INTERSECTIONS OF HIRZEBRUCH-ZAGIER DIVISORS AND CM CYCLES

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Abstract. We study relations between Fourier coefficients of automorphic forms and special cycles on the integral model of a Hilbet modular surface. The main result shows that the Fourier coefficients of a particular Hilbert modular form (the central derivative of an Eisenstein series) agree with the degrees of certain zero cycles on the Hilbert modular surface. As a corollary, the Fourier coefficients of the diagonal restriction of the Hilbert modular form are related to the intersections multiplicities of Hirzebruch-Zagier divisors with a fixed codimension two cycle of CM points. These results are examples of arithmetic Siegel-Weil formulas, in the sense of Kudla.

Contents

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1. Introduction

Let F be a real quadratic field of discriminant d_F and different $\mathfrak{D}_F$, and denote by $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ the nontrivial Galois automorphism of F. Associated to F is a Hilbert modular surface $\mathcal{M}$. The algebraic stack $\mathcal{M}$ is defined as the moduli space of abelian surfaces $\mathbf{A}$ equipped with an action of $\mathcal{O}_F$, and with an $\mathcal{O}_F$-linear principal polarization; see Section 3 for more details. We refer to such $\mathbf{A}$ as *$\mathcal{O}_F$-polarized RM abelian surfaces*. It is known that $\mathcal{M}$ is regular, flat over $\mathrm{Spec}(\mathbb{Z})$ of relative dimension two, and smooth over $\mathrm{Spec}(\mathbb{Z}[1/d_F])$.

For every positive integer m, Hirzebruch and Zagier constructed a divisor on the complex fiber $\mathcal{M}(\mathbb{C})$, and in [24] Kudla and Rapoport gave a moduli-theoretic description of this divisor. Define $\mathcal{T}(m)$ to be the the moduli space of pairs $(\mathbf{A}, j)$ in which $\mathbf{A}$ is an $\mathcal{O}_F$-polarized RM abelian surface, and j is a Rosati fixed endomorphism satisfying $j \circ j = m$ and $j \circ x = x^\sigma \circ j$ for all $x \in \mathcal{O}_F$. The morphism $\mathcal{T}(m) \to \mathcal{M}$ defined by "forget j" is finite and unramified, and the image is a codimension one cycle whose complex fiber equal to the divisor constructed by Hirzebruch and Zagier. As the subring of endomorphisms of $\mathbf{A}$ generated by $\mathcal{O}_F$ and j is an order in an indefinite quaternion algebra, the stacks $\mathcal{T}(m)$ are essentially integral models of quaternionic Shimura curves. The *Hirzebruch-Zagier divisors* $\mathcal{T}(m)$ are studied by Kudla and Rapoport in [24], and by Terstiege in [45, 44]. In those papers the goal, following the conjectures of [23], is to relate the intersection multiplicity of three Hirzebruch-Zagier divisors on $\mathcal{M}$ to the Fourier coefficients of the central derivative of a Siegel Eisenstein series of genus three.

On the other hand, the second author proved a formula in [49, 50], first conjectured by Bruinier and Yang [5], which, under technical conditions, relates the intersection multiplicity of $\mathcal{T}(m)$ with a fixed codimension two cycle of complex multiplication points on $\mathcal{M}$, to the diagonal restriction of the central derivative of a Hilbert modular Eisenstein series of weight one. Our Theorem E is a generalization of this result. The central problem of the current work is to find an arithmetic interpretation of the Hilbert modular Eisenstein series itself, *before* one takes the diagonal restriction. This is Theorem C below, from which Theorem E follows easily.

Let E be a quartic CM field with real quadratic subfield F. There are three mutually exclusive possibilities:

- (**cyclic**) $E/\mathbb{Q}$ is a Galois field extension and $\mathrm{Gal}(E/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$,
- (**biquad**) $E \cong E_1 \otimes_{\mathbb{Q}} E_2$ for quadratic imaginary fields $E_1 \not\cong E_2$,
- (**nongal**) $E/\mathbb{Q}$ is a non-Galois field extension,

and in all cases we denote by $a \mapsto \overline{a}$ the complex conjugation on E. Given a CM type Σ of E, let E_Σ be the reflex field of Σ, and let $\mathcal{O}_\Sigma$ be the ring of integers of E_Σ. There is a quartic $\mathbb{Q}$-algebra $E^\sharp$, the *reflex algebra* of E, characterized up to isomorphism by the existence of an $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-equivarient bijection

$$\{\text{CM types of } E\} \cong \mathrm{Hom}(E^\sharp, \mathbb{C}),$$

denoted $\Sigma \mapsto \phi_\Sigma$ and satisfying $\phi_\Sigma(E^\sharp) = E_\Sigma$. In cases (**cyclic**) and (**nongal**) $E^\sharp$ is a quartic CM field; in case (**biquad**) $E^\sharp \cong E_1 \times E_2$. In all cases we denote by $F^\sharp$ the maximal totally real subalgebra of $E^\sharp$, so that $F^\sharp$ is a real quadratic field in cases (**cyclic**) and (**nongal**), and $F^\sharp \cong \mathbb{Q} \times \mathbb{Q}$ in case (**biquad**).

To the data (E, Σ) we attach an algebraic stack $\mathcal{CM}_\Sigma$, étale and proper over $\mathrm{Spec}(\mathcal{O}_\Sigma)$. This stack is defined as the moduli space of principally polarized abelian surfaces over $\mathcal{O}_\Sigma$-schemes with complex multiplication by $\mathcal{O}_E$, and satisfying the Σ-Kottwitz condition of Section 3.2. The obvious forgetful morphism $\mathcal{CM}_\Sigma \to \mathcal{M}_{/\mathcal{O}_\Sigma}$ is finite and unramified, and its image is a codimension two cycle on $\mathcal{M}_{/\mathcal{O}_\Sigma}$.

In Section 2.3 we define an *$\mathcal{O}_F$-polarized CM module* $\mathbf{T}$ to be a projective $\mathcal{O}_E$-module of rank one equipped with a perfect $\mathbb{Z}$-valued symplectic form (suitably compatible with the $\mathcal{O}_E$-action). To each such $\mathbf{T}$ there is an associated CM type, and we denote by X_Σ the finite set of isomorphism classes of $\mathcal{O}_F$-polarized CM modules with CM type Σ. Taking the first homology of an $\mathcal{O}_F$-polarized CM abelian surface over $\mathbb{C}$ defines a bijection $\mathcal{CM}_\Sigma(\mathbb{C}) \to X_\Sigma$, but we prefer to think of elements of X_Σ as purely linear algebraic (as opposed to algebro-geometric) objects. In Section 4.5 we attach to each $\mathbf{T} \in X_\Sigma$ an *incoherent* quadratic space $\mathcal{C}(\mathbf{T})$ of rank two over the adele ring $\mathbb{A}_{F^\sharp}$. The incoherence condition means that $\mathcal{C}(\mathbf{T})$ does not arise as the adelization of any quadratic space over $F^\sharp$. Using the Weil representation, we then associate to $\mathcal{C}(\mathbf{T})$ an *incoherent* Hilbert modular Eisenstein series $E(\tau, s, \mathbf{T})$ of parallel weight one for the group $\mathrm{GL}_2(\mathbb{A}_{F^\sharp})$. Here $\tau = \mathbf{u} + i\mathbf{v}$ is an element of the $F^\sharp$ upper half-plane

$$\mathcal{H}_{F^\sharp} = \{\mathbf{u} + i\mathbf{v} : \mathbf{u}, \mathbf{v} \in F^\sharp_{\mathbb{R}}, \mathbf{v} \gg 0\} \subset F^\sharp_{\mathbb{C}}.$$

A choice of isomorphism $F^\sharp_{\mathbb{R}} \cong \mathbb{R}\times\mathbb{R}$ identifies $\mathcal{H}_{F^\sharp} \cong \mathcal{H}\times\mathcal{H}$ with a product of two complex upper half-planes. Define

$$E(\tau, s, \Sigma) = \sum_{\mathbf{T}\in X_\Sigma} E(\tau, s, \mathbf{T}).$$

The incoherence condition implies that $E(\tau, s, \Sigma)$ vanishes at $s = 0$, and the derivative at $s = 0$ is a nonholomorphic Hilbert modular form on $\mathcal{H}_{F^\sharp}$ with a Fourier expansion

$$E'(\tau, 0, \Sigma) = \sum_{\alpha\in F^\sharp} c_\Sigma(\alpha, \mathbf{v}) \cdot q^\alpha,$$

in which $e(x) = e^{2\pi i x}$ and

$$q^\alpha = e(\mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha\tau)).$$

It is worth remarking that in cases (**cyclic**) and (**nongal**) the Eisenstein series $E(\tau, s, \Sigma)$ is independent of the CM type Σ. This is not obvious from the definitions, and is a consequence of the proof of Theorem 5.3.3.

Roughly speaking, the main results of [50, 49] relate the intersection multiplicity of $\mathcal{T}(m)_{/\mathcal{O}_\Sigma}$ and $\mathcal{CM}_\Sigma$ with the Fourier coefficients $c_\Sigma(\alpha, \mathbf{v})$, under some restrictive hypotheses on the extension $E/\mathbb{Q}$. These hypotheses exclude cases (**biquad**) and (**cyclic**), and imply that $E(\tau, s, \mathbf{T})$ is independent of $\mathbf{T}$. The results assert that the intersection multiplicity of $\mathcal{T}(m)_{/\mathcal{O}_\Sigma}$ and $\mathcal{CM}_\Sigma$ is equal to (up to a simple explicit constant, see Theorem D)

$$\sum_{\substack{\alpha\in F^\sharp,\ \alpha\gg 0\\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha)=m}} c_\Sigma(\alpha, \mathbf{v}).$$

The formula suggests that one should look for a decomposition of the scheme theoretic intersection

$$\mathcal{T}(m) \cap \mathcal{CM}_\Sigma = \mathcal{T}(m)_{/\mathcal{O}_\Sigma} \times_{\mathcal{M}_{/\mathcal{O}_\Sigma}} \mathcal{CM}_\Sigma,$$

into a disjoint union zero cycles indexed by totally positive $\alpha \in F^\sharp$ of trace m, in such a way that the arithmetic degree of the α^{th} zero cycle is essentially $c_\Sigma(\alpha, \mathbf{v})$.

To find such a decomposition, first reconsider the definition of $\mathcal{T}(m)$. If S is a connected scheme and $\mathbf{A} \in \mathcal{M}(S)$, let $L(\mathbf{A})$ be the space of *special endomorphisms* of $\mathbf{A}$ in the sense of Section 3.1. Thus $L(\mathbf{A})$ consists of those endomorphisms j of $\mathbf{A}$ that are fixed by the Rosati involution and satisfy $j \circ x = x^\sigma \circ j$ for all $x \in \mathcal{O}_F$. The $\mathbb{Z}$-modules $L(\mathbf{A})$ is free of rank at most four, and carries a positive definite quadratic form $Q_{\mathbf{A}}(j) = j \circ j$. Thus an S-valued point of $\mathcal{T}(m)$ consists of a pair $(\mathbf{A}, j)$ with $\mathbf{A} \in \mathcal{M}(S)$, and $j \in L(\mathbf{A})$ satisfying $Q_{\mathbf{A}}(j) = m$. Now suppose S is a connected $\mathcal{O}_\Sigma$-scheme, and we are given an S-valued point of $\mathcal{T}(m) \cap \mathcal{CM}_\Sigma$. Such a point consists of a pair $(\mathbf{A}, j)$ as above, but now $\mathbf{A}$ has complex multiplication by $\mathcal{O}_E$. We will see in Section 3.2 that this complex multiplication endows

$V(\mathbf{A}) = L(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the structure of an $E^\sharp$-module, and that there is a unique totally positive definite $F^\sharp$-quadratic form $Q^\sharp_{\mathbf{A}}$ on $V(\mathbf{A})$ satisfying

$$Q_{\mathbf{A}} = \mathrm{Tr}_{F^\sharp/\mathbb{Q}} \circ Q^\sharp_{\mathbf{A}}.$$

It follows that there is a decomposition

$$\mathcal{T}(m) \cap \mathcal{CM}_\Sigma = \bigsqcup_{\substack{\alpha \in F^\sharp \\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha) = m}} \mathcal{CM}_\Sigma(\alpha)$$

in which $\mathcal{CM}_\Sigma(\alpha)$ is the moduli space of pairs $(\mathbf{A}, j)$ over $\mathcal{O}_\Sigma$-schemes, with $\mathbf{A}$ an $\mathcal{O}_F$-polarized CM abelian surface satisfying the Σ-Kottwitz condition, and $j \in L(\mathbf{A})$ satisfying $Q^\sharp_{\mathbf{A}}(j) = \alpha$.

The following result will be proved in Section 3.3.

Theorem A. *If $\alpha \in F^\sharp$ is totally positive then $\mathcal{CM}_\Sigma(\alpha)$ has dimension zero. Furthermore, all geometric points have the same nonzero residue characteristic, and represent supersingular abelian surfaces.*

Suppose we are in case (**cyclic**) or (**nongal**), so that $F^\sharp$ is a field. If $\alpha \in F^\sharp$ has trace $m > 0$ then certainly $\alpha \in (F^\sharp)^\times$. If such an α is not totally positive then $\mathcal{CM}_\Sigma(\alpha) = \emptyset$, as $Q^\sharp_{\mathbf{A}}$ is totally positive definite. Thus Theorem A implies that $\mathcal{T}(m) \cap \mathcal{CM}_\Sigma$ is zero dimensional, and $\mathcal{T}(m)_{/\mathcal{O}_\Sigma}$ and $\mathcal{CM}_\Sigma$ intersect properly on $\mathcal{M}_{/\mathcal{O}_\Sigma}$. In case (**biquad**) this argument breaks down. In this situation $F^\sharp \cong \mathbb{Q} \times \mathbb{Q}$, and so there are $\alpha \in F^\sharp$ of trace $m > 0$ with $\alpha \notin (F^\sharp)^\times$. For such an α the stack $\mathcal{CM}_\Sigma(\alpha)$ may have irreducible components of dimension one, which implies that $\mathcal{CM}_\Sigma$ has irreducible components contained in $\mathcal{T}(m)_{/\mathcal{O}_\Sigma}$.

For each totally positive $\alpha \in F^\sharp$ define the arithmetic degree

$$\widehat{\deg}\, \mathcal{CM}_\Sigma(\alpha) = \sum_{\mathfrak{q}} \log(\mathrm{Nm}(\mathfrak{q})) \sum_{z \in \mathcal{CM}_\Sigma(\alpha)(\mathbb{F}^{\mathrm{alg}}_{\mathfrak{q}})} \frac{\mathrm{length}\big(\mathcal{O}^{\mathrm{sh}}_{\mathcal{CM}_\Sigma(\alpha),z}\big)}{\#\mathrm{Aut}(z)}$$

where the outer sum is over all primes $\mathfrak{q}$ of $\mathcal{O}_\Sigma$, $\mathcal{O}^{\mathrm{sh}}_{\mathcal{CM}_\Sigma(\alpha),z}$ is the strictly Henselian local ring at z (*i.e.* the local ring for the étale topology), and the automorphism group $\mathrm{Aut}(z)$ is computed in the category $\mathcal{CM}_\Sigma(\alpha)(\mathbb{F}^{\mathrm{alg}}_{\mathfrak{q}})$. For each prime p define a subfield of $\mathbb{Q}_p^{\mathrm{alg}}$

$$\mathbb{E}_p = \mathbb{Q}_p\big(\{\pi(x) : x \in E \text{ and } \pi \in \mathrm{Hom}(E, \mathbb{Q}_p^{\mathrm{alg}})\}\big).$$

Thus $\mathbb{E}_p$ is the smallest extension of $\mathbb{Q}_p$ containing the image of every $\mathbb{Q}$-algebra map $\pi : E \to \mathbb{Q}_p^{\mathrm{alg}}$.

Hypothesis B. *Consider the following conditions on a prime p.*

1. *The degree of the extension $\mathbb{E}_p/\mathbb{Q}_p$ is less than or equal to 4.*
2. *The ramification degree of $\mathbb{E}_p/\mathbb{Q}_p$ is strictly less than p.*

Hypothesis B is fairly mild. For example it holds if p is unramified in E, or if $p \geq 5$ and $E/\mathbb{Q}$ is Galois.

