LECTURE NOTES OF INTRODUCTION TO LIE GROUPS, ALGEBRAIC GROUPS AND ARITHMETIC GROUPS

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Abstract. This is lecture notes of the course *Introduction to Lie Groups, Algebraic Groups and Arithmetic Groups*, which is a summer course of MSC Tsinghua university. These four lectures are given by Lizhen Ji at Room 1112 of new sciences building in the following four days 8 Jul, 15 Jul, 29 Jul and 5 Aug 2010. The notes of lecture 1 and 3 are taken by Zhijie Huang (Email: *hzj010102@126.com*), and notes of lecture 2 and 4 are taken by Duanyang Zhang (Email: *zdy_880618@sina.com*).

Lie groups, algebraic groups and arithmetic groups occur naturally in many subjects of modern mathematics such as differential geometry, algebraic geometry, number theory, representation theories, geometric and algebraic topology etc. Many unexpected applications and relations between them make them some of the most important and beautiful objects in mathematics.

In this series of 4 lectures, we will start with definitions of Lie groups, algebraic groups and arithmetic groups, and discuss some of their basic properties, structures and selected applications from the topics mentioned above. We will emphasize motivations and applications, hoping to provide some answers to questions such as "what, why and how" about Lie groups, algebraic groups and arithmetic groups.

Contents

1. Definitions of the Objects in Title

1.1. Lie Groups.

Definition 1.1. A **Lie group** G is a group G that is also a smooth manifold such that the group operations

(1) Multiplication:

$$\begin{aligned} G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 g_2; \end{aligned}$$

(2) Inverse map:

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1}; \end{aligned}$$

are smooth(C^∞), i.e. the product and inverse are smooth maps.

We can say that a Lie group = a group + a manifold in a **compatible** way.

Definition 1.2. A **topological group** G is a group which is a topological space such that the group operations are continuous.

It's obvious that $\{\textit{Lie groups}\} \hookrightarrow \{\textit{topological groups}\}$.

Q: What we are interested in is Lie groups, then why do we study topological groups, even though some of them are not Lie groups?

Example : natural example of topological group.

Let X be a topological space[1]. Let $\mathrm{Hemeo}(X)$ denote the group of all homeomorphisms of X.

Claim: $\mathrm{Hemeo}(X)$ with the topology of uniform convergence over compact subsets is a topological group.

Define that f_n converges to f, denoting $f_n \to f$, if and only if $\forall K \subset X$ compact,f_n convergence uniform to f. Then we call a subset of X is closed if and only if it's closed under the convergence defined above. It's not hard to check that all such closed sets satisfies the axioms of closed sets, thus it gives a topology.

It's obvious that $Id_X \in \mathrm{Hemeo}(X)$, the connected component of $\mathrm{Hemeo}(X)$ contain Id is called the identity component, denoted by $\mathrm{Hemeo}^0(X)$. It is a normal subgroup. The quotient $\mathrm{Hemeo}(X)/\mathrm{Hemeo}^0(X)$ is also a group, called the mapping class group of X, denoted by $\mathrm{Mod}(X)$. It's naturally a discrete group.

Q: When is a topological group a Lie group?

[1] In order to define the uniform convergence, here we may assume that X is a metric space.

- Necessary condition: If a topological group is a Lie groups, then it must be a topological manifold.
- The converse is also true, which is known as **Hilbert's 5th problem**.

Theorem 1.3. *If a topological group is a topological manifold, (i.e. locally homeomorphism to* $\mathbb{R}^n$*,* $n = \dim G$*.) Then* G *admit a smooth structure such that* G *is a Lie group.*

The difference between Lie groups and topological groups is the smoothness of group operation.

Q: Is there a characterization of Lie groups among topological groups in terms of groups structures?

Theorem 1.4. *A topological group* G *is a Lie group* $\Longleftrightarrow$ *It's locally compact and does not contain small groups.(i.e.* $\exists$ *a neighborhood of the identity element* e *that does not contain any nontrivial subgroups.)*

Example of non-Lie groups.

For every prime number p, there exist a p-adic field $\mathbb{Q}_p$ ($\exists$ a p-adic norm $||\cdot||_p$ on $\mathbb{Q}$, the completion of $\mathbb{Q}$ with respect to $||\cdot||_p$ is $\mathbb{Q}_p$). Then $\mathbb{Q}_p$ is locally compact, but totally disconnected. We define **general linear** p**-adic group** as following:

$$GL(n, \mathbb{Q}_p) = \{n \times n \text{ matrixes with entries in } \mathbb{Q}_p \text{ with determination } \neq 0\}$$

Check: It does contain small groups.

Given p a prime, then for any $x \in \mathbb{Q}$, x can be expressed as the following form

$$x = p^r \cdot \frac{m}{n}, \quad \text{where } r, m, n \in \mathbb{Z}, (p, mn) = 1, (m, n) = 1.$$

It's obvious that when $x \neq 0$, r is uniquely determined by x, then we can define

$$v_p(x) = r, \text{ for } 0 \neq x \in \mathbb{Q}, \text{ and } v_p(0) = \infty.$$

and p-adic norm $|\cdot|_p$ as

$$|x|_p = p^{-v_p(x)} = p^{-r} \text{ for } x \neq 0, \text{ and } |0|_p = 0.$$

and p-adic metric

$$d_p(x, y) = |x - y|_p, \text{ for any } x, y \in \mathbb{Q}.$$

It's not hard to check that: for any $x, y \in \mathbb{Q}$

$$\begin{aligned} |xy|_p &= |x|_p |y|_p \\ |x + y|_p &\leqslant \max\{|x|_p, |y|_p\}. \end{aligned}$$

Thus $(\mathbb{Q}, d_p)$ is a metric space. But it's not complete, we denote $(\mathbb{Q}_p, d_p)$, or simply $\mathbb{Q}_p$, the completion of $(\mathbb{Q}, d_p)$. Just like the case that the completion of $(\mathbb{Q}, |\cdot|)$, where $|\cdot|$ is the usual absolute value, is real number filed. $\mathbb{Q}_p$ is also a field. In fact $\mathbb{Q}_p$ has the following expression

$$\mathbb{Q}_p = \left\{ x = \sum_{n=n_0}^{\infty} a_n p^n \middle| a_n = 0, 1, \cdots, p-1, \forall n \geqslant n_0, n_0 \in \mathbb{Z} \right\}$$

Now let

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p \middle| \ |x|_p \leqslant 1 \right\}$$

be the subring of $\mathbb{Q}_p$ and $U = \mathbb{Z}_p^*$ is the unitary in $\mathbb{Z}_p$. We have

$$\mathbb{Z}_p^* = \left\{ x \in \mathbb{Z}_p \middle| \ 1/x \in \mathbb{Z}_p \right\} = \left\{ x \in \mathbb{Z}_p \middle| \ |x|_p = 1 \right\}$$

Obviously, $U = \mathbb{Z}_p^*$ is a subgroup of $\mathbb{Q}_p^*$. Now denote

$$U_n = 1 + p^n \mathbb{Z}_p, \quad n \geqslant 1$$

We shall show that for any $n \geqslant 1$, U_n is a subgroup of $\mathbb{Q}_p^*$, the group operation is multiply. Firstly, it's not hard to check that for any $a \in \mathbb{Z}_p$, $x = 1 + p^n a \in \mathbb{Z}_p^*$, thus

$$1 = xx^{-1} = (1 + p^n a)x^{-1} = x^{-1} + p^n a x^{-1} \Rightarrow x^{-1} = 1 - p^n a x^{-1} \in U_n.$$

Secondly, for any $x = 1 + p^n a, y = 1 + p^n b$,

$$xy = (1 + p^n a)(1 + p^n b) = 1 + p^n(a + b + p^n ab) \in U_n$$

Thus U_n is closed under multiply and inverse, that is, U_n is a subgroup of $\mathbb{Q}_p^*$.

