On the first eigenvalue of the Witten-Laplacian and the diameter of compact shrinking solitons

Akito Futaki †, Haizhong Li ‡, Xiang-Dong Li ‡

Abstract

We prove a lower bound estimate for the first non-zero eigenvalue of the Witten-Laplacian on compact Riemannian manifolds. As an application, we derive a lower bound estimate for the diameter of compact gradient shrinking Ricci solitons. Our results improve some previous estimates which were obtained by the first author and Y. Sano in [16], and by B. Andrews and L. Ni in [1]. Moreover, we extend the diameter estimate to compact self-similar shrinkers of mean curvature flow.

1 Introduction

During the recent years, the Bakry-Emery Ricci curvature has received a lot of attention in various areas in mathematics. On the one hand, it has been used to establish some functional inequalities which play an important role in the study of the rate of convergence to the equilibrium measure of diffusion processes [2, 3]. On the other hand, it is a good substitute of the Ricci curvature for establishing many interesting theorems in differential geometry, for example, Myers’ theorem, eigenvalues estimates, Li-Yau Harnack inequality, Liouville theorems and the Cheeger-Gromoll splitting theorem [5, 6, 4, 14, 19, 26, 31, 24]. Moreover, it has been an important tool in the optimal transport theory [30] and in Perelman’s work for the entropy formula on Ricci flow [25] (see also [20]). The purpose of this paper is to prove a lower bound estimate for the first non-zero eigenvalue of the Witten-Laplacian on compact Riemannian manifolds with Bakry-Emery Ricci curvature bounded from below by a constant. As an application, we prove a lower bound estimate of the diameter for compact shrinking Ricci solitons. Our results improve some previous estimates which were obtained by the first author and Y. Sano in [16], and by B. Andrews and L. Ni in [1]. Moreover, we extend the diameter estimate to compact self-shrinkers of the mean curvature flow.

Let us first introduce some basic notations. Let \((M, g)\) be a complete Riemannian manifold, \(f \in C^2(M)\) and \(d\mu = e^{-f}dv\), where \(dv\) denotes the Riemannian volume measure on \((M, g)\). For all \(u, v \in C_0^\infty(M)\), the following integration by parts formula holds

\[
\int_M \nabla u \cdot \nabla v \, d\mu = - \int_M (\Delta f u) v \, d\mu = - \int_M u (\Delta f v) \, d\mu.
\]

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where $\Delta_f$ is the so-called Witten-Laplacian on $(M,g)$ with respect to the weighted volume measure $\mu$. More precisely, we have

$$\Delta_f = \Delta - \nabla f \cdot \nabla.$$

In [2], Bakry and Emery proved that for all $u \in C^\infty_0(M)$,

$$\Delta_f |\nabla u|^2 - 2\langle \nabla u, \nabla \Delta_f u \rangle = 2|\nabla^2 u|^2 + 2(Ric + \nabla^2 f)(\nabla u, \nabla u).$$

(1)

The formula (1) can be viewed as a natural extension of the Bochner-Weitzenböck formula. The quantity $Ric + \nabla^2 f$, which is called in the literature the Bakry-Emery Ricci curvature on the weighted Riemannian manifolds $(M,g,f)$, plays as a good substitute of the Ricci curvature in many problems in comparison geometry on weighted Riemannian manifolds. See [2, 3, 4, 5, 6, 14, 19, 20, 26, 31, 24] and reference therein.

Now we state the main results of this paper. The first result of this paper is the following lower bound estimate of the first non-zero eigenvalue of the Witten-Laplacian on compact Riemannian manifolds.

**Theorem 1.1** Let $(M,g)$ be an $n$-dimensional compact Riemannian manifold, and let $\phi \in C^2(M)$. Suppose that there exists a constant $K \in \mathbb{R}$ such that

$$Ric + \nabla^2 \phi \geq Kg.$$  

Then the first non-zero eigenvalue $\lambda_1$ of the Witten-Laplacian $\Delta_\phi$ satisfies

$$\lambda_1 \geq \sup_{s \in (0,1)} \left\{ 4s(1-s)\frac{\pi^2}{d^2} + sK \right\},$$

(2)

where $d$ is the diameter of $(M,g)$.

As an application of the above theorem, we have the following lower bound estimate for the diameter of compact gradient shrinking Ricci solitons. Recall that a complete Riemannian manifold $(M,g)$ is called a gradient shrinking Ricci soliton if there exists a positive constant $\lambda > 0$ and a smooth function $f$ on $M$ such that (see [8] and reference therein)

$$Ric(g) + \nabla^2 f = \lambda g.$$  

(3)

If $f$ is a constant, then $g$ is Einstein. In this case we say that $(M,g,f)$ is trivial.