Theorem C. *Suppose $\alpha \in F^\sharp$ is totally positive, and that $\mathcal{CM}_\Sigma(\alpha)$ is supported in characteristic p for a prime satisfying both conditions of Hypothesis B. Then*

$$\widehat{\deg}\ \mathcal{CM}_\Sigma(\alpha) = -\frac{1}{W_E} \cdot c_\Sigma(\alpha, \mathbf{v})$$

where W_E is the number of roots of unity in E. In particular, the right hand side is independent of $\mathbf{v} \in (F^\sharp_\mathbb{R})^{\gg 0}$.

Theorem C is stated in the text as Theorem 5.1.1, and is proved in Section 5.2. The proof relies heavily on local deformation theory calculations postponed until Section 6. The second condition of Hypothesis B arises from the use of crystalline deformation theory in the proof of Proposition 6.2.3, and would require new ideas to remove from the hypotheses of Theorem C. The first condition of Hypothesis B could probably be removed, but doing so would require adding new cases to the already lengthy local calculations in Section 6; see Remark 5.2.4.

The finite intersection multiplicity $\langle \mathcal{T}(m) : \mathcal{CM}_\Sigma \rangle_{\mathrm{fin}}$ is defined in Section 5.3, and is essentially the sum of the lengths of all local rings of $\mathcal{T}(m) \cap \mathcal{CM}_\Sigma$. Theorem C allows us to relate the finite intersection multiplicity to the Fourier coefficients $c_\Sigma(\alpha, \mathbf{v})$ with $\alpha \gg 0$. To give arithmetic meaning to those coefficients with $\alpha \not\gg 0$, we construct in Section 3.4, following ideas of Kudla [21] and Bruinier [3], a Green function $\mathbf{Gr}(m, \mathbf{v}, \cdot)$ on $\mathcal{M}(\mathbb{C})$ with a logarithmic singularity along $\mathcal{T}(m)(\mathbb{C})$. The Green function depends on the choice of a positive parameter $\mathbf{v} \in \mathbb{R}$, and is defined for all nonzero $m \in \mathbb{Z}$; if $m < 0$ then $\mathcal{T}(m) = \emptyset$ and $\mathbf{Gr}(m, \mathbf{v}, \cdot)$ is a smooth function on $\mathcal{M}(\mathbb{C})$. In (3.4.2) we define $\mathbf{Gr}(m, \mathbf{v}, \mathcal{CM}_\Sigma)$ by summing the values of $\mathrm{Gr}(m, \mathbf{v}, \cdot)$ at the complex points of $\mathcal{CM}_\Sigma$. This sum is finite, provided that we are either in case (**cyclic**) or (**nongal**) so that the complex fibers of $\mathcal{T}(m)$ and $\mathcal{CM}_\Sigma$ are disjoint.

The following theorem is proved in Section 5.3.

Theorem D. *Suppose we are either in case (**cyclic**) or (**nongal**). For any nonzero $m \in \mathbb{Z}$,*

$$\frac{1}{2} \cdot \mathbf{Gr}(m, \mathbf{v}, \mathcal{CM}_\Sigma) = -\frac{1}{W_E} \sum_{\substack{\alpha \in F^\sharp,\ \alpha \not\gg 0 \\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha) = m}} c_\Sigma(\alpha, \mathbf{v}).$$

If Hypothesis B holds for every prime p, then

$$\langle \mathcal{T}(m) : \mathcal{CM}_\Sigma \rangle_{\mathrm{fin}} = -\frac{1}{W_E} \sum_{\substack{\alpha \in F^\sharp,\ \alpha \gg 0 \\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha) = m}} c_\Sigma(\alpha, \mathbf{v}).$$

Let $i_\Delta : \mathcal{H} \to \mathcal{H}_{F^\sharp}$ be the diagonal embedding of the usual complex upper half-plane. The pullback of $E'(\tau, 0, \Sigma)$ to $\mathcal{H}$ is a nonholomorphic modular form of weight two with a Fourier expansion

$$E'(i_\Delta(\tau), 0, \Sigma) = \sum_{m \in \mathbb{Z}} b_\Sigma(m, \mathbf{v}) \cdot q^m$$

(now $\tau = \mathbf{u} + i\mathbf{v} \in \mathcal{H}$ and $q^m = e(m\tau)$ as usual) in which

$$b_\Sigma(m, \mathbf{v}) = \sum_{\substack{\alpha \in F^\sharp \\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha) = m}} c_\Sigma(\alpha, \mathbf{v}).$$

For every nonzero $m \in \mathbb{Z}$ define an arithmetic divisor

$$\widehat{\mathcal{T}}(m, \mathbf{v}) = \big(\mathcal{T}(m), \mathrm{Gr}(m, \mathbf{v}, \cdot)\big)$$

on $\mathcal{M}$. The right hand side is simply the formal pair consiting of $\mathcal{T}(m)$ with its Green function $\mathrm{Gr}(m, \mathbf{v}, \cdot)$. The arithmetic intersection of $\mathcal{CM}_\Sigma$ with $\widehat{\mathcal{T}}(m, \mathbf{v})$ is defined, following [2], by

$$\langle \widehat{\mathcal{T}}(m, \mathbf{v}) : \mathcal{CM}_\Sigma \rangle = \langle \mathcal{T}(m) : \mathcal{CM}_\Sigma \rangle_{\mathrm{fin}} + \frac{1}{2} \cdot \mathrm{Gr}(m, \mathbf{v}, \mathcal{CM}_\Sigma).$$

Theorem D immediately implies the following result

Theorem E. *Suppose that we are either in case (**cyclic**) or (**nongal**), and that Hypothesis B holds for all primes p. For any positive $\mathbf{v} \in \mathbb{R}$ and any nonzero $m \in \mathbb{Z}$,*

$$\langle \widehat{\mathcal{T}}(m, \mathbf{v}) : \mathcal{CM}_\Sigma \rangle = -\frac{1}{W_E} \cdot b_\Sigma(m, \mathbf{v}).$$

For a choice of toroidal compactification $\mathcal{M} \hookrightarrow \mathcal{M}^*$, one would like to view $\widehat{\mathcal{T}}(m, \mathbf{v})$ as an element of the codimension one arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{M}^*_{/\mathcal{O}_\Sigma})$ of Gillet-Soulé [12], or of a modified arithmetic Chow group as in [6, 4], and view $\langle \cdot : \mathcal{CM}_\Sigma \rangle$ as a linear functional on this Chow group. Unfortunately, the behavior of the Green function $\mathbf{Gr}(m, \mathbf{v}, \cdot)$ near the boundary of $\mathcal{M}^*$ is not well understood, unlike that of the Green function $\Psi(m, \cdot)$ for $\mathcal{T}(m)$ constructed by Bruinier and studied in [3, 4]. For this reason, the arithmetic intersection theory of Theorem E takes place on the open Hilbert modular surface $\mathcal{M}$.

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The following notation will be used throughout. If ℓ is a place of $\mathbb{Q}$ and M is a $\mathbb{Z}$-module (or $\mathbb{Z}$-algebra) we abbreviate $M_\ell = M \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$; we also abbreviate $\widehat{M} = M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$. Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and let $\mathbb{A}_f \cong \widehat{\mathbb{Q}}$ be the ring of finite adeles. If L is a number field we set $\mathbb{A}_L = L \otimes_{\mathbb{Q}} \mathbb{A}$. For a $\mathbb{Q}$-module X abbreviate $X_{\mathbb{R}} = X \otimes_{\mathbb{Q}} \mathbb{R}$ and $X_{\mathbb{C}} = X \otimes_{\mathbb{Q}} \mathbb{C}$.

2. Linear algebra

Fix a fractional $\mathcal{O}_F$-ideal $\mathfrak{c} \supset \mathcal{O}_F$. In this section we introduce the linear algebraic notions of $\mathfrak{c}$-polarized RM modules and $\mathfrak{c}$-polarized CM modules, and show that certain spaces of special endomorphisms of these objects carry natural quadratic forms. The modules themselves will reappear in Section 3 as the first homology of abelian surfaces over $\mathbb{C}$ with real and complex multiplication, and the quadratic spaces of special endomorphisms will underlie the the construction of Hilbert modular Eisenstein series in Section 4.5.

2.1. **The reflex algebra.** A *CM type* of E is an unordered pair $\Sigma = \{\pi_1, \pi_2\}$ of $\mathbb{Q}$-algebra homomorphisms $\pi_1, \pi_2 : E \to \mathbb{C}$ whose restrictions to F are related by

$$\pi_1|_F = \pi_2|_F \circ \sigma.$$

By Galois theory, $B \mapsto \mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(B, \mathbb{Q}^{\mathrm{alg}})$ establishes an equivalence between the category of étale $\mathbb{Q}$-algebras and the category of finite sets with a continuous action of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q})$. If we fix an embedding $\mathbb{Q}^{\mathrm{alg}} \to \mathbb{C}$, the set of all CM types of E becomes a $G_{\mathbb{Q}}$-set, and so determines an étale $\mathbb{Q}$-algebra which we call $E^\sharp$. Thus there is a $G_{\mathbb{Q}}$-equivariant bijection $\Sigma \mapsto \phi_\Sigma$

$$\{\text{CM types of } E\} \cong \mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(E^\sharp, \mathbb{C}). \tag{2.1.1}$$

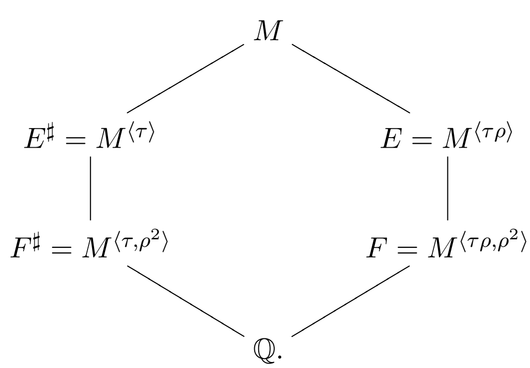
The algebra $E^\sharp$ and the bijection (2.1.1) can be made more explicit as follows. Consider the commutative $\mathbb{Q}$-algebra

$$M = E \otimes_{\mathrm{id}, F, \sigma} E.$$

On the left we view E as an F-algebra via the inclusion $x \mapsto x$ of F into E, and on the right we view E as an F-algebra via the conjugate embedding $x \mapsto x^\sigma$. Thus for any $a, b \in E$ and $x \in F$ we have the relation $(xa) \otimes b = a \otimes (x^\sigma b)$. Define $\mathbb{Q}$-algebra automorphisms $\rho, \tau \in \mathrm{Aut}(M)$ by

$$\rho(a \otimes b) = \overline{b} \otimes a \qquad \tau(a \otimes b) = b \otimes a.$$

Viewing E as a subalgebra of M via the embedding $a \mapsto a \otimes 1$, we define $\mathbb{Q}$-algebras $E^\sharp$ and $F^\sharp$ by



The $\mathbb{Q}$-algebra $E^\sharp$ is the *reflex algebra* of E. The *reflex homomorphism* $\phi_\Sigma : E^\sharp \to \mathbb{C}$ associated to the CM type $\Sigma = \{\pi_1, \pi_2\}$ is defined as the restriction to $E^\sharp$ of the $\mathbb{Q}$-algebra homomorphism $M \to \mathbb{C}$ defined by

$$a \otimes b \mapsto \pi_1(a) \cdot \pi_2(b).$$

The *reflex field* of Σ is $E_\Sigma = \phi_\Sigma(E^\sharp)$, and $\mathcal{O}_\Sigma$ denotes the ring of integers of E_Σ. For a prime $\mathfrak{q}$ of $\mathcal{O}_\Sigma$ let $\mathbb{F}_\mathfrak{q}$ be the residue field of $\mathfrak{q}$.

Let $x \mapsto x^\dagger$ denote the restriction to $E^\sharp$ of the automorphism $a \otimes b \mapsto \overline{a} \otimes \overline{b}$ of M. The subalgebra fixed by $x \mapsto x^\dagger$ is $F^\sharp$.

Lemma 2.1.1.

(1) *In case (**cyclic**) $E^\sharp$ is isomorphic to E, and $x \mapsto x^\dagger$ is complex conjugation.*
(2) *In case (**biquad**) $E^\sharp$ is isomorphic to $E_1 \times E_2$, and $x \mapsto x^\dagger$ is the product of the complex conjugations.*
(3) *In case (**nongal**) $E^\sharp$ is a quartic CM field which is not Galois over $\mathbb{Q}$ and is not isomorphic to E. The automorphism $x \mapsto x^\dagger$ is complex conjugation.*

*In particular in case (**biquad**) $F^\sharp \cong \mathbb{Q} \times \mathbb{Q}$, and in cases (**cyclic**) and (**nongal**) $F^\sharp$ is a real quadratic field.*

Proof. This as an easy exercise in Galois theory, and is left to the reader. □

A *Hermitian form* on an $E^\sharp$-module V is a pairing $\langle\ ,\ \rangle : V \times V \to E^\sharp$ that is $E^\sharp$-linear in the first variable and satisfies $\langle v, w\rangle = \langle w, v\rangle^\dagger$.

2.2. Polarized RM modules.

Definition 2.2.1. An *RM module* is a pair (T, κ_T) in which T is a $\mathbb{Z}$-module, and $\kappa_T : \mathcal{O}_F \to \mathrm{End}_\mathbb{Z}(T)$ is a ring homomorphism making T into a projective $\mathcal{O}_F$-module of rank 2.

The *polarization module* $P(T, \kappa_T)$ is the $\mathcal{O}_F$-module of alternating $\mathbb{Z}$-bilinear forms $\lambda_T : T \times T \to \mathbb{Z}$ satisfying

$$\lambda_T(\kappa_T(x)t_1, t_2) = \lambda_T(t_1, \kappa_T(x)t_2)$$

for every $x \in \mathcal{O}_F$. A $\mathfrak{c}$-*polarization* of (T, κ_T) is a $\lambda_T \in P(T, \kappa_T)$ satisfying

$$\mathfrak{c}T = \{t_1 \in T \otimes_\mathbb{Z} \mathbb{Q} : \lambda_T(t_1, t_2) \in \mathbb{Z} \text{ for all } t_2 \in T\}.$$

The $\mathcal{O}_F$-module $P(T, \kappa_T)$ is projective of rank one. Given a $\mathfrak{c}$-polarized RM module $\mathbf{T} = (T, \kappa_T, \lambda_T)$, let $j \mapsto j^*$ be the involution of $\mathrm{End}_\mathbb{Z}(T) \otimes_\mathbb{Z} \mathbb{Q}$ determined by

$$\lambda_T(jt_1, t_2) = \lambda_T(t_1, j^* t_2).$$

A *special endomorphism* of $\mathbf{T}$ is a $j \in \mathrm{End}_\mathbb{Z}(T)$ satisfying

$$\kappa_T(x) \circ j = j \circ \kappa_T(x^\sigma)$$

for all $x \in \mathcal{O}_F$, and satisfying $j^* = j$. The $\mathbb{Z}$-module of all special endomorphisms of $\mathbf{T}$ is denoted $L(\mathbf{T})$, and we set

$$V(\mathbf{T}) = L(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For a prime ℓ, abbreviate $L_\ell(\mathbf{T}) = L(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ and $V_\ell(\mathbf{T}) = V(\mathbf{T}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

Let $J \mapsto J^\iota$ be the main involution on $M_2(F)$, characterized by $JJ^\iota = \det(J)$, and define a $\mathbb{Q}$-vector space

$$\begin{aligned} W_{M_2(\mathbb{Q})} &= \{J \in M_2(F) : J^\sigma = J^\iota\} \\ &= \left\{ \begin{pmatrix} a & \delta b \\ \delta c & a^\sigma \end{pmatrix} \in M_2(F) : a \in F \text{ and } b, c \in \mathbb{Q} \right\}. \end{aligned}$$

Here $\delta \in F$ is any nonzero element satisfying $\delta^\sigma = -\delta$. The determinant det is a quadratic form on $W_{M_2(\mathbb{Q})}$.

Proposition 2.2.2.

