations of the finite group $\mathcal{S}_{\phi_v}$.

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- Conjectural multiplicity formula for a representation $\pi = \otimes \pi_v$ in the packet Π_ϕ

$$m(\pi, \phi) = |\mathcal{S}_\phi|^{-1} \sum_{\epsilon \in \mathcal{S}_\phi} \prod \langle \epsilon_v, \pi_v \rangle.$$

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- For real groups, Shelstad proved remarkable identities between the sum of character of representations of H in a packet and certain linear combination of character of representations of G in the corresponding packet.
- Langlands suggested a strategy for proving these by using the trace formula : stabilization of the trace formula.

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- For $n = n_1 + n_2$, $U(n_1) \times U(n_2)$ is an endoscopic of $U(n)$.
- Endoscopy theory consists in proving the functoriality from an endoscopic group H to G by comparison of trace formulas. The description of packets is a bonus.

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- It also include the case of base change which is important in number theory.
- With base change, Langlands and Tunnel proved many cases of Artin conjecture. These case were used as a starting point in the proof of Shimura-Taniyama-Weil conjecture of Wiles and Taylor.

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- He also proved the multiplicity formula in global packet.

Construction of Galois representations attached to a selfdual automorphic representation

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- the component at infinity has the same infinitesimal character as some algebraic representation satisfying some regularity condition.

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- Clozel, Harris, Taylor, Yoshida, Labesse, Morel, Shin.

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- The Kottwitz formula for the number of points of modulo p on this Shimura variety needs to be stabilized in the same way as the trace formula does.

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- The trace formula focuses our attention on the orbital integrals O_γ.

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- If $F = F_v$ is a local non archimedean field, A_γ is a finite abelian group.

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- Stable distribution is a weak limit of finite combination of stable orbital integral.

Local versus global stable conjugacy classes

- Let $\gamma \in G(F)$ be a strongly regular element. For each place v, let $\gamma'_v \in G(F_v)$ be stably conjugate to γ. There might not be $\gamma' \in G(F)$ such that γ' in conjugate to $\gamma'_v \in G(F_v)$.

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- The trace formula is not a stable distribution because of the local versus global obstruction.

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- The problem can be reduced to a comparison between κ-orbital integrals on G with stable orbital integrals on endoscopic groups H.

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- The reason is that G and H share the maximal torus and $W_H \subset W$.

Transfer conjecture and the fundamental lemma

- Transfer conjecture : For every smooth function f with compact support on $G(F_v)$, there exists a smooth function f^H with compact support on $H(F_v)$ such that

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- Fundamental lemma : If both G and H are unramified at v, the above identity holds for $f = 1_{G(\mathcal{O}_v)}$ and $f^H = 1_{H(\mathcal{O}_v)}$.

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- Waldspurger, Cluclers-Hales-Loeser proved by different method that the p-adic case of the fundamental lemma is equivalent to the case of formal series case, more accessible to the geometric methods.

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- a traceless twisted endomorphism $\phi : E \to E \otimes K$, K being the canonical bundle of X.
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- The endoscopy theory can be entirely translated to properties of the action of this groups.

Main geometric result

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- The fundamental lemma follows.