Now for any $x = 1 + p^n a \in U_n$, for $a \in \mathbb{Z}_p$, $|a|_p \leqslant 1$, thus

$$d_p(x, 1) = |x - 1|_p = |p^n a|_p = |p^n|_p |a|_p \leqslant |p^n|_p = p^{-n}$$

So we have $U_n \subset B(1, 2p^{-n})$, then it's not hard to check that $GL(n, \mathbb{Q}_p)$ does contain small group by using this fact.

Example of Lie groups.

(1) $\mathbb{R}^n$, group operation given by addition; $\mathbb{R}^*$ group operation given by multiplication.

(2) the general linear group

$$GL(n, \mathbb{R}) = \{n \times n \text{ invertible matrix}\}$$

It's an open set of $M_{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$, so it's a manifold. Obviously, the multiplication is smooth, and by Crammer's rule, the inverse is also smooth.

(3) the special linear group

$$SL(n, \mathbb{R}) = \{g \in GL(n, \mathbb{R}) | \det(g) = 1\}$$

$\det : GL(n, \mathbb{R}) \to \mathbb{R}$ is a smooth map, and $SL(n, \mathbb{R}) = \det^{-1}(1)$. To show that $SL(n, \mathbb{R})$ is indeed a manifold, we just need to show that 1 is a regular value of det, we leave this as a homework.

(4) the orthogonal group

$$O(n) = \{g \in GL(n,\mathbb{R}) | g^t g = Id\}$$

It can be checked that $O(n)$ is a closed Lie subgroup of $GL(n,\mathbb{R})$. We leave it as homework without using the following fact: *(any closed subgroup of a Lie group is a Lie subgroup.)*

(5) the group of all isometries of $\mathbb{R}^n$. It's also called the group of motions of $\mathbb{R}^n$.

(6) the group of all affine transformation of $\mathbb{R}^n$.

Theorem 1.5. *(Myers-Steenrod Theorem) For any Riemannian manifold (M, g), its group of isometries* $\mathrm{Iso}(M)$ *is a Lie group.*

1.2. **Algebraic Groups.**

1.2.1. *Zariski Topology.*

Definition 1.6. Let k be a field. An **algebraic set** means the set $V(\Sigma)$, which is the zero locus of some subset Σ of $k[x^1, \cdots, x^n]$, where $k[x^1, \cdots, x^n]$ is n variable polynomial ring over k.

Proposition 1.7. *(1) The union of two algebraic sets is an algebraic set.*

(2) The intersection of an arbitrary family of algebraic sets is an algebraic set.

Thus the algebraic sets satisfies the axioms defining closed sets in topology theory. We call this topology **Zariski Topology**.

1.2.2. *Varieties and Morphisms.*

Definition 1.8. An **affine algebraic variety** is a subset of $\mathbb{C}^n$, defined by

$$V_0 = \left\{x \in \mathbb{C}^n \middle| f_1(x) = \cdots = f_k(x) = 0\right\}$$

where $f_i, i = 1, \cdots, k$ are polynomials with complex coefficients of n variables $x^1, \cdots, x^n$. That is V_0 is the zero locus of $\{f_1, \cdots, f_k\}$.

Definition 1.9. A **projective algebra variety** (or simply, algebraic variety) is a subset of $\mathbb{C}P^n$, defined by

$$V = \left\{\xi \in \mathbb{C}P^n \middle| F_1(\xi) = \cdots = F_k(\xi) = 0\right\}$$

where $F_i, \cdots, F_k$ are homogeneous polynomials in $n+1$ variables $\xi^0, \xi^1, \cdots, \xi^n$. So V is the zero locus of $\{F_1, \cdots, F_k\}$ in $\mathbb{C}P^n$.

A **rational function** of $\mathbb{C}P^n$ is the ratio f/g of two homogeneous polynomials f and $g \neq 0$ of the same degree (unless $f = 0$). The condition of degree ensures the function is well-defined on the open set $\mathbb{C}P^n - \{g = 0\}$. Such a function on open set $\mathbb{C}P^n - \{g = 0\}$ is said to be **regular**. More generally, a function $f : V \to \mathbb{C}$ on a projective variety $V \subset \mathbb{C}P^n$ is called **regular** if for every point there is a neighborhood (in the *Zariski Topology*), where it is the restriction of a regular function defined above.

Every regular function on an open subset $U \subset \mathbb{C}P^n$ is given by rational functions f/g such that $U \subset \mathbb{C}P^n - \{g = 0\}$.

Similarly, any function which is regular on $\mathbb{C}^n$ can be given by polynomial in affine coordinates.

A map $f : V \to W$ of projective varieties $V \subset \mathbb{C}^n, W \subset \mathbb{C}P^m$ is said to be **regular** if it is given locally by regular functions. That means that, in some neighborhood U of any point $p \in U$, one can write f in coordinate form as $y^1 = f_1(p), \cdots, y^m = f_m(p)$ where $y^1, \cdots, y^m$ are affine coordinates in $\mathbb{C}P^m$ and $f_1, \cdots, f_m$ are regular functions on U. A regular map is also called **a morphism between varieties**.

1.2.3. *Linear algebraic groups.*

Definition 1.10. A **linear algebraic group** is a subgroup of $GL(n, \mathbb{C})$, such that it's a subvariety and the group operation is a morphisms between varieties.

Another definition.

Definition 1.11. A **linear algebraic group** G is a group endowed with the structure of an affine abstract variety (i.e. a closed subvariety of some $\mathbb{C}^N$), such that the group operations are morphism between (abstract) varieties.

Remark 1.12. Based on the definition of topological groups and Lie groups, the following definition is natural.

Definition 1.13. An **algebraic group** G is a group with the structure of an algebraic variety such that the group operations are morphisms.

Theorem 1.14. *If the algebraic variety G is projective, then G is an* ***abelian variety****.*

Roughly, there are two important classes of algebraic groups:

1. linear algebraic groups(the 'affine' theory);
2. abelian varieties (the 'projective' theory).

Example Show $GL(n, \mathbb{C})$ is a linear algebraic group according to the **1.11**.