(1) *Up to isomorphism there is a unique* $\mathfrak{c}$*-polarized RM module,* $\mathbf{T}$*.*
(2) *The function* $Q_{\mathbf{T}}(j) = j \circ j$ *defines a quadratic form on* $L(\mathbf{T})$*.*
(3) *There is an isomorphism of* $\mathbb{Q}$*-quadratic spaces*

$$\big(V(\mathbf{T}), Q_{\mathbf{T}}\big) \cong \big(W_{M_2(\mathbb{Q})}, \det\big).$$

(4) *The* $\mathbb{Q}$*-quadratic space* $\big(V(\mathbf{T}), Q_{\mathbf{T}}\big)$ *has rank* 4*, signature* $(2, 2)$*, determinant* d_F*, and Hasse invariant (normalized as in* [27]*)*

$$\mathrm{hasse}(V(\mathbf{T}), Q_{\mathbf{T}}) = \left(\frac{-d_F, -1}{\mathbb{Q}} \right) \in \mathrm{Br}_2(\mathbb{Q}).$$

Here $\mathrm{Br}_2(\mathbb{Q})$ *is the* 2*-torsion subgroup of the Brauer group of* $\mathbb{Q}$*.*

Proof. Let $\mathbf{T}$ be a $\mathfrak{c}$-polarized RM module. The polarization λ_T has the form $\lambda_T = \mathrm{Tr}_{F/\mathbb{Q}} \circ \Lambda_T$ for a unique $\mathcal{O}_F$-symplectic form $\Lambda_T : T \times T \to \mathfrak{D}_F^{-1}$. As T is projective of rank two as an $\mathcal{O}_F$-module we may fix an $\mathcal{O}_F$-linear isomorphism $T \cong \mathcal{O}_F \oplus \mathfrak{a}$ for some fractional $\mathcal{O}_F$-ideal $\mathfrak{a}$ whose image in $\mathrm{Pic}(\mathcal{O}_F)$ is traditionally called the *Steinitz class* of T. Writing elements of $\mathcal{O}_F \oplus \mathfrak{a} \subset F \oplus F$ as column vectors, the fractional ideal $\mathfrak{a}$ and the isomorphism may be chosen in such a way that

$$\Lambda_T(a, b) = {}^t a \cdot \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right) \cdot b.$$

The condition that λ_T is a $\mathfrak{c}$-polarization is then equivalent to $\mathfrak{a} \cdot \mathfrak{c} = \mathfrak{D}_F^{-1}$. This proves the uniqueness of $\mathbf{T}$.

Using the above isomorphism $T \cong \mathcal{O}_F \oplus \mathfrak{a}$ to view elements of T as column vectors, any $j \in V(\mathbf{T})$ can be written uniquely in the form $t \mapsto J \cdot t^\sigma$ for some $J \in M_2(F)$. The condition $j = j^*$ translates to the condition $J^\sigma = J^\iota$, and the rule $j \mapsto J$ establishes a bijection $V(\mathbf{T}) \cong W_{M_2(\mathbb{Q})}$ identifying $Q_{\mathbf{T}}$ with det. All of the remaining claims are now elementary calculations. □

Let Λ_T be the F-symplectic form on $T_{\mathbb{Q}} = T \otimes_{\mathbb{Z}} \mathbb{Q}$ determined by $\lambda_T = \mathrm{Tr}_{F/\mathbb{Q}} \circ \Lambda_T$, and define algebraic groups over $\mathbb{Q}$

$$G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{Sp}(T_{\mathbb{Q}}, \Lambda_T)$$
$$H = \mathrm{SO}(V(\mathbf{T}), Q_{\mathbf{T}}).$$

The group G acts on $V(\mathbf{T})$ through orthogonal transformations by the rule $g \bullet j = g \circ j \circ g^{-1}$, and this defines a homomorphism $G \to H$. In this way one sees that the construction of $V(\mathbf{T})$ from $\mathbf{T}$ gives a concrete way of realizing the exceptional isomorphism of real Lie algebras $\mathfrak{sp}(2) \times \mathfrak{sp}(2) \to \mathfrak{so}(2,2)$. For any choice of $j \in V(\mathbf{T})$ with $Q_{\mathbf{T}}(j) > 0$ the inclusion $H_j \to H$ of the isotropy subgroup of j in H gives a concrete way of realizing the inclusion of real Lie algebras $\mathfrak{so}(1,2) \to \mathfrak{so}(2,2)$. The above exceptional isomorphism will allow us to identify a Hilbert modular surface with an orthogonal Shimura variety. The inclusions $\mathfrak{so}(1,2) \to \mathfrak{so}(2,2)$ for varying j will then have a moduli-theoretic incarnation in the form of a family of special cycles of codimension one, the Hirzebruch-Zagier divisors, on this Shimura variety.

2.3. **Polarized CM modules.**

Definition 2.3.1. A *CM module* is a pair (T, κ_T) in which T is a $\mathbb{Z}$-module and $\kappa_T : \mathcal{O}_E \to \mathrm{End}_{\mathbb{Z}}(T)$ is a ring homomorphism making T into a projective $\mathcal{O}_E$-module of rank 1.

A $\mathfrak{c}$*-polarization* of (T, κ_T) is a $\mathfrak{c}$-polarization λ_T of the underlying RM module. Let $\mathbf{T} = (T, \kappa_T, \lambda_T)$ be a $\mathfrak{c}$-polarized CM module. Elementary linear algebra shows that the $\mathfrak{c}$-polarization λ_T satisfies

$$\lambda_T(\kappa_T(x)t_1, t_2) = \lambda_T(t_1, \kappa_T(\overline{x})t_2)$$

for all $x \in \mathcal{O}_E$. If $\Sigma = \{\pi_1, \pi_2\}$ is a CM type of E then the homomorphism of $\mathbb{Q}$-vector spaces $E \to \mathbb{C} \times \mathbb{C}$ defined by $x \mapsto (\pi_1(x), \pi_2(x))$ extends to an isomorphism of real vector spaces $E_{\mathbb{R}} \cong \mathbb{C} \times \mathbb{C}$. We therefore acquire an action $\kappa_{T,\Sigma}$ of $\mathbb{C} \times \mathbb{C}$ on $T_{\mathbb{R}}$, and in particular the diagonal embedding $\mathbb{C} \to \mathbb{C} \times \mathbb{C} \cong E_{\mathbb{R}}$ makes $T_{\mathbb{R}}$ into a $\mathbb{C}$-vector space. There is a unique choice of CM type Σ for which the Hermitian form on $T_{\mathbb{R}}$

$$H_T(x, y) = \lambda_T(i \cdot x, y) + i\lambda_T(x, y) \tag{2.3.1}$$

(the scalar multiplication $i \cdot x$ of $\mathbb{C}$ on $T_{\mathbb{R}}$ depends on Σ, as just explained) is positive definite.

Definition 2.3.2. Given a $\mathfrak{c}$-polarized CM module $\mathbf{T}$ the *CM type* of $\mathbf{T}$ is the unique CM type $\Sigma = \Sigma(\mathbf{T})$ for which the Hermitian form (2.3.1) has positive definite real part.

Remark 2.3.3. Let $\mathbf{T}$ be a $\mathfrak{c}$-polarized CM module. If we fix an isomorphism of E-modules $E \cong T_{\mathbb{Q}}$, then there is a unique $\omega_{\mathbf{T}} \in E^{\times}$ such that $\overline{\omega}_{\mathbf{T}} = -\omega_{\mathbf{T}}$ and

$$\lambda_T(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}(\omega_{\mathbf{T}} x \overline{y}).$$

If one makes a different choice of isomorphism $E \cong T_{\mathbb{Q}}$ then $\omega_{\mathbf{T}}$ is multiplied by an element of $\mathrm{Nm}_{E/F}(E^{\times})$. The CM type of $\mathbf{T}$ is characterized as the unique CM type for which the induced $\mathbb{C}$-module structure on $E_{\mathbb{R}}$ makes $i \cdot \omega_{\mathbf{T}} \in F_{\mathbb{R}}$ totally positive.

Now fix a $\mathfrak{c}$-polarized CM module $\mathbf{T}$ and recall the $\mathbb{Q}$-quadratic space $\big(V(\mathbf{T}), Q_{\mathbf{T}}\big)$ of Section 2.2 associated to the underlying RM module. We will use the action of $\mathcal{O}_E$ on $\mathbf{T}$ to make $V(\mathbf{T})$ into a Hermitian $E^{\sharp}$-module. First define an action of the $\mathbb{Q}$-algebra M of Section 2.1 on

$$\tilde{V}(\mathbf{T}) = \{j \in \mathrm{End}_{\mathbb{Z}}(T) \otimes_{\mathbb{Z}} \mathbb{Q} : \kappa_T(x) \circ j = j \circ \kappa_T(x^{\sigma}) \text{ for all } x \in \mathcal{O}_F\}$$

by

$$(a \otimes b) \bullet j = \kappa_T(a) \circ j \circ \kappa_T(\overline{b}).$$

The subspace $V(\mathbf{T}) \subset \tilde{V}(\mathbf{T})$ of $*$-fixed endomorphisms is stable under the action of the subalgebra $E^{\sharp} \subset M$, although it is generally false that the $\mathbb{Z}$-lattice $L(\mathbf{T}) \subset V(\mathbf{T})$ is stable under the action of $\mathcal{O}_{E^{\sharp}}$. If $\mathfrak{l}$ is a place of $F^{\sharp}$ abbreviate $V_{\mathfrak{l}}(\mathbf{T}) = V(\mathbf{T}) \otimes_{F^{\sharp}} F^{\sharp}_{\mathfrak{l}}$.

Lemma 2.3.4. *The $\mathbb{Q}$-bilinear form on $V(\mathbf{T})$ defined by*

$$[j_1, j_2]_{\mathbf{T}} = Q_{\mathbf{T}}(j_1 + j_2) - Q_{\mathbf{T}}(j_1) - Q_{\mathbf{T}}(j_2)$$

satisfies $[x \bullet j_1, j_2]_{\mathbf{T}} = [j_1, x^{\dagger} \bullet j_2]_{\mathbf{T}}$ for every $x \in E^{\sharp}$.

Proof. We may assume that $x = a \otimes b + b \otimes a$ for some $a, b \in E$, as elements of this form generate $E^{\sharp}$ as a $\mathbb{Q}$-module. In the interest of simplifying the notation we suppress κ_T, and simply view E as embedded in $\mathrm{End}_{\mathbb{Z}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. The essential point is that $F = \{f \in \mathrm{End}_{\mathcal{O}_F}(T) \otimes_{\mathbb{Z}} \mathbb{Q} : f^* = f\}$. In particular, as $j_1 \circ \overline{b} \circ j_2 + j_2 \circ b \circ j_1$ is both $*$-fixed and F-linear it belongs to F, and so commutes with a. Thus

$$a \circ j_1 \circ \overline{b} \circ j_2 - j_1 \circ \overline{b} \circ j_2 \circ a = j_2 \circ b \circ j_1 \circ a - a \circ j_2 \circ b \circ j_1$$

and similar reasoning shows that

$$j_2 \circ a \circ j_1 \circ \overline{b} - \overline{b} \circ j_2 \circ a \circ j_1 = \overline{b} \circ j_1 \circ \overline{a} \circ j_2 - j_1 \circ \overline{a} \circ j_2 \circ \overline{b}.$$

Using these relations, direct calculation shows

$$[x \bullet j_1, j_2]_{\mathbf{T}} - [j_1, x^{\dagger} \bullet j_2]_{\mathbf{T}} = 0.$$

□

It follows from Lemma 2.3.4 that there is a unique $E^{\sharp}$-Hermitian form $\langle j_1, j_2 \rangle_{\mathbf{T}}$ on $V(\mathbf{T})$ satisfying

$$[j_1, j_2]_{\mathbf{T}} = \mathrm{Tr}_{E^{\sharp}/\mathbb{Q}} \langle j_1, j_2 \rangle_{\mathbf{T}},$$

and that $Q^{\sharp}_{\mathbf{T}}(j) = \langle j, j \rangle_{\mathbf{T}}$ is the unique $F^{\sharp}$-quadratic form on $V(\mathbf{T})$ satisfying

$$Q_{\mathbf{T}} = \mathrm{Tr}_{F^{\sharp}/\mathbb{Q}} \circ Q^{\sharp}_{\mathbf{T}}.$$

For any CM type Σ of E the restriction of ϕ_{Σ} to $F^{\sharp}$ is an archimedean place of $F^{\sharp}$ denoted ∞_{Σ}^{-}. Let ∞_{Σ}^{+} be the other archimedean place of $F^{\sharp}$.

Proposition 2.3.5. *Suppose* $\mathbf{T}$ *has CM type* Σ. *The* $F^\sharp$*-quadratic space* $(V(\mathbf{T}), Q^\sharp_{\mathbf{T}})$ *has signature* $(2,0)$ *at* ∞^+_Σ, *and has signature* $(0,2)$ *at* ∞^-_Σ.

Proof. Abbreviate $\infty^\pm = \infty^\pm_\Sigma$. Let $\Sigma = \{\pi_1, \pi_2\}$ be the CM type of $\mathbf{T}$, and identify $E_\mathbb{R} \cong \mathbb{C} \times \mathbb{C}$ using the isomorphism $z \mapsto (\pi_1(z), \pi_2(z))$. This makes $T_\mathbb{R}$ into a $\mathbb{C} \times \mathbb{C}$-module, and the idempotents $e_1, e_2 \in F_\mathbb{R}$ induce a decomposition

$$T_\mathbb{R} \cong T_1 \oplus T_2$$

in which each T_k is a one-dimensional $\mathbb{C}$-vector space on which E acts through $\pi_k : E \to \mathbb{C}$. Each T_k comes with an $\mathbb{R}$-symplectic form λ_k (the restriction of λ_T to T_k) for which $x \mapsto \lambda_k(ix, x)$ is positive definite. For any $f \in \mathrm{Hom}_\mathbb{R}(T_1, T_2)$ define $f^\vee \in \mathrm{Hom}_\mathbb{R}(T_2, T_1)$ by the relation

$$\lambda_1(t_1, f^\vee(t_2)) = \lambda_2(f(t_1), t_2)$$

for all $t_k \in T_k$. Using the relation $e_1^\sigma = e_2$, we see that $j \mapsto (j|_{T_1}, j|_{T_2})$ defines an injection

$$V(\mathbf{T})_\mathbb{R} \to \mathrm{Hom}_\mathbb{R}(T_1, T_2) \times \mathrm{Hom}_\mathbb{R}(T_2, T_1),$$

whose image is the space of pairs $(f, f^\vee)$. The quadratic form on $Q_\mathbf{T}$ is identified with $f^\vee \circ f$. In particular restriction to T_1 defines an isomorphism

$$\begin{aligned} V(\mathbf{T})_\mathbb{R} &\cong \mathrm{Hom}_\mathbb{R}(T_1, T_2) \\ &\cong \mathrm{Hom}_\mathbb{C}(T_1, T_2) \oplus \mathrm{Hom}_{\overline{\mathbb{C}}}(T_1, T_2), \end{aligned}$$

where the two spaces in the direct sum are the spaces of $\mathbb{C}$-linear and $\mathbb{C}$-conjugate-linear maps. Tracing through these isomorphisms, one sees that the action of $E^\sharp$ is through the reflex homomorphism

$$\phi_{\{\overline{\pi}_1, \pi_2\}} : E^\sharp \to \mathbb{C}$$

on the first summand and through the reflex homomorphism

$$\phi_{\{\pi_1, \pi_2\}} : E^\sharp \to \mathbb{C}$$

on the second summand. The first of these reflex homomorphisms restricts to the place ∞^+ of $F^\sharp$, while the second restricts to the place ∞^-. In other words

$$V(\mathbf{T}) \otimes_{F^\sharp, \infty^+} \mathbb{R} \cong \mathrm{Hom}_\mathbb{C}(T_1, T_2) \tag{2.3.2}$$

$$V(\mathbf{T}) \otimes_{F^\sharp, \infty^-} \mathbb{R} \cong \mathrm{Hom}_{\overline{\mathbb{C}}}(T_1, T_2). \tag{2.3.3}$$

Fix isomorphisms of $\mathbb{C}$-vector spaces $T_1 \cong \mathbb{C} \cong T_2$ in such a way that the $\mathbb{R}$-symplectic forms λ_1 and λ_2 are each identified with the form

$$\lambda_k(x, y) = -\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(ix\overline{y})$$

(this is possible because $\lambda_k(ix, x)$ is positive definite). Every

$$f \in \mathrm{Hom}_\mathbb{C}(T_1, T_2) \cong \mathrm{Hom}_\mathbb{C}(\mathbb{C}, \mathbb{C})$$

then has the form $f(t_1) = z \cdot t_1$ for some $z \in \mathbb{C}$, and $f^\vee(t_1) = \overline{z} \cdot t_2$. Thus $f^\vee \circ f = z\overline{z}$ proving that (2.3.2) is a positive definite $\mathbb{R}$-quadratic space of rank 2. Similarly every

$$f \in \mathrm{Hom}_{\overline{\mathbb{C}}}(T_1, T_2) \cong \mathrm{Hom}_{\overline{\mathbb{C}}}(\mathbb{C}, \mathbb{C})$$

then has the form $f(t_1) = z \cdot \overline{t}_1$ for some $z \in \mathbb{C}$, and $f^\vee(t_1) = -\overline{z} \cdot \overline{t}_2$. Thus $f^\vee \circ f = -z\overline{z}$ proving that (2.3.3) is negative definite of rank 2. $\square$

Propositions 2.2.2 and 2.3.5 imply that $V(\mathbf{T})$ is free of rank one over $E^\sharp$, and that the $E^\sharp$-Hermitian form $\langle \cdot, \cdot \rangle_{\mathbf{T}}$ on $V(\mathbf{T})$ is nondegenerate. It follows that there is an $E^\sharp$-linear isomorphism of $F^\sharp$-quadratic spaces

$$\big(V(\mathbf{T}), Q^\sharp_{\mathbf{T}}\big) \cong \big(E^\sharp, \beta(\mathbf{T}) x x^\dagger\big), \tag{2.3.4}$$

for some $\beta(\mathbf{T}) \in (F^\sharp)^\times$.