**Theorem 1.2** Let $(M,g,f)$ be a non-trivial compact shrinking Ricci soliton with

$$Ric + \nabla^2 f = \lambda g,$$

where $\lambda$ is a positive constant. Then the diameter of $(M,g)$ satisfies

$$d \geq \frac{2(\sqrt{2} - 1)\pi}{\sqrt{\lambda}}.$$  

(4)
**Corollary 1.3** Let \((M, g, f)\) be a compact shrinking Ricci soliton with
\[
\text{Ric} + \nabla^2 f = \lambda g,
\]
where \(\lambda\) is a positive constant. If the diameter of \((M, g)\) satisfies
\[
d < \frac{2(\sqrt{2} - 1)\pi}{\sqrt{\lambda}},
\]
then \((M, g)\) must be Einstein.

**Remark 1.4** The study of lower bound estimate of the first eigenvalue on Riemannian manifolds has a long time history. See [22, 7, 10, 21, 29, 32, 11, 12, 5] and reference therein. In the case where \((M, g)\) is a compact Riemannian manifold with non-negative Ricci curvature, Zhong and Yang [32] obtained the optimal lower bound estimate of the first eigenvalue of the Laplacian, i.e.,
\[
\lambda_1 \geq \frac{\pi^2}{d^2},
\]
where \(d\) is the diameter of \((M, g)\). In [28], Shi and Zhang proved that on compact Riemannian manifolds with Ricci curvature bounded below by a constant \(K \in \mathbb{R}\), i.e.,
\[
\text{Ric} \geq Kg,
\]
the first non-zero eigenvalue \(\lambda_1\) of the Laplacian \(\Delta\) satisfies
\[
\lambda_1 \geq \sup_{s \in (0, 1)} \left\{ 4s(1-s)\frac{\pi^2}{d^2} + sK \right\}.
\]
See also Qian-Zhang-Zhu [27] for its extension to compact Alexandrov spaces with Ricci curvature bounded from below by \(K\). Note that, by direct calculation, we have
\[
\sup_{s \in (0, 1)} \left\{ 4s(1-s)\frac{\pi^2}{d^2} + sK \right\} = \begin{cases} 0 & \text{if } Kd^2 < -4\pi^2, \\ \left(\frac{\pi}{2} + \frac{Kd}{4\pi} \right)^2 & \text{if } Kd^2 \in [-4\pi^2, 4\pi^2], \\ \frac{K}{K} & \text{if } Kd^2 \in (4\pi^2, (n-1)\pi^2].
\end{cases}
\]

**Theorem 1.1** is a natural extension of the above estimate of Shi and Zhang to the Witten-Laplacian via the Bakry-Emery Ricci curvature on compact Riemannian manifolds, and it is a general principle that this kind of extension is possible. A typical such result can also be found in [23].

**Remark 1.5** In [16], under the same condition as in Theorem 1.1, the first author and Y. Sano proved that the first non-zero eigenvalue of the Witten-Laplacian \(\Delta_{\phi}\) satisfies
\[
\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{31}{100} K.
\]
One can easily check that our new lower bound estimate (2) for \(\lambda_1\) is better than (5). In the case where \((M, g, f)\) is a non-trivial compact gradient shrinking Ricci solitons with Ric +
\( \nabla^2 f = \lambda g \), the estimate (5) led to the following lower bound estimate for the diameter of \((M, g)\):

\[
d \geq \frac{10 \pi}{13 \sqrt{\lambda}}.
\]  

(6)

One can easily check that

\[
2(\sqrt{2} - 1) > \frac{10}{13}.
\]

Hence, the lower bound estimate (4) in Theorem 1.2 is sharper than (6).

**Remark 1.6** Taking \( s = \frac{1}{2} \) in Theorem 1.1, we obtain the following lower bound estimate

\[
\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{K}{2}.
\]  

(7)

This recaptures the lower bound estimate of \( \lambda_1 \) due to Andrews and Ni (Proposition 3.1 in [1]). In the case where \((M, g, f)\) is a non-trivial compact gradient shrinking Ricci solitons with \( \text{Ric} + \nabla^2 f = \lambda g \), Andrews and Ni (Corollary 3.1 in [1]) used the estimate (7) to derive the following lower bound estimate for the diameter of \((M, g)\):

\[
d \geq \sqrt{\frac{2}{3\lambda}} \pi,
\]  

(8)

which is better than (6) obtained in [16]. One can easily check that

\[
2(\sqrt{2} - 1) > \sqrt{\frac{2}{3}}.
\]

Hence, the lower bound estimate (4) in Theorem 1.2 is sharper than (8).