We know that there exist a natural embedding

$$GL(n, \mathbb{C}) \overset{open}{\hookrightarrow} M_{n\times n}(\mathbb{C}) = \mathbb{C}^{n^2}.$$

However this embedding does not give us a variety structure of $GL(n, \mathbb{C})$. So we consider

$$\begin{aligned} i : GL(n, \mathbb{C}) &\hookrightarrow M_{n\times n}(\mathbb{C}) = \mathbb{C}^{n^2} \times \mathbb{C} \\ g &\hookrightarrow \left(g, (\det g)^{-1}\right); \end{aligned}$$

Thus we have $i\left(GL(n, \mathbb{C})\right) = \{(g, z) \in M_{n\times n}(\mathbb{C}) \times \mathbb{C}, \det g \cdot z = 1\}$, which is the zero locus of a polynomial $\det g \cdot z - 1$, thus it's a subvariety of $\mathbb{C}^{n^2+1}$.

The product is given by polynomials, and by $A^{-1} = (\det A)^{-1}A^*$, where A^* is the adjugate matrix (or classical adjoint matrix) of A, we know that the inverse is also regular. Thus $GL(n, \mathbb{C})$ is an algebraic group.

Example Symplectic Group

$$Sp(2n, \mathbb{C}) = \left\{ g \in GL(2n, \mathbb{C}) \middle| g^t J g = J, J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \right\}$$

Example $G = \mathbb{C}, \mathbb{C}^*$ are all linear algebraic groups

Using the **1.10**. We observe that

$$i : \mathbb{C} \cong \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{C} \right\} = G_a; \text{ given by } i(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

This embedding is a homomorphism as topological group. Because

$$i(a+b) = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = i(a) \cdot i(b)$$

Thus i is a group isomorphism, and it's obviously also a homeomorphism of the underlying topological spaces, that it's a homomorphism. This implies that $\mathbb{C}$ is also a linear algebraci group. For $\mathbb{C}^*$ we just need to observe that

$$\mathbb{C}^* = GL(1, \mathbb{C}).$$

and $GL(1, \mathbb{C})$ is a linear algebraic group.

Generalization: Instead of $\mathbb{C}$, we can also use other algebraical closed filed $\mathbb{K}$. Replace $GL(n, \mathbb{C})$ with $GL(n, \mathbb{K})$.

Definition 1.15. Let $\mathbb{K} \subset \mathbb{C}$ be a subfield except $\mathbb{K} = \mathbb{Q}, \mathbb{R}$. A linear algebraic group $G \subset GL(n, \mathbb{C})$ is said to be **defined over** $\mathbb{K}$, if G is defined over $\mathbb{K}$ as a variety and the group operations as morphisms are also defined over $\mathbb{K}$.

Fact: If a linear algebraic group $G \subset GL(n, \mathbb{C})$ is define over $\mathbb{R}$, then its real (zero) locus $G(\mathbb{R})$ is a Lie group.

Why does it make sense? We just need to check the following two condition:

(1) $G(\mathbb{R})$ forms a (topological) group;
(2) $G(\mathbb{R})$ is also a smooth manifold, with finitely many connected components.

Q: How do algebraic groups arise?

The polynomial equations that defines the varieties often come from the fact that the algebraic groups preserve some structure. For example:

$$SL(n, \mathbb{C}) = \left\{ g \in GL(n, \mathbb{C}) \middle| \det g = 1 \right\}$$

and det : $GL(n,\mathbb{C}) \to \mathbb{C}$ is homomorphism. Then the condition $\det g = 1$ for the special linear group can be viewed as that g preserves the volume form $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

1.3. **Arithmetic Groups.**

Definition 1.16. A **discrete group** Γ is a topological group with the discrete topology.

Definition 1.17. Let G be a Lie group with finitely many connected component. A subgroup Γ of G is called a **discrete subgroup** if the induced subspace topology is discrete.

Example

(1) $G = \mathbb{R}, \Gamma = \mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$.
(2) $G = \mathbb{R}^n, \Gamma = \mathbb{Z}^n$. More generally, a lattice $\Gamma = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \cdots \oplus \mathbb{Z}v_n$, where $v_1, v_2, \cdots, v_n$ is a basis of $\mathbb{R}^n$, is a discrete subgroup of $\mathbb{R}^n$.

All the above examples of discrete subgroups are related to integers.

Definition 1.18. Let $G \subset GL(n,\mathbb{C})$ be a linear algebraic group that is defined over $\mathbb{Q}$. A subgroup $\Gamma \subset G(\mathbb{Q})$ is called an arithmetic subgroup if it is commensurable with $G(\mathbb{Q}) \cap GL(n,\mathbb{Z})$, i.e. $\Gamma \cap G(\mathbb{Q}) \cap GL(n,\mathbb{Z})$ is of finite index in both Γ and $G(\mathbb{Q}) \cap GL(n,\mathbb{Z})$.

Example Show $\mathbb{Z}$ is an arithmetic subgroup in the above sense.

$$\mathbb{Z} \cong \Gamma \subset G = \mathbb{C} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{C} \right\}$$

In this case $n = 2$ and $G(\mathbb{Q})$ has the following expression

$$G(\mathbb{Q}) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{Q} \right\}$$

and $G(\mathbb{Q}) \cap GL(2,\mathbb{Z})$ is just

$$G(\mathbb{Q}) \cap GL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{Z} \right\}$$

which is just Γ. Thus $\Gamma \cap G(\mathbb{Q}) \cap GL(z,\mathbb{Z}) = \Gamma$ is of finite index 1 in both Γ and $G(\mathbb{Q}) \cap GL(2,\mathbb{Z})$. This shows that $\mathbb{Z}$ is an arithmetic group in this sense.

Example $G = GL(n,\mathbb{Q}), \Gamma = GL(n,\mathbb{Z})$ and $G = SL(2,\mathbb{C}), \Gamma = SL(2,\mathbb{Z})$.

Example Principal Congruence Subgroups

An importance class of congruence subgroups is given by reduction of the ring of entries: in general given a group such as the special linear group $SL(n,\mathbb{Z})$ we can reduce the entries to modular arithmetic in $\mathbb{Z}/N\mathbb{Z}$ for any $N > 1$, which gives a homomorphism

$$SL(n,\mathbb{Z}) \to SL(n,\mathbb{Z}/N\mathbb{Z})$$

of groups. The kernel of this reduction map is an example of a congruence subgroup — the condition is that the diagonal entries are congruent to 1 mod N, and the off-diagonal entries be congruent to $0 \mod N$ (divisible by N), and is known as a principal congruence subgroup.

For any integer N,

$$\Gamma(N) = \left\{ g \in SL(2, \mathbb{Z}) \middle| \, g \equiv Id \mod N \right\},$$

then we have

$$[SL(2, \mathbb{Z}) : \Gamma(N)] < +\infty.$$

Any subgroup Γ of $SL(2, Z)$ containing some $\Gamma(N)$ is called a congruence subgroup.

Fact: Given a linear algebraic group $G \subset GL(n, \mathbb{C})$ defined over $\mathbb{Q}$, $G(\mathbb{R})$ is a Lie group and arithmetic subgroup Γ of $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{R})$.