The importance of the $F^\sharp$-quadratic space structure on the space $V(\mathbf{T})$ may be understood by considering the algebraic group over $\mathbb{Q}$

$$H^\sharp = \mathrm{Res}_{F^\sharp/\mathbb{Q}} \mathrm{SO}(V(\mathbf{T}), Q^\sharp_{\mathbf{T}}).$$

This group is naturally a subgroup of

$$H = \mathrm{SO}(V(\mathbf{T}), Q_{\mathbf{T}}),$$

and the inclusion $H^\sharp \to H$ gives a concrete way of realizing the inclusion of real Lie algebras $\mathfrak{so}(2) \times \mathfrak{so}(2) \to \mathfrak{so}(2,2)$. In the discussion of moduli problems in Section 3, this inclusion will have a moduli-theoretic incarnation in the form of a codimension two cycle on a Hilbert modular surface: the cycle of points with complex multiplication by $\mathcal{O}_E$.

2.4. **Algebraic groups and class groups.** In this subsection we construct generalized class groups

$$C_0(E) \subset C_+(E) \subset C(E)$$

that act on the set of all $\mathfrak{c}$-polarized CM modules, and algebraic groups S_E and T_E that act on the space of special endomorphisms of a $\mathfrak{c}$-polarized CM module. Let S_E be the algebraic group over $\mathbb{Q}$ whose functor of points is

$$S_E(A) = \{x \in (E^\sharp \otimes_{\mathbb{Q}} A)^\times : x x^\dagger = 1\}$$

for any $\mathbb{Q}$-algebra A. Let T_E be the algebraic group over $\mathbb{Q}$ with functor of points

$$T_E(A) = \{x \in (E \otimes_{\mathbb{Q}} A)^\times : x\overline{x} \in A^\times\}.$$

Let $\mathbb{G}_m$ be the multiplicative over $\mathbb{Q}$, and view $\mathbb{G}_m$ as a subgroup of T_E using the inclusion $A^\times \to (E \otimes_{\mathbb{Q}} A)^\times$. There is a natural group homomorphism $E^\times \to (E^\sharp)^\times$ defined by $x \mapsto x \otimes x$. This homomorphism may be modified, as in the following lemma, to yield a homomorphism of algebraic groups $T_E \to S_E$.

Lemma 2.4.1. *Define a homomorphism* $\nu_E : T_E \to S_E$ *by*

$$\nu_E(x) = \frac{x \otimes x}{x\overline{x}}.$$

If k *is a field of characteristic* 0*, or* $k = \mathbb{A}$*, or* $k = \mathbb{A}_f$*, then the sequence*

$$1 \to \mathbb{G}_m(k) \to T_E(k) \xrightarrow{\nu_E} S_E(k) \to 1$$

is exact.

Proof. See the proof of [18, Proposition 2.13] □

For every prime $\ell < \infty$ define a compact open subgroup $U_E = \prod_\ell U_{E,\ell}$ of $T_E(\mathbb{A}_f)$ by

$$U_E = T_E(\mathbb{A}_f) \cap \widehat{\mathcal{O}}_E^\times.$$

The map $\nu_E : T_E \to S_E$ of Lemma 2.4.1 induces an isomorphism

$$T_E(\mathbb{Q})\backslash T_E(\mathbb{A}_f)/U_E \cong S_E(\mathbb{Q})\backslash S_E(\mathbb{A}_f)/\nu_E(U_E).$$

Let $I(E)$ be the set of all pairs $\mathbf{Z} = (\mathfrak{z}, \zeta)$ in which $\mathfrak{z}$ is a fractional ideal of $\mathcal{O}_E$ and $\zeta \in F^\times$ satisfies $\mathfrak{z}\overline{\mathfrak{z}} = \zeta\mathcal{O}_E$. Then $I(E)$ is a group under componentwise multiplication, and $P(E) = \{(z\mathcal{O}_E, z\overline{z}) : z \in E^\times\}$ is a subgroup. Define a generalized class group

$$C(E) = I(E)/P(E)$$

and let $C_+(E) \subset C(E)$ be the subgroup consisting of those $(\mathfrak{z}, \zeta)$ for which ζ is totally positive. The function $(\mathfrak{z}, \zeta) \mapsto \mathfrak{z}$ defines a homomorphism $C(E) \to \mathrm{Pic}(\mathcal{O}_E)$ with finite kernel, and so $C(E)$ is finite. Given a $t \in T_E(\mathbb{A}_f)$ let ζ be the unique positive rational number that satisfies $\zeta\widehat{\mathbb{Z}} = (t\bar{t})\widehat{\mathbb{Z}}$, and let $\mathfrak{z}$ be the fractional $\mathcal{O}_E$-ideal defined by $\mathfrak{z}\widehat{\mathcal{O}}_E = t\widehat{\mathcal{O}}_E$. Then $t \mapsto (\mathfrak{z}, \zeta)$ determines an injective homomorphism

$$T_E(\mathbb{Q})\backslash T_E(\mathbb{A}_f)/U_E \to C_+(E) \tag{2.4.1}$$

whose image is denoted $C_0(E) \subset C_+(E)$.

Let $\mathbf{T} = (T, \kappa_T, \lambda_T)$ be a $\mathfrak{c}$-polarized CM module. Given a pair $\mathbf{Z} = (\mathfrak{z}, \zeta) \in I(E)$ define a new $\mathfrak{c}$-polarized CM module

$$(T, \kappa_T, \lambda_T) \otimes \mathbf{Z} = (S, \kappa_S, \lambda_S)$$

as follows. The underlying $\mathbb{Z}$-module is $S = T \otimes_{\mathcal{O}_E} \mathfrak{z}$, the action $\kappa_S : \mathcal{O}_E \to \mathrm{End}(S)$ is $\kappa_S(x)(t \otimes z) = t \otimes (xz)$, and λ_S is defined by

$$\lambda_S\big(t_1 \otimes z_1, t_2 \otimes z_2\big) = \lambda_T(\kappa_T(\zeta^{-1} z_1 \overline{z}_2)t_1, t_2).$$

The right hand side makes sense as $\zeta^{-1}z_1\overline{z}_2 \in \mathcal{O}_E$. The construction $\mathbf{T} \mapsto \mathbf{T} \otimes \mathbf{Z}$ defines an action of $C(E)$ on the set of isomorphism classes of $\mathfrak{c}$-polarized CM modules. Using the notation of Remark 2.3.3, a simple calculation shows that $\omega_{\mathbf{T}\otimes\mathbf{Z}} = \zeta^{-1} \cdot \omega_{\mathbf{T}}$ from which it follows that

$$\Sigma(\mathbf{T} \otimes \mathbf{Z}) = \Sigma(\mathbf{T}) \iff \mathbf{Z} \in C_+(E). \tag{2.4.2}$$

Proposition 2.4.2.

(1) *The set X of isomorphism classes of $\mathfrak{c}$-polarized CM modules is a simply transitive $C(E)$-set.*
(2) *The set X_Σ of isomorphism classes of $\mathfrak{c}$-polarized CM modules with a fixed CM type Σ is either empty or is a simply transitive $C_+(E)$-set. If there is a finite prime of F ramified in E then X_Σ is nonempty.*

Proof. First we show that the set of $\mathfrak{c}$-polarized CM modules is nonempty. Let $\mathfrak{A}$ be any fractional $\mathcal{O}_E$-ideal, and fix an $\omega \in E^\times$ such that $\overline{\omega} = -\omega$. Define a $\mathbb{Z}$-bilinear alternating form

$$\lambda(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}(\omega x \overline{y})$$

on $\mathfrak{A}$. If $\kappa : \mathcal{O}_E \to \mathrm{End}_{\mathbb{Z}}(\mathfrak{A})$ is the natural action, the triple $(\mathfrak{A}, \kappa, \lambda)$ is a $\mathfrak{b}$-polarized CM module, where

$$\mathfrak{b}^{-1} = \omega \mathfrak{A} \overline{\mathfrak{A}} \mathfrak{D}_{E/\mathbb{Q}}.$$

The Hilbert class field of F is linearly disjoint from E (as E is ramified at the archimedean places), and so class field theory implies that the norm map from the ideal class group of E to the ideal class group of F is surjective. Therefore we may factor $\mathfrak{c}\mathfrak{b}^{-1} = y\mathfrak{Y}\overline{\mathfrak{Y}}$ for some $y \in F^\times$ and some fractional $\mathcal{O}_E$-ideal $\mathfrak{Y}$. If ω is replaced by $y\omega$ and $\mathfrak{A}$ is replaced by $\mathfrak{Y}\mathfrak{A}$, then $\mathbf{T} = (\mathfrak{A}, \kappa, \lambda)$ is a $\mathfrak{c}$-polarized CM module. In the notation of Remark 2.3.3, $\omega = \omega_{\mathbf{T}}$.

The proof that the action of $C(E)$ on X is simply transitive is a routine exercise, which we leave to the reader. This, together with (2.4.2), implies that X_Σ is either empty or a simply transitively $C_+(E)$-set.

Now use Σ to view $E_{\mathbb{R}}$ as a $\mathbb{C}$-vector space, as in Section 2.3. We may repeat the argument of the first paragraph, but choose the initial the traceless $\omega \in E^\times$ so that $i\omega \in F_{\mathbb{R}}$ is totally positive. If there is at least one finite prime of F that is ramified in E then the narrow Hilbert class field of F is linearly disjoint from E, and class field theory implies that the norm map from the ideal class group of E to the narrow ideal class group of F is surjective. This allows us to choose y to be totally positive, and Remark 2.3.3 then shows that the $\mathbf{T}$ constructed above has CM type Σ. □

The remainder of this subsection is devoted to the proof of the following proposition, which will be a crucial ingredient in the proof of Theorem 5.3.3. For a $\mathfrak{c}$-polarized CM module $\mathbf{T}$ set

$$\widehat{L}(\mathbf{T}) = L(\mathbf{T}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \qquad \widehat{V}(\mathbf{T}) = V(\mathbf{T}) \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}.$$

Proposition 2.4.3. *Assume either (**cyclic**) or (**nongal**). There is a $\mathbf{Z} = (\mathfrak{Z}, \zeta) \in C(E)$ such that $\mathrm{Nm}_{F/\mathbb{Q}}(\zeta) < 0$, and such that for any $\mathfrak{c}$-polarized CM module $\mathbf{T}$ there is an isomorphism of $\widehat{F}^\sharp$-quadratic spaces*

$$\big(\widehat{V}(\mathbf{T}), Q^\sharp_{\mathbf{T}}\big) \cong \big(\widehat{V}(\mathbf{T} \otimes \mathbf{Z}), Q^\sharp_{\mathbf{T}\otimes\mathbf{Z}}\big)$$

identifying $\widehat{L}(\mathbf{T})$ with $\widehat{L}(\mathbf{T} \otimes \mathbf{Z})$. For any such $\mathbf{Z}$ the reflex homomorphisms

$$\phi_{\Sigma(\mathbf{T}\otimes\mathbf{Z})}, \phi_{\Sigma(\mathbf{T})} : E^\sharp \to \mathbb{C}$$

have distinct restrictions to $F^\sharp$ *(equivalently, the CM types* $\Sigma(\mathbf{T}\otimes\mathbf{Z})$ *and* $\Sigma(\mathbf{T})$ *are neither equal nor complex conjugates).*

Before the proof, we need some technical preparation. Letting ∞ denote the archimedean place of $\mathbb{Q}$, define finite groups of exponent 2

$$\mathrm{Gen}_\infty(E/F) = F_\infty^\times/\mathrm{Nm}_{E/F}(E_\infty^\times)$$
$$\mathrm{Gen}_f(E/F) = \widehat{\mathcal{O}}_F^\times/\mathrm{Nm}_{E/F}(\widehat{\mathcal{O}}_E^\times),$$

and the *genus group*

$$\mathrm{Gen}(E/F) = \mathrm{Gen}_\infty(E/F)\times\mathrm{Gen}_f(E/F).$$

The projections to the two factors are denoted $\mathbf{z}\mapsto\mathbf{z}_\infty$ and $\mathbf{z}\mapsto\mathbf{z}_f$. Given $\mathbf{Z}=(\mathfrak{z},\zeta)\in I(E)$ we may choose an idele $z\in\mathbb{A}_E^\times$ such that $z\widehat{\mathcal{O}}_E=\mathfrak{z}\widehat{\mathcal{O}}_E$. Then $\mathrm{gen}(\mathbf{Z})=\zeta^{-1}z\overline{z}$ defines the *genus invariant*

$$\mathrm{gen}: C(E)\to\mathrm{Gen}(E/F).$$

The subgroup $C_+(E)\subset C(E)$ is precisely the kernel of $\mathbf{Z}\mapsto\mathrm{gen}(\mathbf{Z})_\infty$. If $\chi:\mathbb{A}_F^\times\to\{\pm1\}$ denotes the idele class character corresponding to the extension E/F, a brief exercise in class field theory shows that the sequence

$$C(E)\xrightarrow{\mathrm{gen}}\mathrm{Gen}(E/F)\xrightarrow{\chi}\{\pm1\}\to 1 \tag{2.4.3}$$

is exact, where the arrow labeled χ is the composition

$$\mathrm{Gen}(E/F)\to\mathbb{A}_F^\times/\mathrm{Nm}_{E/F}(\mathbb{A}_E^\times)\xrightarrow{\chi}\{\pm1\}.$$

Lemma 2.4.4. *Assuming either (**cyclic**) or (**nongal**), there is a* $\mathbf{Z}\in C(E)$ *and a* $u\in\widehat{\mathbb{Z}}^\times$ *such that* $\mathrm{Nm}_{F/\mathbb{Q}}(\mathbf{z}_\infty)<0$ *and*

$$u^2\cdot\mathrm{Nm}_{F/\mathbb{Q}}(\mathbf{z}_f)\in\mathrm{Nm}_{E/\mathbb{Q}}(\widehat{\mathcal{O}}_E^\times),$$

where $\mathbf{z}=\mathrm{gen}(\mathbf{Z})$*.*

Proof. If we choose a totally negative $\Delta\in F^\times$ such that $E=F(\sqrt{\Delta})$ then our hypothesis that $E/\mathbb{Q}$ is not a biquadratic extension implies

$$\mathrm{Nm}_{F/\mathbb{Q}}(\Delta)\notin(\mathbb{Q}^\times)^2.$$

Let p be any prime such that $\mathrm{ord}_p(\mathrm{Nm}_{F/\mathbb{Q}}(\Delta))$ is odd. Then p is either split or ramified in F, and in either case there is a place v_0 of F above p for which $\mathrm{ord}_{v_0}(\Delta)$ is odd. The place v_0 is necessarily ramified in E, and if w_0 denotes the place of E above v_0 then we may choose a $\mathbf{z}_{v_0}\in\mathcal{O}_{F,v_0}^\times$ that is not a norm from $\mathcal{O}_{E,w_0}^\times$.