As another application of Theorem 1.1, we can also obtain a lower bound estimate for the diameter of compact self-shrinkers of the mean curvature flow (Theorem 4.3). In fact Theorem 1.2 above follows from Theorem 1.1 and the fact that the Witten-Laplacian on the Ricci soliton with (3) has eigenvalue \( 2\lambda \) ([16], see Lemma 3.1 below), but we can show that the Witten-Laplacian takes the same eigenvalue \( 2\lambda \) on a compact self-shrinker

\[
x^\perp = -\frac{1}{\lambda} \bar{H}.
\]  

(9)

2 Proof of Theorem 1.1

To prove Theorem 1.1, we use the following comparison theorem due to Chen and Wang [11, 12], Bakry and Qian [5], also Andrews and Ni [1].

**Theorem 2.1** (Chen-Wang [11, 12], Bakry-Qian [5], Andrews-Ni [1]) Let \((M, g)\) be an \( n \)-dimensional compact Riemannian manifold, and let \( \phi \in C^2(M) \). Suppose that there exists a constant \( K \in \mathbb{R} \) such that

\[
\text{Ric} + \nabla^2 \phi \geq Kg.
\]
Then the first non-zero Neumann eigenvalue of the Witten-Laplacian \( \Delta \phi \) satisfies
\[
\lambda_1 \geq \lambda_1(L),
\]
where \( \lambda_1(L) \) denotes the first non-zero Neumann eigenvalue of the following one-dimensional Ornstein-Uhlenbeck operator on \((-d/2, d/2)\):
\[
L = \frac{d^2}{dx^2} - Kx \frac{d}{dx}.
\]
More precisely, \( \lambda_1(L) \) is the first non-zero eigenvalue of the problem
\[
\begin{align*}
v''(x) - Kx v'(x) &= -\lambda v(x), & x \in (-\frac{d}{2}, \frac{d}{2}), \\
v'(-\frac{d}{2}) &= 0, & v'(\frac{d}{2}) = 0.
\end{align*}
\]

**Proof of Theorem 1.1.** By Theorem 4.3, we need only to prove that
\[
\lambda_1(L) \geq \sup_{s \in (0,1)} \left\{ 4s(1-s) \frac{\pi^2}{d^2} + sK \right\}.
\]
To prove (12), we modify the argument used in the proof of Theorem 1.1 in [28], cf. also the one of Corollary 4.3 in [27]. Denote \( D = \frac{d}{2}, f = v' \). Then \( f \) is the eigenfunction of the first non-zero eigenvalue \( \lambda - K \) of the Ornstein-Uhlenbeck operator \( L \) on \((-D, D)\) with the Dirichlet boundary condition. More precisely, we have
\[
f'' - Kx f' = -(\lambda - K)f, \tag{13}
\]
and \( f(-D) = f(D) = 0 \). By maximum principle, we can prove that \( f \) must have fixed sign on \((-D, D)\). So we can assume that \( f(x) > 0 \) for all \( x \in (-D, D) \).

Fix a constant \( a > 1 \). Multiplying \( f^{a-1} \) to the both sides of (13) and integrating on \((-D, D)\), we have
\[
\int_{-D}^{D} f^{a-1}(x)f''(x)dx = -(\lambda - K) \int_{-D}^{D} f^a(x)dx + \int_{-D}^{D} Kxf^{a-1}(x)f'(x)dx.
\]
Integrating by parts and using the fact \( f(\pm D) = 0 \) and \( f(x) > 0 \) on \((-D, D)\), we get
\[
\int_{-D}^{D} f^{a-1}(x)f''(x)dx = -(a-1) \int_{-D}^{D} f^{a-2}(x)f'(x)dx = -\frac{4(a-1)}{a^2} \int_{-D}^{D} [(f^{a/2}(x))^2]'dx,
\]
Similarly, we have
\[
\int_{-D}^{D} Kxf^{a-1}(x)f'(x)dx = - \int_{-D}^{D} f(Kf^{a-1} + (a-1)Kxf^{a-2}f')dx
\]
\[
= -K \int_{-D}^{D} f^{a}(x)dx - (a-1) \int_{-D}^{D} Kxf^{a-1}(x)f'(x)dx,
\]
which yields
\[
\int_{-D}^{D} Kxf^{a-1}(x)f'(x)dx = -\frac{K}{a} \int_{-D}^{D} f^{a}(x)dx.
\]