Given a linear algebraic group $G \subset GL(n, \mathbb{C})$, we can get the following groups:

(1) Lie group $G(\mathbb{R})$
(2) arithmetic group Γ
(3) p-adic groups $G(\mathbb{Q}_p)$
(4) finite groups $G(F_q)$, where F_q is a finite field.
(5) the combinations, such as $G(\mathbb{R}) \times G(\mathbb{Q}_p)$

2. Main Contributors to the Subjects and More Facts on Lie Group

2.1. **Main Contributors.** Following are the main contributors to the development of Lie groups, algebraic groups or arithmetic groups.

- Claude Chevalley (11 February 1909 – 28 June 1984) : There is a kind of algebraic groups named after him, he also wrote a famous book on Lie groups;
- Armand Borel (21 May 1923 – 11 August 2003) : His most famous work are on the Borel subgroup (defined in the next section) and reduction theory of arithmetic groups;
- Jacques Tits (12 August 1930 –): He gave a method, named **Tits building**, to construct the geometry of simple complex Lie algebras, the method has many unexpected applications which refines our understanding of algebraic groups and finite simple groups. **Tits alternative**, another terminology named after him, is an important theorem about the structure of finitely generated linear groups;
- Carl Ludwig Siegel (31 December 1896 – 4 April 1981): He did foundational work on arithmetic groups, **Siegel upper half-space** a nd **Siegel modular form** are named in honor of him;
- Andr Weil (6 May 1906 – 6 August 1998): He determined all arithmetic subgroups of $SL(2, \mathbb{R})$;
- Harish-Chandra (11 October 1923 – 16 October 1983): He single-handedly built up the representation theory of semisimple Lie groups.

Sophus Lie (17 December 1842 – 18 February 1899), Wilhelm Killing (10 May 1847 – 11 February 1923) , Henri Cartan (8 July 1904 – 13 August 2008), Hermann Weyl (9 November 1885 – 8 December 1955), Charles Picard (24 July 1856 – 12 December 1941), Robert Steinberg (25 May 1922 –), Ellis Kolchin (18 April 1916 – 30 October 1991) , Atle Selberg (14 June 1917 – 6 August 2007) , Gregori Margulis (24 February 1946 –) , George Mostow and Robert Langlands (6 October 1936 –) also did much important work on these subjects.

2.2. **More facts on Lie group.** Let G be a Lie group (connected or with finitely many connected components).

Q: How to understand G?

We will understand it from the following two aspects:

- internal structures: subgroups;
- external structures: relations with other groups.

Today, we will give some general results on subgroups and homogenuous spaces.

Definition 2.1. A subgroup H of G is called a **Lie subgroup** if it is also a submanifold of G.

Proposition 2.2. *A Lie subgroup is a Lie group.*

Example *Some examples of the Lie subgroups of* $GL(n, \mathbb{R})$*:*

(1) *the special linear group* $SL(n, \mathbb{R})$*, which is a semi-simple Lie group;*
(2) *the upper triangle matrices, which is also called the* ***Borel subgroup****;*
(3) *the diagonal matrices, which is also called the* ***Cartan subgroup / algebra****.*

Proposition 2.3. *Any Lie subgroup H of G is a closed subgroup, i.e.*$\overline{H} = H$.

Proposition 2.4. *For any subgroup H of G, the closure $\overline{H}$ is also a subgroup.*

Theorem 2.5. ***(Cartan)*** *Any closed subgroup of a Lie group G is a Lie subgroup.*

To prove this theorem, it requires the notion of ***exponential map*** *from Lie algebras to Lie groups, and also requires the following proposition:*

Proposition 2.6. *Let H be a subgroup of G, if there exists a neighborhood U of the identity element $e \in G$ such that $U \cap H$ is a submanifold, then H is a Lie subgroup.*

An important way to produce Lie subgroups is to use Lie subalgebras.

Definition 2.7. Let G, H be Lie groups. A map $f : G \to H$ is called a **Lie group homomorphism** if it is both a homomorphism as abstract groups and a smooth map as smooth manifolds. It is called an **isomorphism** if there exists an inverse Lie group homomorphism f^{-1}.

Example Some Lie group homomorphisms:

(1) $\mathbb{R} \to \mathbb{R}_+, x \mapsto e^x$;
(2) $GL(n, \mathbb{R}) \to \mathbb{R}^\times = \mathbb{R} \setminus \{0\}, g \mapsto \det g$.

There is a consequence of the Cartan theorem:

Corollary 2.8. *Let f be a continuous homomorphism between Lie groups, then f is a smooth map, and hence is a Lie group homomorphism.*

Example *Any continuous homomorphism* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is linear.*

Definition 2.9. Let V be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, $GL(V)$ be the group of invertible linear transformations on V. Then any homomorphism $\rho : G \to GL(V)$ is called a **linear representation** of G. If ρ is injective, it is called a **faithful representation**.

Definition 2.10. Let X be a smooth manifold, $\mathrm{Diff}(X)$ be the group of diffeomorphisms of X (this is not a Lie group). A homomorphism $\rho : G \to$

Diff(X) is called a **(smooth) G-action on** X if the map

$$G \times X \to X,\ (g, x) \mapsto \rho(g)(x)$$

is a smooth map.

Such G-actions on X are sometimes called **nonlinear representations**.

Example G acts on G both on the left and right. In fact, $\forall g \in G$, we can define

$$l_g : G \to G,\ x \mapsto gx;\ r_g : G \to G,\ x \mapsto xg^{-1}.$$

$\mathrm{Inn}_g = l_g \circ r_g : G \to G$ is also a diffeomorphism of G.

A very important subclass of representations consists of **character** $\rho : G \to \mathbb{R}^{\times}$(or $\mathbb{C}^{\times}$).

Given a G-action $\alpha : G \to \mathrm{Diff}(X)$. For $x \in X$, the image

$$\alpha(G)(x) = \{\alpha(g)(x) : g \in G\}$$

is called the **orbit through** x. The set

$$G_x = \{g \in G : \alpha(g)(x) = x\}$$

is called the **stabilizer of x in G**.

Fact: In general, an orbit is not a manifold.

Proposition 2.11. *If G is a compact Lie group, then every orbit of G in X is a submanifold.*

Problem in transformation group theory is to understand structures of orbits.

Proposition 2.12. *Let G be a Lie group, H be a Lie subgroup of G. Then there exists a unique differential structure of G/H, such that the natural map $G \to G/H$ is smooth and the left G-multiplication on G/H is smooth. If H is a normal subgroup, then G/H is a Lie group.*

Definition 2.13. A G-action on X is called **transitive** if $\forall\ x, x' \in X$, there exists some $g \in G$, such that $g \cdot x = x'$.

Proposition 2.14. *If a G-action on X is transitive, then all the stabilizers are conjugate. Pick any $x \in X$, G/G_x is canonically diffeomorphic to X.*

Definition 2.15. A smooth manifold X with a transitive action of G is called a **homogeneous manifold**.

The previous result implies that any homogeneous manifold of G is of the form G/H, where H is a closed subgroup of G.

Conclusion: It is important to study subgroups.

3. Symmetric Spaces and Lie Algebras of Lie Groups

3.1. **Homogeneous Spaces and Symmetric Spaces.** Recall homogeneous spaces G/H.

If G acts transitively on N and for a point $x \in X$, its stability G_x is a closed subgourp (hence a Lie subgroup). We have $X \simeq G/G_x$.