If p is split in F then let $v_1\neq v_0$ be the other place above p. Then $\mathrm{ord}_{v_1}(\Delta)$ is even, and class field theory (or a Hilbert symbol calculation) gives the first equality in

$$\mathbb{Z}_p^\times=\mathrm{Nm}_{E_{v_0}/\mathbb{Q}_p}(\mathcal{O}_{E,v_0}^\times)\cdot\mathrm{Nm}_{E_{v_1}/\mathbb{Q}_p}(\mathcal{O}_{E,v_1}^\times)=\mathrm{Nm}_{E_p/\mathbb{Q}_p}(\mathcal{O}_{E,p}^\times).$$

Thus

$$\mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0})\in\mathrm{Nm}_{E_p/\mathbb{Q}_p}(\mathcal{O}_{E,p}^\times).$$

If v is a finite place of F with $v \neq v_0$ then set $\mathbf{z}_v = 1 \in \mathcal{O}_{F,v}^\times$. Now define

$$\mathbf{z}_f = \prod_v \mathbf{z}_v \in \mathrm{Gen}_f(E/F)$$

and set

$$\mathbf{z}_\infty = (1, -1) \in \{\pm 1\} \times \{\pm 1\} \cong \mathrm{Gen}_\infty(E/F).$$

and $\mathbf{z} = (\mathbf{z}_\infty, \mathbf{z}_f) \in \mathrm{Gen}(E/F)$. By construction $\chi(\mathbf{z}) = 1$, and so by the exactness of 2.4.3 there is a $\mathbf{Z} \in C(E)$ such that $\mathrm{gen}(\mathbf{Z}) = \mathbf{z}$. This choice of $\mathbf{Z}$ has the desired properties.

Now assume that p is totally ramified in E. If $E_{w_0}/\mathbb{Q}_p$ is a biquadratic field extension then $\mathrm{Nm}_{F/\mathbb{Q}}(\Delta) \in (\mathbb{Q}_p^\times)^2$, contradicting the choice of p. Thus either $E_{w_0}/\mathbb{Q}_p$ is not Galois, or $E_{w_0}/\mathbb{Q}_p$ is Galois with cyclic Galois group. Assume first that $E_{w_0}/\mathbb{Q}_p$ is Galois with cyclic Galois group. The Artin symbol $[\mathbf{z}_{v_0}; E_{w_0}/F_{v_0}]$ is the nontrivial element of $\mathrm{Gal}(E_{w_0}/F_{v_0})$. By local class field theory the inclusion $\mathrm{Gal}(E_{w_0}/F_{v_0}) \to \mathrm{Gal}(E_{w_0}/\mathbb{Q}_p)$ satisfies

$$[\mathbf{z}_{v_0}; E_{w_0}/F_{v_0}] \mapsto [\mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}); E_{w_0}/\mathbb{Q}_p]$$

and we deduce that the element

$$\mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}) \in \mathbb{Z}_p^\times / \mathrm{Nm}_{E_{w_0}/\mathbb{Q}_p}(\mathcal{O}_{E,w_0}^\times) \cong \mathrm{Gal}(E_{w_0}/\mathbb{Q}_p)$$

has order 2, and hence is a square. Thus for some $u_p \in \mathbb{Z}_p^\times$ we have

$$u_p^2 \cdot \mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}) \in \mathrm{Nm}_{E_p/\mathbb{Q}_p}(\mathcal{O}_{E,p}^\times).$$

We now set $\mathbf{z}_v = 1$ for every finite place $v \neq v_0$ and construct $\mathbf{z}$ and $\mathbf{Z}$ exactly as in the previous paragraph. It remains to treat the case in which $E_{w_0}/\mathbb{Q}_p$ is not Galois. In this case if we set $L = F_{v_0}(\sqrt{\Delta^\sigma})$ then $L \ncong E_{w_0}$, and so class field theory implies

$$F_{v_0}^\times = \mathrm{Nm}_{E_{w_0}/F_{v_0}}(E_{w_0}^\times) \cdot \mathrm{Nm}_{L/F_{v_0}}(L^\times).$$

If we now factor

$$\mathbf{z}_{v_0} = \mathrm{Nm}_{E_{w_0}/F_{v_0}}(a) \cdot \mathrm{Nm}_{L/F_{v_0}}(b)$$

with $a \in E_{w_0}^\times$ and $b \in L^\times$ then

$$\mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}) = \mathrm{Nm}_{E_{w_0}/\mathbb{Q}_p}(a) \cdot \mathrm{Nm}_{L/\mathbb{Q}_p}(b).$$

By construction of L the norm maps $E_{w_0}^\times \to \mathbb{Q}_p^\times$ and $L^\times \to \mathbb{Q}_p^\times$ have the same image, and so

$$\mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}) \in \mathrm{Nm}_{E_p/\mathbb{Q}_p}(E_{w_0}^\times).$$

But $\mathbf{z}_{v_0} \in \mathbb{Z}_p^\times$, and hence

$$\mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}) \in \mathrm{Nm}_{E_p/\mathbb{Q}_p}(\mathcal{O}_{E,p}^\times).$$

The construction of $\mathbf{z}$ and $\mathbf{Z}$ now proceeds as in the previous paragraph. $\square$

Lemma 2.4.5. *Fix a* $\mathbf{Z} \in C(E)$ *and a* $\mathfrak{c}$*-polarized CM module* $\mathbf{T}$*. If we set* $\mathbf{S} = \mathbf{T} \otimes \mathbf{Z}$*, then there is an isomorphism of* $\widehat{F}^\sharp$*-quadratic spaces*

$$\big(\widehat{V}(\mathbf{T}), Q^\sharp_{\mathbf{T}}\big) \cong \big(\widehat{V}(\mathbf{S}), \mathrm{Nm}_{F/\mathbb{Q}}(\mathbf{z}_f) \cdot Q^\sharp_{\mathbf{S}}\big)$$

identifying $\widehat{L}(\mathbf{T})$ *with* $\widehat{L}(\mathbf{S})$*. Here* $\mathbf{z}_f \in \widehat{\mathcal{O}}_F^\times$ *is any representative of the finite part of* $\mathbf{z} = \mathrm{gen}(\mathbf{Z})$*.*

Proof. This is a simple calculation. Fix a representative $(\mathfrak{z}, \zeta) \in I(E)$ of $\mathbf{Z}$ and let $z \in \mathbb{A}_E^\times$ satisfy $z \cdot \widehat{\mathcal{O}}_E = \mathfrak{z}\widehat{\mathcal{O}}_E$. There is an $\widehat{\mathcal{O}}_E$-linear isomorphism $\psi : \widehat{T} \to \widehat{S}$ defined by $\psi(t) = t \otimes z_f$. Given a $j \in \widehat{L}(\mathbf{T})$ one checks directly that

$$\psi_* j = \psi \circ \kappa_T(\mathbf{z}_f^{-1}) \circ j \circ \psi^{-1}$$

defines an element of $\widehat{L}(\mathbf{S})$, and that $j \mapsto \psi_* j$ is the desired isomorphism. □

Proof of Proposition 2.4.3. Let $\mathbf{Z}$ be as in Lemma 2.4.4, and set $\mathbf{S} = \mathbf{T} \otimes \mathbf{Z}$. By Lemma 2.4.5 there is an $r \in \widehat{\mathcal{O}}_E^\times$, and an isomorphism

$$(\widehat{V}(\mathbf{T}), Q^\sharp_{\mathbf{T}}) \cong (\widehat{V}(\mathbf{S}), \mathrm{Nm}_{E/\mathbb{Q}}(r) \cdot Q^\sharp_{\mathbf{S}})$$

identifying the $\widehat{\mathbb{Z}}$-lattices $\widehat{L}(\mathbf{T})$ and $\widehat{L}(\mathbf{S})$. If we set $s = r \otimes r \in (\widehat{E}^\sharp)^\times$ then $\mathrm{Nm}_{E^\sharp/F^\sharp}(s) = \mathrm{Nm}_{E/\mathbb{Q}}(r)$, and

$$s \bullet \widehat{L}(\mathbf{S}) = \kappa_S(r) \circ \widehat{L}(\mathbf{S}) \circ \kappa_S(\overline{r}) = \widehat{L}(\mathbf{S}).$$

Using the relation $Q^\sharp_{\mathbf{S}}(s \bullet x) = \mathrm{Nm}_{E^\sharp/F^\sharp}(s) \cdot Q^\sharp_{\mathbf{S}}(x)$ we see that $x \mapsto s \bullet x$ defines an isomorphism

$$(\widehat{V}(\mathbf{S}), \mathrm{Nm}_{E/\mathbb{Q}}(r) \cdot Q^\sharp_{\mathbf{S}}) \cong (\widehat{V}(\mathbf{S}), Q^\sharp_{\mathbf{S}}),$$

which preserves $\widehat{L}(\mathbf{S})$.

If we represent $\mathbf{Z} \in C(E)$ by a pair $(\mathfrak{z}, \zeta) \in I(E)$ then $\mathrm{Nm}_{F/\mathbb{Q}}(\mathbf{z}_\infty) < 0$ implies that $\zeta \in F^\times$ is neither totally positive nor totally negative. From Remark 2.3.3 and the discussion preceding (2.4.2), it follows that the CM types of $\mathbf{S}$ and $\mathbf{T}$ are neither equal nor complex conjugates. □

3. Moduli spaces of abelian surfaces

Let $\mathfrak{c} \supset \mathcal{O}_F$ be a fractional $\mathcal{O}_F$-ideal. In this section we define $\mathfrak{c}$-polarized RM abelian surfaces and $\mathfrak{c}$-polarized CM abelian surfaces. The moduli space of all $\mathfrak{c}$-polarized RM abelian surfaces is a classical Hilbert modular surface, and the moduli space of all $\mathfrak{c}$-polarized CM abelian surfaces determines a codimension two cycle on the Hilbert modular surface. Useful references for Hilbert modular surfaces include [10], [13], [15], [38], [46], and [48].

Throughout Section 3, "scheme" always means locally Noetherian scheme. We impose these hypotheses because they are imposed in [34], our primary reference for abelian schemes.

3.1. **Abelian surfaces with real multiplication.**

Let S be a connected scheme. For any $x \in \mathcal{O}_F$ define a polynomial

$$c_x(T) = (T - x)(T - x^\sigma) \in \mathbb{Z}[T].$$

Definition 3.1.1. An *RM abelian surface* over S is a pair (A, κ_A) in which A is an abelian scheme over S of relative dimension two, and $\kappa_A : \mathcal{O}_F \to \mathrm{End}(A)$ is an action satisfying the *Kottwitz determinant condition*: every point of S admits an open affine neighborhood $\mathrm{Spec}(R) \to S$ over which

- $\mathrm{Lie}(A_{/R})$ is a free R-module of rank two,
- for all $x \in \mathcal{O}_F$ the characteristic polynomial of $\kappa_A(x)$ acting on $\mathrm{Lie}(A_{/R})$ is $c_x(T) \in R[T]$.

A $\mathfrak{c}$-*polarization* of an RM abelian surface (A, κ_A) over S is an $\mathcal{O}_F$-linear polarization $\lambda_A : A \to A^\vee$ whose kernel is the $\mathfrak{c}^{-1}$-torsion subgroup scheme of A. The condition of $\mathcal{O}_F$-linearity means that $\lambda \circ \kappa_A(t) = \kappa_A(t)^\vee \circ \lambda$ for all $t \in \mathcal{O}_F$.

Fix a $\mathfrak{c}$-polarized RM abelian surface $\mathbf{A} = (A, \kappa_A, \lambda_A)$ over S and let $j \mapsto j^*$ be the Rosati involution on $\mathrm{End}(A) \otimes_\mathbb{Z} \mathbb{Q}$ induced by λ_A. A *special endomorphism* of $\mathbf{A}$ is a $j \in \mathrm{End}(A)$ such that

$$\kappa_A(x) \circ j = j \circ \kappa_A(x^\sigma)$$

for every $x \in \mathcal{O}_F$, and such that $j^* = j$. The $\mathbb{Z}$-module of all special endomorphisms is denoted $L(\mathbf{A})$, and we set

$$V(\mathbf{A}) = L(\mathbf{A}) \otimes_\mathbb{Z} \mathbb{Q}.$$

If ℓ is a rational prime we abbreviate $L_\ell(\mathbf{A}) = L(\mathbf{A}) \otimes_\mathbb{Z} \mathbb{Z}_\ell$ and $V_\ell(\mathbf{A}) = V(\mathbf{A}) \otimes_\mathbb{Q} \mathbb{Q}_\ell$. For any $j \in L(\mathbf{A})$ define $Q_\mathbf{A}(j) = j \circ j$.

Proposition 3.1.2. *If $S = \mathrm{Spec}(\mathbb{C})$ then $\mathrm{rank}_\mathbb{Z} L(\mathbf{A}) \le 2$, and $Q_\mathbf{A}$ is a positive definite $\mathbb{Z}$-valued quadratic form on $L(\mathbf{A})$.*

Proof. Let $\epsilon_1, \epsilon_2 \in F_\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ be the orthogonal idempotents. The polarization λ_A induces an alternating $\mathbb{C}$-bilinear form on the complexified homology $H_1(A(\mathbb{C}), \mathbb{C})$. The natural map

$$L(\mathbf{A}) \otimes_\mathbb{Z} \mathbb{C} \to \mathrm{End}_\mathbb{C}(H_1(A(\mathbb{C}), \mathbb{C}))$$

is injective, and every j in the image satisfies

(1) $j \circ \epsilon_2 = \epsilon_1 \circ j$ and $j \circ \epsilon_1 = \epsilon_2 \circ j$;
(2) $\lambda_A(jx, y) = \lambda_A(x, jy)$;
(3) j preserves the Hodge decomposition
$$H_1(A(\mathbb{C}), \mathbb{C}) \cong \mathrm{Lie}(A^\vee)^* \oplus \mathrm{Lie}(A).$$

The Kottwitz determinant condition implies that each $\epsilon_i\mathrm{Lie}(A)$ has complex dimension one. If we pick a generator $e_i \in \epsilon_i\mathrm{Lie}(A)$ and define $f_i \in \epsilon_i\mathrm{Lie}(A^\vee)^*$ by the relation $\lambda_A(e_i, f_i) = 1$, then the above three conditions imply that the matrix of j with respect to the basis $\{e_1, e_2, f_1, f_2\}$ has the form
$$\begin{pmatrix} 0 & y & & \\ x & 0 & & \\ & & 0 & x \\ & & y & 0 \end{pmatrix}.$$
From this it is clear that $L(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{C}$ has complex dimension at most 2, and that $j \circ j$ is a scalar. It follows that $\mathrm{rank}_{\mathbb{Z}} L(\mathbf{A}) \leq 2$, and that $Q_{\mathbf{A}}(j) \in \mathbb{Q}$. The positive definiteness of $Q_{\mathbf{A}}(j) = j \circ j^*$ follows from the positivity of the Rosati involution [35, Section 21]. $\square$

Let H be a rational quaternion algebra. The quaternion algebra $H_F = H \otimes_{\mathbb{Q}} F$ over F is equipped with the automorphism $v \mapsto v^\sigma$ whose restriction to H is the identity, and whose restriction to F is σ. It is also equipped with its main involution $v \mapsto v^\iota$, which restricts to the main involution on H and restricts to the identity on F. Define a rational quadratic space

$$W_H = \{v \in H_F : v^\iota = v^\sigma\} \tag{3.1.1}$$

with quadratic form $\mathrm{Nm}(v) = vv^\iota$. Routine calculations show that this quadratic space has rank 4, determinant d_F, Hasse invariant

$$\mathrm{hasse}(W_H, \mathrm{Nm}) = H \otimes \left(\frac{-d_F, -1}{\mathbb{Q}}\right) \in \mathrm{Br}_2(\mathbb{Q}), \tag{3.1.2}$$

and signature
$$\mathrm{sig}(W_H, \mathrm{Nm}) = \begin{cases} (2,2) & \text{if } H \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \\ (4,0) & \text{otherwise.} \end{cases}$$

Proposition 3.1.3. *Suppose $S = \mathrm{Spec}(\mathbb{F}_p^{\mathrm{alg}})$ for some prime p. Suppose further that there is an elliptic curve A_0 over S and an isogeny $A \sim A_0 \times A_0$.*

(1) *If A_0 is ordinary then set $K_1 = \mathrm{End}(A_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and let K_2 be the other quadratic imaginary subfield of $K_1 \otimes_{\mathbb{Q}} F$. There is an isomorphism of $\mathbb{Q}$-quadratic spaces*
$$\big(V(\mathbf{A}), Q_{\mathbf{A}}\big) \cong \big(K_2, \beta(\mathbf{A}) \cdot \mathrm{Nm}_{K_2/\mathbb{Q}}\big)$$
for some positive $\beta(\mathbf{A}) \in \mathbb{Q}^\times$.