Set \( u = f^{a/2} \). Then we can deduce that

\[
\frac{4(a - 1)}{a^2} \int_{-D}^D |u'|^2 \, dx = \left( \lambda - K \left( 1 - \frac{1}{a} \right) \right) \int_{-D}^D u^2 \, dx.
\]

Let \( s = 1 - \frac{1}{a} \). We get

\[
4s(1 - s) \int_{-D}^D |u'|^2 \, dx = (\lambda - Ks) \int_{-D}^D u^2 \, dx.
\]

Hence

\[
\frac{\lambda - Ks}{4s(1 - s)} = \frac{\int_{-D}^D |u'|^2 \, dx}{\int_{-D}^D u^2 \, dx}.
\]

Using the Wirtinger inequality, we have

\[
\frac{\lambda - Ks}{4s(1 - s)} \geq \frac{\pi^2}{4D^2} = \frac{\pi^2}{d^2}.
\]

Hence, for all \( s \in (0, 1) \), we have

\[
\lambda \geq 4s(1 - s) \frac{\pi^2}{d^2} + Ks.
\]

The proof of Theorem 1.1 is completed. \( \square \)

### 3 Proof of Theorem 1.2

To prove Theorem 1.2, we need the following

**Lemma 3.1** ([16]) Let \((M, g, f)\) be a non-trivial gradient shrinking Ricci solitons with

\[\text{Ric} + \nabla^2 f = \lambda g.\] (14)

Then \( f \) is an eigenfunction of the Witten-Laplacian \( \Delta_f \) with eigenvalue equal to \( 2\lambda \).

**Proof of Theorem 1.2.** By Theorem 1.1 and Lemma 3.1, for all \( s \in (0, 1) \), we have

\[
2\lambda \geq 4s(1 - s) \frac{\pi^2}{d^2} + s\lambda,
\]

which yields

\[
\lambda \geq \frac{4s(1 - s) \pi^2}{2 - s} \frac{1}{d^2}.
\]

But elementary computations show

\[
\frac{4s(1 - s)}{2 - s} \leq 12 - 8\sqrt{2},
\]
where the equality is attained for $s = 2 - \sqrt{2} \in (0, 1)$. Thus we obtain

$$\lambda \geq 4(3 - 2\sqrt{2}) \frac{\pi^2}{d^2}.$$ 

Equivalently, the diameter of $(M, g)$ must satisfy the following lower bound

$$d \geq \frac{2(\sqrt{2} - 1)\pi}{\sqrt{\lambda}}.$$ 

The proof of Theorem 1.2 is completed.

□

4 The diameter of compact self-shrinkers for mean curvature flow

Let $x : M \to \mathbb{R}^{n+p}$ be an $n$-dimensional submanifold in the $(n+p)$-dimensional Euclidean space. If we let the position vector $x$ evolve in the direction of the mean curvature $\vec{H}$, then it gives rise to a solution to the mean curvature flow:

$$x : M \times [0, T) \to \mathbb{R}^{n+p}, \quad \frac{\partial x}{\partial t} = \vec{H}.$$ 

We call the immersed manifold $M$ a self-shrinker if it satisfies the quasilinear elliptic system (see [17], or [13]): for some positive constant $\lambda$,

$$\vec{H} = -\lambda x^\perp,$$

where $\perp$ denotes the projection onto the normal bundle of $M$.

We have (see [18])

$$\frac{1}{2\lambda} |\vec{H}|^2 + \frac{1}{4} \Delta |x|^2 = \frac{n}{2}.$$ 

Put

$$\phi := 2\lambda \left(\frac{|x|^2}{4} - \frac{n}{4\lambda}\right).$$

Define the Witten-Laplacian by

$$\Delta_\phi = \Delta - \nabla \phi \cdot \nabla.$$ 

From above formulas, we can check

$$\Delta_\phi \left(\frac{1}{2} |x|^2\right) = \Delta \left(\frac{1}{2} |x|^2\right) - \frac{1}{2} \nabla |x|^2 \cdot \nabla |x|^2$$

$$= \frac{n}{2} - \frac{1}{2\lambda} |\vec{H}|^2 - \frac{\lambda}{2} |x|^2$$

$$= \frac{n}{2} - \frac{\lambda}{2} |x|^2.$$ 

Thus we have

$$\Delta_\phi \left(\frac{1}{4} |x|^2 - \frac{n}{4\lambda}\right) = -2\lambda \left(\frac{|x|^2}{4} - \frac{n}{4\lambda}\right).$$