Example

(1) $\mathbb{R}^2$;
$\mathbb{R}^2$ is itself a Lie group, thus a homogeneous space.
(2) $S^2 = SO(3)/SO(2)$;
Consider S^2 as a natural subset of R^3, $SO(3)$ act naturally on S^2, for any $p \in S^2$, $O \in SO(3)$, group action is given by $O \cdot p = Op$. The stabilizer of $e_1 = (1, 0, 0)$ is just the x-rotation, which is $SO(2)$.
(3) $\mathbb{H}^2 = SL(2, R)/SO(2)$.
Recall that for any $z \in \mathbb{H}^2$, i.e. $z \in \mathbb{C}$ such that $\mathrm{Im}(z) > 0$,and for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$, the action of g on z is given by:

$$\begin{aligned} g \cdot z &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d} \\ &= \frac{(ac|z|^2 + bd + (ad + bc)\mathrm{Re}(z)) + i\mathrm{Im}(z)}{|cz + d|^2} \end{aligned}$$

The stabilizer of i is the rotation group $SO(2)$.

Basic examples of symmetric spaces.

Let (M, g) be a Riemannian manifold, for every point $x \in M$, there exist a local geodesic symmetry, denoted by s_x, define by reversing geodesics passing through x, i.e., for any geodesic $\gamma(t), t \in \mathbb{R}$, with $\gamma(0) = x$,

$$s_x(\gamma(t)) = \gamma(-t).$$

Definition 3.1. A Riemannian manifold (M, g) is called a **locally symmetric space** if every local geodesic symmetry is a local isometry.

Example Find an example of locally symmetric space (non-trivial) and also an example of non-locally symmetric space.

Definition 3.2. (M, g) is called a symmetric space if it's locally symmetric and $\forall x \in M$ the geodesic symmetry s_x extend to a global isometry of M.

Example A locally symmetric but not symmetric space: $\mathbb{R}^2 \backslash \{0\}$.

Proposition 3.3. *Every symmetric space is a complete Riemannian manifold.*

The proof is left as an exercise.

Proposition 3.4. *Let (M, g) be a complete locally symmetric space. Then it's universal covering space, i.e. $\tilde{M}$ with the lifted Riemannian metric, is symmetric space.*

The fundamental group $\Gamma = \pi_1(M)$ acts properly and fixed point freely and isometrically on M, and

$$\Gamma \backslash \tilde{M} = M$$

Conclusion: Given a symmetric space X and a group Γ, which acts properly and fixed point freely and isometrically on X, then $\Gamma \backslash X$ is locally symmetric space.

Remark 3.5. If Γ is torsion free, the properness of action $\Longrightarrow$ fixed point freely.

But more generally, even if Γ contains torsion elements, we still call $\Gamma \backslash X$ a locally symmetric space.

The reason is that many natural and important discrete subgroups are not torsion free. Here is an example.

$\Gamma = SL(2, \mathbb{Z})$ acts properly on $\mathbb{H}^2$

$SL(2, \mathbb{Z}) \backslash \mathbb{H}^2$ moduli curve = moduli spaces of elliptic curves.

However $SL(2, \mathbb{Z})$ is not torsion free. We leave which as a exercise for reader to find torsion elements of $SL(2, \mathbb{Z})$.

Proposition 3.6. *If X is a symmetric space, then its group of isometries,* $\mathrm{Is}(X)$*, acts transitively on X.*

Proof. By Proposition 3.3, we know that X is a complete manifold. Then for any pair of points $p, q \in X$, there exists a geodesic γ connecting p, q, say $\gamma(0) = p, \gamma(1) = q$. Take $x = \gamma(1/2)$. Consider the geodesic symmetry s_x define around x. Since X is symmetric, it can be extended to a global isometry s of X. Then we have $s(p) = q$. This shows that $\mathrm{Is}(X)$ acts transitively on X. □

Remark 3.7. If one is not sure whether the extend s of s_x satisfies $s(p) = q$ or not. He can do it in the following way. For every $t \in [0, 1]$, $\gamma(t) \in X$, thus there exists a small neighborhood of $\gamma(t)$ such that $s_{\gamma(t)}$ can be well defined. Extend it to a global isometry s_t. Then we can choose suitable $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ such that $\gamma(t_{i-1})$ and $\gamma(t_i)$ are in the small neighborhood of $s_{\gamma(r_i)}$, where $r_i = (t_{i-1} + t_i)/2$. So we have $s_{r_i}(\gamma(t_{i-1})) = \gamma(t_i)$. Thus $s_{r_n} \circ s_{r_{n-1}} \circ \cdots s_{r_1}(\gamma(t_0)) = \gamma(t_n)$. i.e., define $s = s_{r_n} \circ s_{r_{n-1}} \circ \cdots s_{r_1} \in \mathrm{Is}(X)$, we have $s(p) = q$.

Recall $\mathrm{Is}(X)$ is Lie group $\Longrightarrow X$ is a homogeneous space.

The converse of proposition is not true: many homogeneous space are not symmetric.

Proposition 3.8. *Assume X is symmetric, let $G = \mathrm{Is}^0(X)$ be the identity component of* $\mathrm{Is}(X)$*. Then G also acts transitively on X.*

Conclusion: a symmetric space is of the form G/K, where G is a Lie group, K is a Lie subgroup.

If $\Gamma \subset G$ is a discrete subgroup, then $\Gamma\backslash(G/K)$ is a locally symmetric space.

Example Let Q_n=space of all positive defines quadratic form in n-vatiables. ($n \times n$ symmetric positive matrix.) $GL(n, \mathbb{R})$ acts on Q_n by for $A \in Q_n, g \in GL(n, \mathbb{R})$

$$g \cdot A = g^t A g$$

Proposition 3.9. *$GL(n, \mathbb{R})$ acts transitively on Q_n and the stability of I_n is $O(n)$. Thus $Q_n \simeq GL(n, \mathbb{R})/O(n)$.*

Let $SQ_n = \{A \in Q_n \big| \det A = 1\}$.

Proposition 3.10. *$SL(n, \mathbb{R})$ acts transitively on SQ_n, and*

$$SQ_n \simeq SL(n, \mathbb{R})/O(n).$$

Let G be s semi-simple Lie Group, with finitely many connected components, $K \subset G$ be a maximal compact subgroup.

Theorem 3.11. *The homogeneous space $X = G/K$ with any G-invariant metric is a symmetric space.*

(Hint: Basically, start with a norm on $T_e X$, where $e = K$ is the identity coset.)

Remark 3.12. For a pair of Lie groups (G, K), K is compact. So that its homogeneous space G/K is a symmetric space $\iff$ $\exists$ an involution σ on G, $(\sigma : G \to G, \sigma^2 = id)$. The fixed point set G^σ satisfies $G^{\sigma,0} \subset K \subset G^\sigma$.

Such a pair (G, K) with σ is called a **symmetric pair**.

Corollary 3.13. *$SL(2, \mathbb{R})$ with any invariant Riemannianian metric is not symmetric space.*

Establish correspondence:

$$\begin{array}{ccc} \{\text{Lie groups}\} & \longleftrightarrow & \{\text{symmetric spaces}\} \\ (G, K) & & X \end{array}$$

Remark 3.14. A Riemannian manifold M is called *locally symmetric* if its curvature tensor is parallel, i.e., $\nabla R = 0$.