(2) *If A_0 is supersingular then there is an isomorphism of quadratic spaces*

$$\big(V(\mathbf{A}), Q_{\mathbf{A}}\big) \cong \big(W_H, \mathrm{Nm}\big)$$

where H is the rational quaternion algebra of discriminant p.

If A is not isogenous to the square of an elliptic curve then $L(\mathbf{A}) = 0$.

Proof. Up to isogeny there are two p-divisible groups over $\mathbb{F}_p^{\mathrm{alg}}$ of dimension one and height two. One of them, $\mathfrak{g}_{\mathrm{ss}}$, is connected and is isomorphic to $E[p^\infty]$ for any supersingular elliptic curve E. The other, $\mathfrak{g}_{\mathrm{ord}} = \mathbb{Q}_p/\mathbb{Z}_p \times \mu_{p^\infty}$, is isomorphic to $E[p^\infty]$ for any ordinary elliptic curve. The only possibilities for the isogeny type of the p-divisible group $A[p^\infty]$ are

(1) $\mathfrak{g}_{\mathrm{ss}} \times \mathfrak{g}_{\mathrm{ord}}$,
(2) $\mathfrak{g}_{\mathrm{ord}} \times \mathfrak{g}_{\mathrm{ord}}$,
(3) $\mathfrak{g}_{\mathrm{ss}} \times \mathfrak{g}_{\mathrm{ss}}$.

First suppose that $A[p^\infty] \sim \mathfrak{g}_{\mathrm{ss}} \times \mathfrak{g}_{\mathrm{ord}}$. The endomorphism algebra $H_p = \mathrm{End}(\mathfrak{g}_{\mathrm{ss}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a quaternion division algebra, and the existence of the action

$$\kappa_A : F_p \to \mathrm{End}(A[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H_p \times \mathbb{Q}_p \times \mathbb{Q}_p$$

then implies that $F_p \cong \mathbb{Q}_p \times \mathbb{Q}_p$. The orthogonal idempotents $\epsilon_1, \epsilon_2 \in F_p$ are interchanged by the Galois automorphism σ, and hence for any $j \in V(\mathbf{A})$

$$j \circ \epsilon_1 = \epsilon_2 \circ j$$
$$j \circ \epsilon_2 = \epsilon_1 \circ j.$$

It follows that the image of j in $\mathrm{End}(A[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is determined by its restrictions

$$j_1 \in \mathrm{Hom}(\mathfrak{g}_{\mathrm{ss}}, \mathfrak{g}_{\mathrm{ord}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$
$$j_2 \in \mathrm{Hom}(\mathfrak{g}_{\mathrm{ord}}, \mathfrak{g}_{\mathrm{ss}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

But both of the above Hom spaces are trivial, and hence $V(\mathbf{A}) = 0$.

Next suppose that A is ordinary, so that $A[p^\infty] \sim \mathfrak{g}_{\mathrm{ord}} \times \mathfrak{g}_{\mathrm{ord}}$. By [7, Lemma 3], the endomorphism ring of a simple ordinary abelian variety over $\mathbb{F}_p^{\mathrm{alg}}$ is commutative, and so if A is simple then $L(\mathbf{A}) = 0$. If $A \sim A_1 \times A_2$ with A_1 and A_2 non-isogenous ordinary elliptic curves, then again $\mathrm{End}(A)$ is commutative and so $L(\mathbf{A}) = 0$. Assume now that $A \sim A_0^2$ with A_0 an ordinary elliptic curve, so that $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M_2(K_1)$ with $K_1 = \mathrm{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ a quadratic imaginary field. Use the embedding $\kappa_A : F \to M_2(K_1)$ to view F as a subalgebra of $M_2(K_1)$, and let L be the $\mathbb{Q}$-subalgebra of $M_2(K_1)$ generated by K_1 and F. The algebra L is a biquadratic field, and is equal to the commutant of F in $M_2(K_1)$. We let K_2 be the quadratic imaginary subfield of L that is not isomorphic to K_1. The Rosati involution on $M_2(K_1)$ induced by λ_A must preserve the center K_1, and as the Rosati involution is positive it must restrict to complex conjugation on K_1. As F is fixed by the Rosati involution, the restriction of the Rosati involution to L

agrees with complex conjugation, and in particular is complex conjugation on K_2. Define

$$\tilde{V} = \{j \in M_2(K_1) : \forall x \in F,\ x \circ j = j \circ x^\sigma\}.$$

It follows from the Noether-Skolem theorem that

$$M_2(K_1) \cong L \oplus \tilde{V}$$

with each summand stable under $*$. In particular $\dim_{\mathbb{Q}} \tilde{V} = 4$. The ± 1 eigenspaces for the operator $*$ on $\tilde{V}$ are interchanged by the action of any trace zero element of K_1, and so each eigenspace must be a two-dimensional $\mathbb{Q}$-vector space. It follows that the subspace

$$V = \{j \in \tilde{V} : j^* = j\}$$

has $\mathbb{Q}$-dimension two, and it is easy to see that V is stable under the action of K_2. Hence V is a K_2-vector space of dimension one. Fixing any nonzero $j \in V$ we now find that $\lambda \mapsto \lambda \cdot j$ defines an isomorphism

$$\big(K_2, \beta \cdot \mathrm{Nm}_{K_2/\mathbb{Q}}\big) \cong \big(V(\mathbf{A}), Q_{\mathbf{A}}\big)$$

where $\beta = jj^*$. As β commutes with F, it must lie in L. As β is fixed by the Rosati involution it must further lie in F. But $\beta = jj$ shows that β commutes with j, and the only elements of F that commute with j are the rational numbers. Therefore $\beta \in \mathbb{Q}$. Of course $\beta > 0$, by the positivity of the Rosati involution.

Finally suppose that $A[p^\infty] \sim \mathfrak{g}_{\mathrm{ss}} \times \mathfrak{g}_{\mathrm{ss}}$. By [36, Theorem 4.2] there is an isogeny $f : A \sim A_0^2$ with A_0 a supersingular elliptic curve. Fix any $\mathbb{Z}$-algebra embedding $\iota : \mathcal{O}_F \to M_2(\mathbb{Z})$ and write $A_0 \otimes \mathcal{O}_F$ for the abelian surface A_0^2 with the action

$$\kappa_{A_0 \otimes \mathcal{O}_F} : \mathcal{O}_F \to \mathrm{End}(A_0 \otimes \mathcal{O}_F)$$

determined by ι (compare with [8, Theorem 7.5]). If $\lambda_{A_0} : A_0 \to A_0^\vee$ is the unique principal polarization of A_0 then, as in [17, Section 3.1], there is an induced $\mathfrak{D}_F^{-1}$-polarization $\lambda_{A_0 \otimes \mathcal{O}_F}$ of $A_0 \otimes \mathcal{O}_F$ defined by

$$A_0 \otimes \mathcal{O}_F \xrightarrow{\lambda_{A_0} \otimes \mathrm{id}} A_0^\vee \otimes \mathfrak{D}_F^{-1} \cong (A_0 \otimes \mathcal{O}_F)^\vee.$$

Thus we obtain a $\mathfrak{D}_F^{-1}$-polarized RM abelian surface

$$\mathbf{B} = (A_0 \otimes \mathcal{O}_F, \kappa_{A_0 \otimes \mathcal{O}_F}, \lambda_{A_0 \otimes \mathcal{O}_F})$$

over $\mathbb{F}_p^{\mathrm{alg}}$. Set $H = \mathrm{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ so that H is a quaternion algebra ramified precisely at p and ∞, and

$$\mathrm{End}(A_0 \otimes \mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M_2(H).$$

Let H_F be the commutant of F in $M_2(H)$, so that $H_F \cong H \otimes_{\mathbb{Q}} F$. The subalgebra $H_F \subset M_2(H)$ is stable under the Rosati involution induced by $\lambda_{A_0 \otimes \mathcal{O}_F}$, and the restriction of the Rosati involution is equal to the main involution on the F-quaternion algebra H_F (as the main involution is the only positive involution of H_F that is the identity on F). The Noether-Skolem theorem implies that any two $\mathbb{Q}$-algebra maps $F \to M_2(H)$ are conjugate, and from this one deduces that the isogeny $f : A \to A_0 \otimes \mathcal{O}_F$ may be chosen to be $\mathcal{O}_F$-linear. By [48, Proposition 1.3] the polarization $f^\vee \circ \lambda_{A_0 \otimes \mathcal{O}_F} \circ f$ of A has the form

$$f^\vee \circ \lambda_{A_0 \otimes \mathcal{O}_F} \circ f = \lambda_A \circ \kappa_A(x)$$

for some totally positive $x \in F^\times$. As the reduced norm $H_F^\times \to F^\times$ surjects onto the totally positive elements, we may choose $\rho \in H_F$ such that $\rho^*\rho = x^{-1}$. Replacing f by a suitable integer multiple of $\rho \circ f$ allows us to assume that $f^\vee \circ \lambda_{A_0 \otimes \mathcal{O}_F} \circ f$ is a positive integer multiple of λ_A. This implies that the isomorphism

$$\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{End}(A_0 \otimes \mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Q}$$

induced by f identifies the Rosati involution induced by λ_A with the Rosati involution induced by $\lambda_{A_0 \otimes \mathcal{O}_F}$, and so restricts to an isomorphism of $\mathbb{Q}$-quadratic spaces

$$\big(V(\mathbf{A}), Q_{\mathbf{A}}\big) \cong \big(V(\mathbf{B}), Q_{\mathbf{B}}\big).$$

The $\mathfrak{D}_F^{-1}$-polarized RM abelian surface $\mathbf{B}$ admits a natural special endomorphism

$$j_0 : A_0 \otimes \mathcal{O}_F \to A_0 \otimes \mathcal{O}_F$$

defined by $j_0(a \otimes x) = a \otimes (x^\sigma)$, which obviously satisfies $j_0 \circ j_0 = 1$. Given any other $j \in V(\mathbf{B})$ the endomorphism $J = j \circ j_0$ is $\mathcal{O}_F$-linear, so lies in $H_F \subset M_2(H)$. The condition $j^* = j$ is equivalent to $J^\iota = J^\sigma$, and $j \mapsto J$ defines the desired isomorphism

$$\big(V(\mathbf{B}), Q_{\mathbf{B}}\big) \cong \big(W_H, \mathrm{Nm}\big).$$

□

Over an arbitrary connected scheme S we obtain the following result.

Corollary 3.1.4. *The $\mathbb{Z}$-module $L(\mathbf{A})$ is free of rank at most 4. For every $j \in L(\mathbf{A})$ the endomorphism $Q_{\mathbf{A}}(j) = j \circ j$ lies in $\mathbb{Z}$, and $Q_{\mathbf{A}}$ is a positive definite quadratic form on $L(\mathbf{A})$.*

Proof. After [34, Corollary 6.2] it suffices to prove this when $S = \mathrm{Spec}(k)$ is the spectrum of a field. As A and $A^\vee$ are of finite type over k, and $\mathrm{Hom}(A, A^\vee)$ and $\mathrm{End}(A)$ are finitely generated $\mathbb{Z}$-modules, we may further reduce to the case of k finitely generated over its prime subfield k_0. Using the theory of Néron models one may extend $\mathbf{A}$ to a $\mathfrak{c}$-polarized RM abelian surface over a local Dedekind domain with fraction field k and residue field k' of transcendence degree (over k_0) one less than that of k. By reducing $\mathbf{A}$ from k to k', applying [34, Corollary 6.2], and repeating, we eventually are reduced to the case in which k is an algebraic extension of k_0. The claim then follows from Propositions 3.1.2 and 3.1.3. □

3.2. **Abelian surfaces with complex multiplication.** Let S be a connected scheme.

Definition 3.2.1. A *CM abelian surface* over S is a pair (A, κ_A) in which A is an abelian scheme over S of relative dimension two, and $\kappa_A : \mathcal{O}_E \to \mathrm{End}(A)$ is an action such that the restriction of κ_A to $\mathcal{O}_F$ satisfies the Kottwitz determinant condition. In other words, a CM abelian surface is an RM abelian surface together with an extension of the $\mathcal{O}_F$-action to $\mathcal{O}_E$. A $\mathfrak{c}$-*polarization* of (A, κ_A) is a $\mathfrak{c}$-polarization λ_A of the underlying RM abelian surface.

Fix a CM type $\Sigma = \{\pi_1, \pi_2\}$ of E, and let $\phi_\Sigma : E^\sharp \to \mathbb{C}$ be the associated reflex map of Section 2.1. Recalling that $\mathcal{O}_\Sigma$ is the ring of integers of $E_\Sigma = \phi_\Sigma(E^\sharp)$, for any $x \in \mathcal{O}_E$ define a polynomial

$$c_{\Sigma,x}(T) = (T - \pi_1(x))(T - \pi_2(x)) \in \mathcal{O}_\Sigma[T].$$

If S is an $\mathcal{O}_\Sigma$-scheme and $\mathbf{A}$ is an $\mathcal{O}_F$-polarized CM abelian surface over S, we say that $\mathbf{A}$ satisfies the Σ-*Kottwitz condition* if every point of S admits an open affine neighborhood $\mathrm{Spec}(R) \to S$ over which $\mathrm{Lie}(A_{/R})$ is a free rank two R-module on which every $x \in \mathcal{O}_E$ acts with characteristic polynomial $c_{\Sigma,x}(T) \in R[T]$.

Fix a $\mathfrak{c}$-polarized CM abelian surface $\mathbf{A} = (A, \kappa_A, \lambda_A)$ over S satisfying the Σ-Kottwitz condition.

Lemma 3.2.2. *The Rosati involution induced by λ_A satisfies $\kappa_A(x)^* = \kappa_A(\overline{x})$ for every $x \in \mathcal{O}_E$.*

Proof. As in the proof of Corollary 3.1.4, it suffices to treat the case where S is the spectrum of an algebraically closed field. Given any nonzero $x \in \mathcal{O}_E$, [48, Proposition 1.3] implies that the pullback of λ_A by the endomorphism $\kappa_A(x)$ is an F-multiple of λ_A. More precisely, there is a $y \in F$ such that

$$\kappa_A(x)^\vee \circ \lambda_A \circ \kappa_A(x) = \lambda_A \circ \kappa_A(y).$$

Of course this is equivalent to $\kappa_A(x)^* \circ \kappa_A(x) = \kappa_A(y)$, which implies that $\kappa_A(x)^*$ lies in the image of $\kappa_A : E \to \mathrm{End}(A) \otimes_\mathbb{Z} \mathbb{Q}$. The Rosati involution therefore stabilizes the image of κ_A, and its restriction to E is a positive involution that fixes F. The only such involution is complex conjugation. □

Lemma 3.2.3.