Thus we have proved
Theorem 4.1 In the above situation we have the eigenvalue $2\lambda$ of the Witten-Laplacian $\Delta_\phi$ with eigenfunction $\phi$:

$$\Delta_\phi \phi = -2\lambda \phi.$$ 

Let $h_{ij}^\alpha$ is the components of the second fundamental form, $H^\alpha = \sum_k h_{kk}^\alpha$, the Gauss equation is (see [9])

$$R_{ij} = \sum_\alpha H^\alpha h_{ij}^\alpha - \sum_{\alpha,k} h_{ik}^\alpha h_{kj}^\alpha.$$ 

Thus we have from the definition of $\phi$ in (15)

$$R_{ij} + \phi_{ij} = \lambda g_{ij} - \sum_{\alpha,k} h_{ik}^\alpha h_{kj}^\alpha 
\geq [\lambda - K_0]g_{ij},$$

where

$$K_0 = \max_{1 \leq i \leq n} \left\{ \sum_{\alpha,k} h_{ik}^\alpha h_{ki}^\alpha \right\},$$

and we have used

$$\phi_{ij} = (\frac{\lambda}{2} |x|^2)_{ij} = \lambda g_{ij} - \sum_\alpha H^\alpha h_{ij}^\alpha.$$ 

By the similar argument as previous sections, we have

Theorem 4.2 Let $X : M \rightarrow R^{n+p}$ be an $n$-dimensional compact self-shrinker. Suppose that there exists a constant $K \in \mathbb{R}$ such that

$$\text{Ric} + \nabla^2 \phi \geq Kg,$$

where

$$K = \lambda - K_0, \quad K_0 = \max_{1 \leq i \leq n} \left\{ \sum_{\alpha,k} h_{ik}^\alpha h_{ki}^\alpha \right\}.$$ 

Then the first non-zero eigenvalue $\lambda_1$ of the Witten-Laplacian $\Delta_\phi$ satisfies

$$\lambda_1 \geq \sup_{s \in (0,1)} \left\{ 4s(1-s) \frac{\pi^2}{d^2} + sK \right\},$$

where $d$ is the diameter of $M$.

Following the arguments of previous sections, we have

$$2\lambda \geq 4s(1-s) \frac{\pi^2}{d^2} + sK$$

for all $s \in (0,1)$.

Thus we obtain a diameter estimate for compact self-shrinker, which are not minimal submanifold of $S^{n+p-1}(\sqrt{n/\lambda})$ (which corresponds $|x|^2 = \text{constant}$, so from (15) we get $\phi = 0$, trivial case). Choosing $s = \frac{1}{2}$ in (16), we have for $n$-dimensional self-shrinkers.
Theorem 4.3 Let $x : M \to R^{n+p}$ be an $n$-dimensional compact self-shrinker such that $x(M)$ is not minimal submanifold in $S^{n+p-1}(\sqrt{n}/\lambda)$, and let $h_{ij}^o$ be the components of the second fundamental form of $M$. Then we have

$$d \geq \frac{1}{\sqrt{\frac{3\lambda}{2} + \frac{1}{2} K_0}} \pi,$$

where

$$K_0 := \max_{1 \leq i \leq n} \left[ \sum_{\alpha, k} h_{ik}^o h_{ki}^o \right].$$

When $p = 1$, we have

Corollary 4.4 Let $x : M \to R^{n+1}$ be an $n$-dimensional compact self-shrinker such that $x(M)$ is not $S^n(\sqrt{n}/\lambda)$, and let $\lambda_i$ be the principal curvatures of $M$. Then we have

$$d \geq \frac{1}{\sqrt{\frac{3\lambda}{2} + \frac{1}{2} K_0}} \pi,$$

where

$$K_0 := \max_{p \in M} \max_{1 \leq i \leq n} \lambda_i^2.$$

Remark 4.5 The similar results hold for self-shrinkers in Riemannian cone manifolds as considered in [15].

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References


Akito Futaki, Department of Mathematics, Tokyo Institute of Technology, O-okayama, Meguro, Tokyo, 152-8551, Japan, E-mail: futaki@math.titech.ac.jp

Haizhong Li, Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China, E-mail: hli@math.tsinghua.edu.cn

Xiang-Dong Li, Academy of Mathematics and System Science, Chinese Academy of Sciences, 55, Zhongguancun East Road, Beijing, 100190, P. R. China, E-mail: xdli@amt.ac.cn