Fact: Irreducible symmetric spaces are classified into 3 types:

$$\text{flat}(\mathbb{R}^n), \text{compact type, noncompact type.}$$

Theorem 3.15. *A symmetric space of compact type has non-negative section curvature and positive Ricci curvature.*

Corollary 3.16. *It is compact.*

Theorem 3.17. *A symmetric space of noncompact type has non-positive section curvature and has strictly negative Ricci curvature. It is diffeomorphism to $T_xX, \forall x \in M \Rightarrow X$ is simply connected.$\Longrightarrow X$ is a Hadamard manifold.*

Remark 3.18. A simply connected non-positively curved Riemannian manifold is called a *Hadamard manifold.*

Theorem 3.19. ***(Cartan fixed point theorem)*** *Let X be a Hadamard manifold, and $K \subset \mathrm{Is}(X)$ a compact isometry group. Then there exists at least one point fixed by K, i.e. $X^K \neq 0$.*

Main idea of proof. For details, one can read related references.

Proof. Consider a special case where $K \subset \mathrm{Is}(X)$ is a finite group. We want to find a fixed point of K.

Take any point $x \in X$, consider the orbit $K \cdot x$. $K \cdot x$ is fixed by $K \Rightarrow$ its "center" is fixed by K. □

Corollary 3.20. *For any semisimple Lie group G with finitely many connected components, every two maximal compact subgroups K, K' of G are conjugate.*

Proof. We only give a sketch of proof. Use K to get $X = G/K$, which is a Hadamard manifold. K' acts on G, thus acts on $X = G/K$. We also know that K' is compact, thus by Cartan fixed point theorem, we know that there exist a point $g \cdot K \in X$, which is invariant under K'. That is for any $k' \in K', k'gK = g \Rightarrow g^{-1}k'gK = K \Rightarrow g^{-1}k'g \in K$, i.e., $g^{-1}K'g \subset K$, by the maximal of K' we know that $g^{-1}K'g = K$, thus K, K' are conjugate. □

Lie groups are important in topology. Topology of Lie groups are concentrated in compact Lie groups.

If $\Gamma \subset G$ is a discrete subgroup. $\Gamma\backslash G/K$ is a locally symmetric space. (since Γ acta properly on G and also on $X = G/K$.) The action of Γ on X is considered for many quotient about Γ. For example, classifying space of Γ.

For many natural Γ, $\Gamma\backslash X$ is not smooth, but has finite quotient singularities. Satake firstly introduced the notion of V-manifolds[2], and it is now called orbifolds[3], which is introduced by Thurstion.

[2]See I.Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. USA, 42, 356-363, 1956.

[3]See J.Ratcliffe, *Foundations of hyperbolic manifolds*, Second edition, GTM 149, Springer, 2006.

3.2. **Lie algebras.** Since Lie groups are so important, then a natural problem is: given Lie groups, how to study them?

A major contribution of Lie is: we can understand Lie group by their Lie algebra.

Fact: A connected Lie group is determined by a small neighborhood of the identity elements. In fact assume $e \in U$ is a small neighborhood, then

$$G = \bigcup_{n=1}^{\infty} U^n.$$

Let G be a Lie group, then G is a smooth manifold. A vector field X is a section of the tangent bundle $X : G \to TG \xrightarrow{\pi} G$, where X is smooth and $\pi \circ X = id$. G acts on G by left multiplication.

Definition 3.21. A vector field is called *left invariant vector filed* on G, if

$$\mathrm{d}L(g)X = X$$

which can be written as

$$\mathrm{d}L(g) \circ X = X \circ L(g)$$

where $L(g)$ denotes the left action.

The Lie algebra $\mathfrak{g}$ of G is the space of all left invariant vector fields with a suitable Lie bracket.

Before we give the Lie bracket of $\mathfrak{g}$, we shall recall the definition of Lie algebra.

Definition 3.22. Let k be a field. $\mathfrak{l}$ a finite dimensional vector space over k. If $\mathfrak{l}$ admits a bracket $[,] : \mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$ satisfying the conditions:

(1) (bilinearity) $\forall X, Y, Z \in \mathfrak{l}, a, b \in k$
$$[aX + bY, Z] = a[X, Z] + b[Y, Z];$$
(2) (skew-symmetry) $\forall X, Y \in \mathfrak{l}$
$$[X, Y] = -[Y, X];$$
(3) (Jacobian property) $\forall X, Y, Z \in \mathfrak{l}$
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

To make $\mathfrak{g}$ a Lie algebra, we still need to assign a Lie bracket on $\mathfrak{g}$={left invariant vector filed on G}.

Let $C^\infty(G)$ be the space of smooth function on G. The derivation of $C^\infty(G)$ is a linear map

$$X : C^\infty(G) \to C^\infty(G)$$

such that $X(ab) = X(a)b + aX(b), \forall a, b \in C^\infty(G)$.

Given any vector field X on G, we get a derivation

$$X : C^\infty(G) \to C^\infty(G)$$

Besides, every derivation of $C^\infty(G)$ arises in this way.

Proposition 3.23. *if D, D' are two derivation of $C^\infty(G)$, then $[D, D'] = DD' - D'D$ is also a derivation. Here, for any $f \in C^\infty(G)$,$[D, D']f$ is define by $DD'f - D'Df$ which does make sense.*

Proposition 3.24. *If X, X' are left invariant vector field on G, then $[X, X']$ is also left invariant.*

Thus the Lie bracket of $\mathfrak{g}$ is just defined as $[X, X'] = XX' - X'X$, and $(\mathfrak{g}, [,])$ forms a Lie algebra.

Example $G = GL(n, \mathbb{R}), \mathfrak{g} = M_{n\times n}(\mathbb{R})$, for any $X, Y \in M_{n\times n}(\mathbb{R})$,

$$[X, Y] = XY - YX$$

where XY is the matrix product of X, Y.

Example exp for matrices.

For $z \in \mathbb{C}$, $e^z = 1 + \dfrac{z}{1!} + \dfrac{z^2}{2!} + \cdots$, define this for matrices:

$$e^A = I_n + \frac{A}{1!} + \frac{A^2}{2!} + \cdots$$

Suppose $G \subset GL(n, \mathbb{R})$ is Lie subgroup. Its Lie algebra $\mathfrak{g}$ can be described as

$$\mathfrak{g} = \{\text{matrices } X \in M_{n\times n}(\mathbb{R}) \big| \exp(tX) \in G, \text{ for all } t \in \mathbb{R}\}.$$

Proposition 3.25. *Given a Lie group homomorphism $F : G \to H$, there is a Lie algebra homomorphism*

$$(\mathrm{d}f)_e : \mathfrak{g} \to \mathfrak{h}$$

Recall that, as a vector space, $\mathfrak{g} = T_eG$.