(1) *The map $\kappa_A : \mathcal{O}_E \to \mathrm{Hom}_{\mathcal{O}_E}(A)$ is an isomorphism.*
(2) *The automorphism group of $\mathbf{A}$ is isomorphic to $\mu(E)$, the group of roots of unity in E.*

Proof. For the first claim, as in the proof of Corollary 3.1.4, it suffices to treat the case where $S = \mathrm{Spec}(k)$ with k either $\mathbb{C}$ or $\mathbb{F}_p^{\mathrm{alg}}$. The case of $k = \mathbb{C}$ is clear from the complex uniformization of $A(\mathbb{C})$. If $k = \mathbb{F}_p^{\mathrm{alg}}$ then fix a prime $\ell \neq \mathrm{char}(k)$. The proof of [16, Proposition 2.1.1] shows that the ℓ-adic Tate module $\mathrm{Ta}_\ell(A)$ is free of rank one over $\mathcal{O}_{E,\ell}$. From this it follows first that $\mathcal{O}_{E,\ell} \to \mathrm{End}_{\mathcal{O}_{E,\ell}}(\mathrm{Ta}_\ell(A))$ is an isomorphism, and then (using the results of [35, Section 19]) that $\mathcal{O}_{E,\ell} \to \mathrm{End}_{\mathcal{O}_E}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is an isomorphism. In particular $\mathrm{End}_{\mathcal{O}_E}(A)$ is a rank 4 $\mathbb{Z}$-algebra, and first claim follows easily. In particular any automorphism of $\mathbf{A}$ is of the form $f = \kappa_A(x)$ for some $x \in \mathcal{O}_E^\times$. The condition that f preserve the polarization λ_A means that $f^\vee \circ \lambda_A \circ f = \lambda_A$, or, equivalently, that $f^* \circ f = 1$. After Lemma 3.2.2 this is equivalent to $\overline{x}x = 1$, and so the obvious injective homomorphism

$$\mu(E) = \{x \in \mathcal{O}_E^\times : x\overline{x} = 1\} \xrightarrow{x \mapsto \kappa_A(x)} \mathrm{Aut}(\mathbf{A})$$

is in fact an isomorphism. □

As in Section 2.3 we will use the action $\kappa_A : \mathcal{O}_E \to \mathrm{End}(A)$ to endow $V(\mathbf{A})$ with an action of the reflex algebra $E^\sharp$. First define a $\mathbb{Q}$-vector space

$$\tilde{V}(\mathbf{A}) = \{j \in \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} : \kappa_A(x) \circ j = j \circ \kappa_A(x^\sigma) \text{ for all } x \in \mathcal{O}_F\ \}$$

and let the $\mathbb{Q}$-algebra M of Section 2.1 act on $\tilde{V}(\mathbf{A})$ by

$$(a \otimes b) \bullet j = \kappa_A(a) \circ j \circ \kappa_A(\overline{b}).$$

The action of $E^\sharp$ leaves invariant the subspace

$$V(\mathbf{A}) = \{j \in \tilde{V}(\mathbf{A}) : j^* = j\},$$

and defines the desired $E^\sharp$-module structure on $V(\mathbf{A})$. If $\mathfrak{l}$ is a place of $F^\sharp$, abbreviate $V_{\mathfrak{l}}(\mathbf{A}) = V(\mathbf{A}) \otimes_{F^\sharp} F^\sharp_{\mathfrak{l}}$.

For any $\mathcal{O}_F$-linear $f \in \mathrm{End}(A)$ satisfying $f^* = f$, the map $\lambda_A \circ f : A \to A^\vee$ is also $\mathcal{O}_F$-linear and satisfies $(\lambda_A \circ f)^\vee = \lambda_A \circ f$. By [48, Proposition 1.3] there is an $x \in F$ such that $\lambda_A \circ f = \lambda_A \circ \kappa_A(x)$. It follows that

$$F = \{f \in \mathrm{End}_{\mathcal{O}_F}(A) \otimes_{\mathbb{Z}} \mathbb{Q} : f^* = f\}.$$

With this fact in hand, imitating the proof of Lemma 2.3.4 shows that the $\mathbb{Q}$-bilinear form on $V(\mathbf{A})$ defined by

$$[j_1, j_2]_{\mathbf{A}} = Q_{\mathbf{A}}(j_1 + j_2) - Q_{\mathbf{A}}(j_1) - Q_{\mathbf{A}}(j_2)$$

satisfies $[x \bullet j_1, j_2]_{\mathbf{A}} = [j_1, x^\dagger \bullet j_2]_{\mathbf{A}}$ for all $x \in E^\sharp$. From this it follows that there is a unique totally positive definite Hermitian form $\langle j_1, j_2 \rangle_{\mathbf{A}}$ on the $E^\sharp$-module $V(\mathbf{A})$ satisfying

$$[j_1, j_2]_{\mathbf{A}} = \mathrm{Tr}_{E^\sharp/\mathbb{Q}} \langle j_1, j_2 \rangle_{\mathbf{A}},$$

and that $Q^\sharp_{\mathbf{A}}(j) = \langle j, j \rangle_{\mathbf{A}}$ is the unique $F^\sharp$-quadratic form on $V(\mathbf{A})$ satisfying

$$Q_{\mathbf{A}}(j) = \mathrm{Tr}_{F^\sharp/\mathbb{Q}} Q^\sharp_{\mathbf{A}}(j). \tag{3.2.1}$$

Proposition 3.2.4. *Suppose that $S = \mathrm{Spec}(\mathbb{F}_p^{\mathrm{alg}})$ for some prime p, and that $\mathbf{A}$ is supersingular. For some totally positive $\beta(\mathbf{A}) \in (F^\sharp)^\times$ there is an $E^\sharp$-linear isomorphism of $F^\sharp$-quadratic spaces*

$$\big(V(\mathbf{A}), Q^\sharp_{\mathbf{A}}\big) \cong \big(E^\sharp, \beta(\mathbf{A}) x x^\dagger\big).$$

Proof. The only thing to prove is that $V(\mathbf{A})$ is free of rank one over $E^\sharp$. In cases (**cyclic**) and (**nongal**) this is obvious, as $E^\sharp$ is a degree four field extension of $\mathbb{Q}$, and $\dim_{\mathbb{Q}} V(\mathbf{A}) = 4$ by Proposition 3.1.3. In case (**biquad**) $E^\sharp \cong E_1 \times E_2$ splits as a product of quadratic imaginary fields, and the freeness of $V(\mathbf{A})$ over $E^\sharp$ is less obvious. We must rule out the possibility that the action of $E^\sharp$ factors through a projection $E^\sharp \to E_i$. To do this it suffices to exhibit a prime ℓ such that $V_\ell(\mathbf{A}) = V(\mathbf{A}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is free over $E^\sharp_\ell$.

Fix a prime $\ell \neq p$ that splits completely in E. Label the four $\mathbb{Q}$-algebra maps

$$\pi_1, \pi_2, \pi_3, \pi_4 : E \to \mathbb{Q}_\ell$$

in such a way that $\pi_3(x) = \pi_1(\overline{x})$ and $\pi_4(x) = \pi_1(\overline{x})$. The proof of Lemma 3.2.3 shows that $\mathrm{Ta}^0_\ell(A) = \mathrm{Ta}_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is free of rank one over E_ℓ, and so there is a $\mathbb{Q}_\ell$-basis $\{e_1, e_2, e_3, e_4\}$ of $\mathrm{Ta}^0_\ell(A)$ with the property that

$$\kappa_A(x) \cdot e_i = \pi_i(x) \cdot e_i$$

for all $x \in E_\ell$. With respect to this basis the Weil pairing

$$e_\ell : \mathrm{Ta}^0_\ell(A) \times \mathrm{Ta}^0_\ell(A) \to \mathbb{Q}_\ell(1)$$

induced by λ_A is given, after fixing an isomorphism of $\mathbb{Q}_\ell$-vector spaces $\mathbb{Q}_\ell(1) \cong \mathbb{Q}_\ell$ and appealing to Lemma 3.2.2, by a matrix of the form

$$\begin{pmatrix} & & \rho_1 & \\ & & & \rho_2 \\ -\rho_1 & & & \\ & -\rho_2 & & \end{pmatrix}$$

for some $\rho_1, \rho_2 \in \mathbb{Q}_\ell^\times$. Every special endomorphism $j \in V(\mathbf{A})$ acts on $\mathrm{Ta}^0_\ell(A)$ by a matrix of the form

$$j = \begin{pmatrix} & \rho_2 b & & \rho_2 c \\ \rho_1 a & & -\rho_1 c & \\ & -\rho_2 d & & \rho_2 a \\ \rho_1 d & & \rho_1 b & \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Q}_\ell$. By counting dimensions, the injection

$$V_\ell(\mathbf{A}) \to \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{Ta}^0_\ell(A))$$

identifies $V_\ell(\mathbf{A})$ with the space of all such matrices. The pair $\{\pi_1, \pi_2\}$ induces a $\mathbb{Q}$-algebra map (the ℓ-adic version of the reflex map of Section 2.1)

$$\phi_{12} : E^\sharp \to \mathbb{Q}_\ell$$

defined by restricting the domain of the map $M \to \mathbb{Q}_\ell$ defined by

$$x \otimes y \mapsto \pi_1(x)\pi_2(y).$$

Define ϕ_{14}, ϕ_{23}, and ϕ_{34} in the same manner. The action of $z \in E^\sharp$ on $V(\mathbf{A})$ now takes the explicit form

$$z \bullet j = \begin{pmatrix} & \rho_2 b \cdot \phi_{14}(z) & & \rho_2 c \cdot \phi_{12}(z) \\ \rho_1 a \cdot \phi_{23}(z) & & -\rho_1 c \cdot \phi_{12}(z) & \\ & -\rho_2 d \cdot \phi_{34}(z) & & \rho_2 a \cdot \phi_{23}(z) \\ \rho_1 d \cdot \phi_{34}(z) & & \rho_1 b \cdot \phi_{14}(z) & \end{pmatrix}.$$

If we define

$$a(z) = \phi_{23}(z)$$
$$b(z) = \phi_{14}(z)$$
$$c(z) = \phi_{12}(z)$$
$$d(z) = \phi_{34}(z)$$

then the product $a \times b \times c \times d : E^\sharp_\ell \to \mathbb{Q}^4_\ell$ is an isomorphism of $\mathbb{Q}_\ell$-vector spaces, and

$$z \mapsto \begin{pmatrix} & \rho_2 b(z) & & \rho_2 c(z) \\ \rho_1 a(z) & & -\rho_1 c(z) & \\ & -\rho_2 d(z) & & \rho_2 a(z) \\ \rho_1 d(z) & & \rho_1 b(z) & \end{pmatrix}$$

defines an $E^\sharp_\ell$-linear isomorphism $E^\sharp_\ell \to V_\ell(\mathbf{A})$. In particular $V_\ell(\mathbf{A})$ is free of rank one over $E^\sharp_\ell$. $\square$

Recall from Section 2.4 that the kernel of the norm map $(E^\sharp)^\times \to (F^\sharp)^\times$, viewed as an algebraic group over $\mathbb{Q}$, is denoted S_E. By restricting the action of $E^\sharp$ on $V(\mathbf{A})$ to the subgroup $S_E(\mathbb{Q}) \subset (E^\sharp)^\times$ we obtain a representation of S_E on the $E^\sharp$-module $V(\mathbf{A})$. Composition with the homomorphism

$$\nu_E : T_E \to S_E$$

of Lemma 2.4.1 yields a representation of T_E on $V(\mathbf{A})$, which is given by the simple formula

$$\nu_E(t) \bullet j = \kappa_A(t) \circ j \circ \kappa_A(t^{-1}).$$

If $S = \mathrm{Spec}(\mathbb{F}_p^{\mathrm{alg}})$ for some prime p, and $\mathbf{A}$ is supersingular, then it follows from Proposition 3.2.4 that the action of S_E on $V(\mathbf{A})$ defines an isomorphism of algebraic groups over $\mathbb{Q}$

$$S_E \cong \mathrm{Res}_{F^\sharp/\mathbb{Q}} \, \mathrm{SO}(V(\mathbf{A}), Q^\sharp_{\mathbf{A}}).$$

We end this subsection by defining an action of the group $C_+(E)$ of Section 2.4 on the set of isomorphism classes of $\mathfrak{c}$-polarized CM abelian surfaces over S. Suppose $\mathbf{A} = (A, \kappa_A, \lambda_A)$ is a $\mathfrak{c}$-polarized CM abelian surface over S and $\mathbf{Z} = (\mathfrak{Z}, \zeta) \in C_+(E)$. As in [8, Theorem 7.5], define a new CM abelian surface $B = A \otimes_{\mathcal{O}_E} \mathfrak{Z}$ over S and let $\kappa_B : \mathcal{O}_E \to \mathrm{End}(B)$ be the action

$$\kappa_B(x)(a \otimes z) = a \otimes (xz).$$

The pair (B, κ_B) is characterized up to isomorphism by its functor of points

$$B(\cdot) = A(\cdot) \otimes_{\mathcal{O}_E} \mathfrak{Z}$$

from the category of S-schemes to the category of $\mathcal{O}_E$-modules. There is an $f \in \mathrm{Hom}_{\mathcal{O}_E}(B, A) \otimes_{\mathbb{Z}} \mathbb{Q}$ defined by

$$f(a \otimes z) = \kappa_A(z)a,$$

and we obtain a $\lambda_B \in \mathrm{Hom}(B, B^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$ defined by

$$\lambda_B = f^\vee \circ \lambda_A \circ f \circ \kappa_B(\zeta^{-1}),$$

which one can show is a $\mathfrak{c}$-polarization of the CM abelian surface (B, κ_B). Thus we have constructed a new $\mathfrak{c}$-polarized CM abelian surface

$$(A, \kappa_A, \lambda_A) \otimes \mathbf{Z} \stackrel{\text{def}}{=} (B, \kappa_B, \lambda_B),$$

and $\mathbf{A} \mapsto \mathbf{A} \otimes \mathbf{Z}$ defines an action of $C_+(E)$ on the set of all isomorphism classes of $\mathfrak{c}$-polarized CM abelian surfaces over S. If $\mathbf{A}$ satisfies the Σ-Kottwitz condition then so does $\mathbf{A} \otimes \mathbf{Z}$ for any $\mathbf{Z} \in C_+(E)$.

Proposition 3.2.5. *Suppose $t \in T_E(\mathbb{A}_f)$, and let $\mathbf{Z} \in C_0(E)$ be the image of t under (2.4.1). There is an isomorphism of $F^\sharp$-quadratic spaces*

$$\big(V(\mathbf{A} \otimes \mathbf{Z}), Q^\sharp_{\mathbf{A}\otimes\mathbf{Z}}\big) \to \big(V(\mathbf{A}), Q^\sharp_{\mathbf{A}}\big)$$

identifying $\widehat{L}(\mathbf{A} \otimes \mathbf{Z}) \cong \nu_E(t) \cdot \widehat{L}(\mathbf{A})$.

Proof. As above, define $f \in \mathrm{Hom}_{\mathcal{O}_E}(A \otimes_{\mathcal{O}_E} \mathfrak{Z}, A) \otimes_{\mathbb{Z}} \mathbb{Q}$ by $f(a \otimes z) = \kappa_A(z)a$. The isomorphism

$$\mathrm{End}(A \otimes_{\mathcal{O}_E} \mathfrak{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

defined by $j \mapsto f \circ j \circ f^{-1}$ identifies $\mathrm{End}(A \otimes_{\mathcal{O}_E} \mathfrak{Z})$ with

$$\{\kappa_A(z) \circ v \circ \kappa_A(z)^{-1} : v \in \mathrm{End}(A) \text{ and } z \in \mathfrak{Z}\}.$$

The restriction of this map to $V(\mathbf{A} \otimes \mathbf{Z}) \to V(\mathbf{A})$ is the desired isomorphism. □

Remark 3.2.6. To every $\mathfrak{c}$-polarized CM module $\mathbf{T}$ we may attach a $\mathfrak{c}$-polarized CM abelian surface $\mathbf{A} = (A, \kappa_A, \lambda_A)$ over $\mathbb{C}$ as follows. As in the discussion surrounding Definition 2.3.2, the CM type $\Sigma(\mathbf{T})$ determines a $\mathbb{C}$-vector space structure on $T_{\mathbb{R}} = T \otimes_{\mathbb{Z}} \mathbb{R}$, and so

$$A(\mathbb{C}) = T_{\mathbb{R}}/T$$

is a complex torus. The Hermitian form (2.3.1) determines a polarization $\lambda_A : A \to A^\vee$, and the action $\kappa_T : \mathcal{O}_E \to \mathrm{End}_{\mathbb{Z}}(T)$ induces an action $\kappa_A : \mathcal{O}_E \to \mathrm{End}(A)$. It is easily seen that the construction $\mathbf{T} \mapsto \mathbf{A}$ establishes a $C_+(E)$-equivariant bijection between the set of isomorphism classes of $\mathfrak{c}$-polarized CM modules with CM type Σ, and the set of isomorphism classes of $\mathfrak{c}$-polarized CM abelian surfaces over $\mathbb{C}$ satisfying the Σ-Kottwitz condition. The inverse of this bijection is "take first homology".