Proposition 3.26. *If H is a normal Lie subgroup of G, then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, i.e., $\mathfrak{h}$ is a linear subspace s.t. $\forall Y \in \mathfrak{g}, [\mathfrak{h}, Y] \subset \mathfrak{h}$.*

Proposition 3.27. *A Lie group homomorphism $f : G \to H$ is determined by the Lie algebra homomorphism $(\mathrm{d}f)_e : \mathfrak{g} \to \mathfrak{h}$. That is, if $\exists f' : G \to H$ s.t. $(\mathrm{d}f')_e = (\mathrm{d}f)_e$, then $f = f'$.* (Hint: use the uniqueness of initial value problem of ODE.)

Definition 3.28. *Exponential map*: $\exp : \mathfrak{g} \to G$ is an analytic map satisfying the following conditions

(1) $\exp(0) = e$;
(2) $(\mathrm{d}\exp)_0 = Id$;($\Rightarrow$ By inverse function theorem exp is a local diffeomorphism.)

(3) For every $X \in \mathfrak{g}$, the map $\mathbb{R} \to G, t \mapsto \exp(tX)$ is a Lie group homomorphism. (called one parameter subgroups.)

Proposition 3.29. *Let $f : G \to G$ a Lie group homomorphism, then we have a commutative diagram*

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{(\mathrm{d}f)_e} & \mathfrak{h} \end{array}$$

$\Rightarrow$ *f is determined by* $\mathrm{d}f_e$.

4. The Exponential Map, Some Kinds of Lie Group and Something About Algebraic Geometry

4.1. **The Exponential Map.** *Recall that the* ***exponential map*** *is a map*

$$\exp : \mathfrak{g} \to G$$

such that (1) $\exp(0) = e$*, (2)*$(\mathrm{dexp})_0 = Id$*, (3)*$\forall X \in \mathfrak{g}$*, the map* $\mathbb{R} \to G, t \mapsto \exp(tX)$ *is a one parameter subgroup.*

The existence of the exponential map is proved by using integrating vector fields.

General comments: Let M *be a smooth manifold. Given a smooth curve* $c : (-\epsilon, \epsilon) \to M$*, we get a vector field along* $c(t)$*, such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} c(t)|_{t=t_0} \in T_{c(t_0)} M.$$

Conversely, given a vector field X *on* M*, an* ***integral curve*** *of* X *through a point* $x \in M$ *is a curve* $c : (-\epsilon, \epsilon) \to M$ *such that* $c(0) = X$*,* $\frac{\mathrm{d}}{\mathrm{d}t} c(t) = X(c(t))$*.*

Theorem 4.1. *Under the above assumption, for all* $x \in M$*, there exists an integral curve of* X *through* x*.*

The existence of the exponential map can be proved by using this idea: $\forall X \in \mathfrak{g}$*, we can get a left invariant vector field. Then by the above theorem, we have an integral curve* c *of this vector field through the identity element, that is,* $c : (-\epsilon, \epsilon) \to G$ *is a smooth curve, such that* $c(0) = e, c'(0) = X$*. We can extend* c *so that* $c(1)$ *is defined, then we define* $\exp X$ *to be* $c(1)$*.*

The details are left to be an exercise.

Proposition 4.2. *Let* $f : G \to H$ *be a group homomorphism, then the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\mathrm{d}f_e} & \mathfrak{h} \end{array}$$

This implies that if $\mathfrak{g} = \mathfrak{h}$*, then* G *and* H *are locally the same.*

Start with a Lie group homomorphism, we get a Lie algebra homomorphism. How about the converse?

Theorem 4.3. *Let* G, H *be Lie groups. Assume* G *is simply connected, then for any Lie algebra homomorphism* $\varphi : \mathfrak{g} \to \mathfrak{h}$*, there exists a Lie group homomorphism* $f : G \to H$ *such that* $\mathrm{d}f_e = \varphi$*.*

Remark 4.4. : If G is not simply connected, then the theorem is false. For example, $G = S^1$, $H = \mathbb{R}$.

Idea of the proof: $\forall g \in G$, connect it to the identity element $e \in G$ by a smooth path $c(t), t \in [0, 1]$ in G. We then get a path $X(t) = c'(t)$ in

$\mathfrak{g} = T_eG$. Under the Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$, we get a path in $\mathfrak{h}$, namely $\varphi(X(t))$. Then solve the equation

$$h : [0,1] \to H, \quad \frac{\mathrm{d}h(t)}{\mathrm{d}t} = \varphi(X(t))h(t)$$

and define $f(g)$ to be $h(1)$.

The definition of $f : G \to H$ is dependent of the choice of the path $c(t)$. But the assumption that G is simply connected implies that the map f is in fact well-defined.

Correspondence between Lie subgroups and Lie subalgebras.

Proposition 4.5. *If G_1, G_2 are two Lie subgroups of G, then $G_1 \subset G_2$ if and only if $\mathfrak{g}_1 \subset \mathfrak{g}_2$.*

We know that any Lie subgroup H of G gives a Lie subalgebra.

Proposition 4.6. *Any Lie subalgebra of $\mathfrak{g}$ is the Lie algebra of an immersed Lie subgroup of G. That is to say, there exists a Lie group $\tilde{H}$ and an immersion $i : \tilde{H} \to G$ such that $i(\tilde{\mathfrak{g}})$ is the given Lie subalgebra.*

Remark 4.7. The Lie subgroup is not always closed (or embedded). For example, take $G = \mathbb{R}^2/\mathbb{Z}^2$, then a line through the origin with irrational slope is a Lie subgroup of G but not closed.

4.2. **Solvable, Semisimple and Nilpotent.** Given a Lie group G, define its **commutator subgroup**

$$G' = \langle G, G \rangle = \langle (xy) = xyx^{-1}y^{-1} | x, y \in G \rangle.$$

This is a normal subgroup of G, and is the smallest normal subgroup such that the quotient is abelian.

Proposition 4.8. *G' is an immersed subgroup. Further, if G is simply connected, then it is an embedded Lie subgroup.*

Let $G^{(1)} = G', G^{(k+1)} = (G^{(k)})'$ by induction. Then we get

$$G \supset G^{(1)} \supset G^{(2)} \supset \cdots.$$

Definition 4.9. G is called a **solvable** Lie group if $G^{(k)} = \{1\}$ for some k.

Example (1)The Borel subgroup is solvable;
(2)Given a partition $(n_1, n_2, \cdots, n_k)$ of n, or a block decomposition, then we get a **block upper triangular subgroup**

$$N = \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ & I_{n_2} & \cdots & * \\ & & \ddots & \vdots \\ & & & I_{n_k} \end{pmatrix} \right\},$$

this subgroup is solvable.

Theorem 4.10. ***(Lie)*** *Let G be a connected solvable Lie group, $R : G \to GL(V)$ is a complex linear representation, where V is a complex vector space. Then there exists a basis of V in which all operators $R(g), g \in G$ are given by upper triangular matrices.*

Similarly, we can define solvable Lie algebras.

Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^{(1)} = \mathfrak{g}', (\mathfrak{k} + \imath) = [(\mathfrak{k}), (\mathfrak{k})]$ by induction. Then $\mathfrak{g}$ is called ***solvable*** *if there exists a k such that $\mathfrak{g}^{(k)} = 0$.*

Fact: *A connected Lie group is solvable if and only if its algebra is solvable.*

Proposition 4.11. *Given any Lie algebra $\mathfrak{g}$, the sum of two solvable ideals of $\mathfrak{g}$ is also a solvable ideal.*

Corollary 4.12. *Any Lie algebra contains a maximal solvable ideal.*

We call the maximal solvable ideal of a Lie algebra $\mathfrak{g}$ the ***radical*** *of $\mathfrak{g}$, and denoted it by* Rad$\mathfrak{g}$.

Proposition 4.13. *In any Lie group G, there is also a largest connected solvable normal Lie subgroup, which is denoted by* Rad$G$*, called the* ***radical*** *of G.*

Definition 4.14. A Lie group G is called **semisimple** if Rad$G = \{1\}$. A Lie algebra $\mathfrak{g}$ is called semisimple if Rad$\mathfrak{g} = 0$.

Proposition 4.15. *A Lie algebra is* ***semisimple*** *if and only if it has no commutative ideals.*

Remark 4.16. We can define the **central series** $C_1(G) = (G, G), C_n(G) = (G, C_{n-1}(G))$ by induction, then

$$G \supset C_1(G) \supset \cdots \supset C_n(G) \supset \cdots .$$

Similarly, for Lie algebra we can define $C_1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}], C_{n+1}(\mathfrak{g}) = [\mathfrak{g}, C_n(\mathfrak{g})]$ by induction.

Definition 4.17. G is **nilpotent** if for some k, $C_k(G) = \{1\}$; $\mathfrak{g}$ is nilpotent if for some k, $C_k(\mathfrak{g}) = 0$.

4.3. **A crash review of algebraic geometry.** Let K be an arbitrary field, $A^n = K^n$ be the affine space.

Definition 4.18. An affine algebraic variety V, or an algebraic variety in A^n, is a subset of A^n defined by a system of equations: $f(x_1, \cdots, x_n) = 0, f \in S$, where S could be finite or infinite.

For any finite collection S of polynomials $S = \{f_1, \cdots, f_m\}$, it generates an ideal of the polynomial ring $K[x_1, \cdots, x_n]$, namely

$$I = \{\sum g_i f_i | g_i \in K[x_1, \cdots, x_n]\}$$

. We say S is a set of generator of I.

Fact: S and I define the same affine variety.

Fact: For any subvariety $V \subset A^n$, the set of polynomials that vanish at V form an ideal.

Given an ideal $I \subset K[x_1, \cdots, x_n]$. How to find a set of generator?

Problem:(1) existence of finite set of generators; (2) explicit generators.

Definition 4.19. An algebra is called **Noetherian** if one of the following two equivalent conditions is satisfied.

(a) every ideal is finitely generated;

(b) every increasing chain of ideals stabilizing after finitely many steps. That is to say, if

$$I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$$

is an increasing chain of ideals, then $I_{m_0} = I_{m_0+1} = \cdots$ for some m_0.

Example :(1)$\mathbb{Z}$ is Noetherian; (2) every field is Noetherian.

Theorem 4.20. ***(Hilbert's Basis Theorem)*** *If R is a Noetherian ring, then so is $R[x_1, \cdots, x_n]$.*

Corollary 4.21. *Every affine algebraic variety is defined by finitely many polynomials.*

Given an ideal $I \subset K[x_1, \cdots, x_n]$, let $V(I)$ be the variety defined by I, let $A = K[x_1, \cdots, x_n]/I$.

Proposition 4.22. *There is a one-to-one correspondence between points of $V(I)$ with coordinates in K and algebra homomorphisms from A to K.*

How to get this map?

Given $(a_1, \cdots, a_n) \in V(I)$, we can define a map

$$K[x_1, \cdots, x_n] \to K, \quad P(x_1, \cdots, x_n) \mapsto P(a_1, \cdots, a_n).$$

The map factors through A, so we get a homomorphism $A \to K$.

Conversely, given a homomorphism $\varphi : A \to K$, how to pick a point in $V(I)$?

Let $\pi : K[x_1, \cdots, x_n] \to A$ be the natural map, then it is easy to verify that the point $(\varphi(\pi(x_1)), \cdots, \varphi(\pi(x_n)))$ is just what we want.

Definition 4.23. Let A be a commutative algebra, then elements of A can be seen as functions on $V(I)$, they are called **polynomials** on $V(I)$.

Definition 4.24. Let $M \subset A^n$, $N \subset A^m$ be two algebraic varieties. A **morphism** $f : M \to N$ is a map given by polynomials. That is to say, there exist $f_1, \cdots, f_m \in K[M]$, such that $f(x) = (f_1(x), \cdots, f_m(x))$, where suppose $M = V(I)$, and let $K[M] = K[x_1, \cdots, x_n]/I$.

Proposition 4.25. *There is a one-to-one correspondence between the morphisms between M and N and the algebra homomorphisms between $K[N]$ and $K[M]$.*

Definition 4.26. **Zariski topology** on A^n is defined as follows: the closed subsets are given by algebraic subvarieties.

Claim: This indeed defines a topology.

Remark 4.27. Open subsets are very large.

Definition 4.28. A topological space is called **irreducible** if it is nonempty and one of the following three conditions is satisfied:

(1) any nonempty open subset is dense in M;

(2) any two nonempty open sets intersect;

(3) M is not the union of two proper closed subset.

Given a variety $V = V(I)$, let $K[V] = K[x_1, \cdots, x_n]/I$, then form the quotient

$$QK[V] = \{\frac{f}{g} | f, g \in K[V]\}.$$

Proposition 4.29. *V is irreducible can imply that $QK[V]$ is a field.*

Definition 4.30. The **projective varieties** are subsets of P^n, the projective space $K^{n+1}/K^{\times}$, that are zero locus of a set of homogeneous polynomials.

Definition 4.31. The **Grassmannian variety** is $\{k$-dim linear subspaces of $K^{n+1}\}$, for $1 \leq k \leq n-1$.

Example $P^n = \{$ 1-dim linear subspaces of $K^{n+1}\}$.

Definition 4.32. The **full flag variety** is

$\{0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset V_{n+1} = K^{n+1} | V_i$ is $i-$dim linear subspace.$\}$

Definition 4.33. The **partial flag variety** is

$$\{V_0 \subset V_1 \subset \cdots \subset V_k \subset K^{n+1} | V_i \text{ is } i\text{−dim linear subspace.}\}$$

Fact: $GL(n, K)$ acts on the full flag variety transitively. The stabilizer of the standard flag is the Borel subgroup B. $GL(n, K)/B$ is a flag variety.

Definition 4.34. Given a linear algebraic group G, an algebraic subgroup P is called **parabolic** if G/P is a projective variety.

Example The block upper triangular subgroup defined above is a parabolic subgroup of $GL(n, K)$.

Every parabolic subgroup of $GL(n, K)$ is conjugate to a block upper triangular subgroup.

4.4. **What's next.** The structure and classification of complex semisimple Lie algebras and Lie groups. This involves Cartan decomposition, Iwasawa decomposition, Brahat decomposition and Langlands decomposition.

The compactification and boundary of a symmetric space X. This has some relations with the classifying space.