3.3. **Moduli spaces.** In this subsection we take $\mathfrak{c} = \mathcal{O}_F$ and fix a CM type Σ of E. We are now ready to define the moduli space $\mathcal{M}$ of all $\mathcal{O}_F$-polarized RM abelian surfaces (a classical Hilbert modular surface) and some special cycles on $\mathcal{M}$. The first special cycle, $\mathcal{CM}_\Sigma$, is the codimension two cycle of points with complex multiplication by $\mathcal{O}_E$ satisfying the Σ-Kottwitz condition. The second type of special cycle is the family of Hirzebruch-Zagier divisors, denoted $\mathcal{T}(m)$, composed of points that admit special endomorphisms of norm m. The third type of special cycle, $\mathcal{CM}_\Sigma(\alpha)$, will appear as a closed substack of the intersection of $\mathcal{CM}_\Sigma$ with $\mathcal{T}(m)$. The definitions are as follows. Recall that a *groupoid* is a category in which all arrows are isomorphisms. For algebraic stacks see [47] or [29].

Definition 3.3.1. Let $\mathcal{M}$ be the algebraic stack over $\mathrm{Spec}(\mathbb{Z})$ whose functor of points assigns to a connected scheme S the groupoid of all $\mathcal{O}_F$-polarized RM abelian surfaces $\mathbf{A}$ over S.

It is known that $\mathcal{M}$ is regular of dimension 3, is flat over $\mathbb{Z}$, and is smooth over $\mathbb{Z}[1/d_F]$. See [10, 38, 48].

Definition 3.3.2. For each nonzero $m \in \mathbb{Z}$, let $\mathcal{T}(m)$ be the algebraic stack over $\mathrm{Spec}(\mathbb{Z})$ whose functor of points assigns to every connected scheme S the groupoid of pairs $(\mathbf{A}, j)$ in which

- $\mathbf{A} \in \mathcal{M}(S)$ is an $\mathcal{O}_F$-polarized RM abelian surface over S;
- $j \in L(\mathbf{A})$ satisfies $Q_{\mathbf{A}}(j) = m$.

Definition 3.3.3. Let $\mathcal{CM}_\Sigma$ be the algebraic stack over $\mathrm{Spec}(\mathcal{O}_\Sigma)$ whose functor of points assigns to every connected $\mathcal{O}_\Sigma$-scheme S the groupoid of all $\mathcal{O}_F$-polarized CM abelian surfaces $\mathbf{A}$ over S satisfying the Σ-Kottwitz condition.

Definition 3.3.4. For every nonzero $\alpha \in F^\sharp$, let $\mathcal{CM}_\Sigma(\alpha)$ be the algebraic stack over $\mathrm{Spec}(\mathcal{O}_\Sigma)$ whose functor of points assigns to a connected $\mathcal{O}_\Sigma$-scheme S the groupoid of pairs $(\mathbf{A}, j)$ in which

- $\mathbf{A} \in \mathcal{CM}_\Sigma(S)$ is an $\mathcal{O}_F$-polarized CM abelian surface over S satisfying the Σ-Kottwitz condition;
- $j \in L(\mathbf{A})$ satisfies $Q^\sharp_{\mathbf{A}}(j) = \alpha$.

For $m \in \mathbb{Z}^+$ let R_m be the ring obtained by adjoining to $\mathcal{O}_F$ a single element j satisfying the relations $j^2 = m$ and $xj = jx^\sigma$ for all $x \in \mathcal{O}_F$. Then R_m is an order in an indefinite quaternion algebra over $\mathbb{Q}$. Given a scheme S and an S-valued point $(\mathbf{A}, j) \in \mathcal{T}(m)(S)$, the subalgebra $\mathcal{O}_F[j] \subset \mathrm{End}(A)$ is isomorphic to R_m. Thus one may think of $\mathcal{T}(m)$ as the moduli space of $\mathcal{O}_F$-polarized abelian surfaces with an extension of the action of $\mathcal{O}_F$ to R_m. If we use the forgetful map $\mathcal{T}(m) \to \mathcal{M}$ to view $\mathcal{T}(m)$ as a cycle on $\mathcal{M}$, then $\mathcal{T}(m)$ is a classical *Hirzebruch-Zagier divisor*. Our moduli-theoretic definition of these divisors follows the characterization given by Kudla-Rapoport [24].

For every nonzero $m \in \mathbb{Z}$ there are evident forgetful morphisms $\mathcal{T}(m) \to \mathcal{M}$ and $\mathcal{CM}_\Sigma \to \mathcal{M}_{/\mathcal{O}_\Sigma}$. Using the definition of the moduli problems and the relation (3.2.1), there is a canonical decomposition of $\mathcal{O}_\Sigma$-schemes

$$\mathcal{T}(m)_{/\mathcal{O}_\Sigma} \times_{\mathcal{M}_{/\mathcal{O}_\Sigma}} \mathcal{CM}_\Sigma \cong \bigsqcup_{\substack{\alpha\in F^\sharp \\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha)=m}} \mathcal{CM}_\Sigma(\alpha). \tag{3.3.1}$$

Note that $Q^\sharp_{\mathbf{A}}$ is totally positive definite for any $\mathcal{O}_F$-polarized CM abelian surface $\mathbf{A}$, and so $\mathcal{CM}_\Sigma(\alpha)=\emptyset$ unless α is totally nonnegative.

Proposition 3.3.5. *The structure morphism $f : \mathcal{CM}_\Sigma \to \mathrm{Spec}(\mathcal{O}_\Sigma)$ is étale and proper.*

Proof. To show that the map is étale, it suffices to show that the induced map

$$f^* : \widehat{\mathcal{O}}^{\mathrm{sh}}_{\Sigma,f(z)} \to \widehat{\mathcal{O}}^{\mathrm{sh}}_{\mathcal{CM}_\Sigma,z}$$

on completed strictly Henselian local rings is an isomorphism for any geometric point $z \in \mathcal{CM}_\Sigma(\mathbb{F}_\mathfrak{q}^{\mathrm{alg}})$, $\mathfrak{q}$ a prime of $\mathcal{O}_\Sigma$. For such a z the ring $\widehat{\mathcal{O}}^{\mathrm{sh}}_{\Sigma,f(z)}$ is isomorphic to the completion of the ring of integers of the maximal unramified extension of $\mathcal{O}_{\Sigma,\mathfrak{q}}$. If z represents the $\mathcal{O}_F$-polarized CM abelian surface $\mathbf{A}$ over $\mathbb{F}_\mathfrak{q}^{\mathrm{alg}}$, then $\widehat{\mathcal{O}}^{\mathrm{sh}}_{\mathcal{CM}_\Sigma,z}$ represents the functor of deformations of $\mathbf{A}$ to complete local Noetherian $\widehat{\mathcal{O}}^{\mathrm{sh}}_{\Sigma,f(z)}$-algebras with residue field $\mathbb{F}_\mathfrak{q}^{\mathrm{alg}}$, where the deformations are again required to satisfy the Σ-Kottwitz condition. By [16, Theorem 2.2.1], such deformations exist and are unique, and so the above map f^* is an isomorphism.

Let R be a discrete valuation ring, and let $\mathbf{A}$ be an $\mathcal{O}_F$-polarized CM abelian surface over $L = \mathrm{Frac}(R)$. We claim (as is well-known) that the underlying abelian variety A has potentially good reduction. After replacing L by a finite extension L' we may assume that the Néron model $\underline{A}'$ of $A' = A_{/L'}$ over R' (the ring of integers in L') has semi-abelian reduction, as in [1, §7.4]. In other words, the identity component of the reduction of $\underline{A}'$ to the residue field of R' is an extension of an abelian variety by a torus, necessarily of dimension at most 2. But the character group of this torus admits an action of $\mathcal{O}_E$, and so the torus has dimension 0. Thus, by [1, Theorem 7.4.5], $\underline{A}'$ is itself an abelian scheme, and it follows that $\mathbf{A}' = \mathbf{A}_{/L'}$ extends to a $\mathcal{O}_F$-polarized CM abelian surface over $\mathrm{Spec}(R')$. The properness of $\mathcal{CM}_\Sigma$ is now clear from the valuative criterion of properness for stacks [29]. □

Proposition 3.3.6. *Let k be an algebraically closed field. For any nonzero $m \in \mathbb{Z}$ and geometric point $z \in \mathcal{T}(m)(k)$, the completed strictly Henselian local ring $\widehat{\mathcal{O}}^{\mathrm{sh}}_{\mathcal{T}(m),z}$ is a complete intersection, and is the quotient of $\widehat{\mathcal{O}}^{\mathrm{sh}}_{\mathcal{M},z}$ by a principal ideal.*

Proof. We use Grothendieck's deformation theory of abelian schemes, as described in [28, Chapter 2.1.6]. This is essentially the Grothendieck-Messing theory of [14, 31], but those references frequently restrict to local rings of nonzero residue characteristic, while we wish to allow characteristic zero.

Let R be the completion of the strictly Henselian local ring of $\mathcal{M}$ at z, so that

$$\widehat{\mathcal{O}}^{\mathrm{sh}}_{\mathcal{T}(m),z} \cong R/I$$

for some ideal I. Let $\mathfrak{m}$ be the maximal ideal of R, and abbreviate $S_0 = R/I$ and $S = R/\mathfrak{m}I$. The point z represents a pair $(\mathbf{A}, j)$ over k. The tautological R-valued point $\mathbf{A}^{\mathrm{univ}} \in \mathcal{M}(R)$ corresponds to the universal formal deformation of $\mathbf{A}$, and S_0 is the largest quotient of R for which the special endomorphism j lifts to a special endomorphism of $\mathbf{A}^{\mathrm{univ}}_{/S_0}$. The de Rham homology

$$H_1^{\mathrm{dR}}(\mathbf{A}^{\mathrm{univ}}_{/S}) = \mathrm{Hom}_S(H^1_{\mathrm{dR}}(\mathbf{A}^{\mathrm{univ}}_{/S}), S)$$

has a Hodge short exact sequence of free S-modules

$$0 \to \mathrm{Fil}^1 H_1^{\mathrm{dR}}(\mathbf{A}^{\mathrm{univ}}_{/S}) \to H_1^{\mathrm{dR}}(\mathbf{A}^{\mathrm{univ}}_{/S}) \to \mathrm{Lie}(\mathbf{A}^{\mathrm{univ}}_{/S}) \to 0,$$

and Grothendieck's theory implies that the lift of j to $L(\mathbf{A}^{\mathrm{univ}}_{/S_0})$ induces an endomorphism (still called j) of $H_1^{\mathrm{dR}}(\mathbf{A}^{\mathrm{univ}}_{/S})$. The composition

$$\mathrm{Fil}^1 H_1^{\mathrm{dR}}(\mathbf{A}^{\mathrm{univ}}_{/S}) \to H_1^{\mathrm{dR}}(\mathbf{A}^{\mathrm{univ}}_{/S}) \xrightarrow{j} H_1^{\mathrm{dR}}(\mathbf{A}^{\mathrm{univ}}_{/S}) \to \mathrm{Lie}(\mathbf{A}^{\mathrm{univ}}_{/S}),$$

which we denote by $\mathrm{Obst}(j)$, becomes trivial after applying $\otimes_S S_0$, precisely because j lifts $L(\mathbf{A}^{\mathrm{univ}}_{/S_0})$. Thus $\mathrm{Obst}(j)$ may be viewed as a map

$$\mathrm{Fil}^1 H_1^{\mathrm{dR}}(\mathbf{A}^{\mathrm{univ}}_{/S}) \to (I/\mathfrak{m}I) \otimes_R \mathrm{Lie}(\mathbf{A}^{\mathrm{univ}}_{/S}).$$

The principal polarization $\lambda^{\mathrm{univ}}_{/S}$ of $\mathbf{A}^{\mathrm{univ}}$ induces a perfect symplectic form on $H_1^{\mathrm{dR}}(\mathbf{A}^{\mathrm{univ}}_{/S})$, under which the Hodge filtration $\mathrm{Fil}^1 H_1^{\mathrm{dR}}(\mathbf{A}^{\mathrm{univ}}_{/S})$ is maximal isotropic. Fix an S-basis $\{e_1, e_2, f_1, f_2\}$ of $H_1^{\mathrm{dR}}(\mathbf{A}^{\mathrm{univ}}_{/S})$ in such a way that $\{e_1, e_2\}$ is a basis of the Hodge filtration, and the symplectic form $\lambda^{\mathrm{univ}}_{/S}$ is given by the matrix $\left(\begin{smallmatrix} & I \\ -I & \end{smallmatrix}\right)$ where I is the 2×2 identity. The condition $j = j^*$ implies that j has the form

$$j = \begin{pmatrix} A & B \\ C & {}^tA \end{pmatrix}$$

for some $A, B, C \in M_2(S)$ satisfying ${}^tB = -B$ and ${}^tC = -C$, and the map $\mathrm{Obst}(j)$ is given by the lower left 2×2 block,

$$C = \begin{pmatrix} & c \\ -c & \end{pmatrix}$$

with $c \in I/\mathfrak{m}I$. In particular $\mathrm{Obst}(j)$ vanishes after applying $\otimes_S S/cS$, and so j lifts to a special endomorphism of the reduction of $\mathbf{A}^{\mathrm{univ}}_{/S}$ to S/cS. It follows that $S/cS = S_0$, and $cS = I/\mathfrak{m}I$. Nakayama's lemma now implies that the ideal $I \subset R$ is generated by a single element, and as R is a regular local ring, R/I is a complete intersection by [30, Theorem 21.2]. □

Proposition 3.3.7. *Suppose* $\alpha \in (F^\sharp)^\times$.

(1) *The category* $\mathcal{CM}_\Sigma(\alpha)(\mathbb{C})$ *is empty.*

(2) *For every prime* $\mathfrak{q}$ *of* $\mathcal{O}_\Sigma$ *and every point* $(\mathbf{A}, j) \in \mathcal{CM}_\Sigma(\alpha)(\mathbb{F}_\mathfrak{q}^{\mathrm{alg}})$ *the underlying* $\mathcal{O}_F$*-polarized CM abelian surface* $\mathbf{A}$ *is supersingular. Moreover, there is an* $E^\sharp$*-linear isomorphism of* $F^\sharp$*-quadratic spaces*

$$\big(V(\mathbf{A}), Q^\sharp_{\mathbf{A}}\big) \cong \big(E^\sharp, \alpha x x^\dagger\big). \tag{3.3.2}$$

Proof. If $\mathbf{A} \in \mathcal{CM}_\Sigma(\mathbb{C})$ then Proposition 3.1.2 tells us that $\dim_\mathbb{Q} V(\mathbf{A}) \le 2$. As $V(\mathbf{A})$ is an $E^\sharp$ module, the only way we can have $V(\mathbf{A}) \neq 0$ is if we are in case (**biquad**), so that $E^\sharp \cong E_1 \times E_2$ (as in Lemma 2.1.1), and one of the two orthogonal idempotents in $E^\sharp$ kills $V(\mathbf{A})$. This implies that $(V(\mathbf{A}), Q^\sharp_{\mathbf{A}})$ can only represent elements of $F^\sharp$ of norm 0, and in particular cannot represent α. Thus $\mathcal{CM}_\Sigma(\alpha)(\mathbb{C}) = \emptyset$.

Suppose $(\mathbf{A}, j) \in \mathcal{CM}_\Sigma(\alpha)(\mathbb{F}_\mathfrak{q}^{\mathrm{alg}})$. If $\mathbf{A}$ is not supersingular then Proposition 3.1.3 implies that $\dim_\mathbb{Q} V(\mathbf{A}) < 4$, and the argument of the preceding paragraph shows that $V(\mathbf{A})$ cannot represent α. As $V(\mathbf{A})$ represents α by hypothesis, we conclude first that $\mathbf{A}$ is supersingular, and then, using Proposition 3.2.4 that there is an isomorphism (3.3.2). ☐

For $\alpha \in (F^\sharp)^\times$ define a quadratic form $Q_\alpha(x) = \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha x x^\dagger)$ on the $\mathbb{Q}$-vector space $E^\sharp$. Define the *global support invariant* of α

$$\mathrm{inv}(\alpha) = \mathrm{hasse}\big(E^\sharp, Q_\alpha\big) \otimes \left(\frac{-d_F, -1}{\mathbb{Q}}\right) \in \mathrm{Br}_2(\mathbb{Q}), \tag{3.3.3}$$

and, for any place $\ell \le \infty$, let the *local support invariant* of α

$$\mathrm{inv}_\ell(\alpha) \in \mathrm{Br}_2(\mathbb{Q}_\ell) \cong \{\pm 1\}$$

be the image of the global invariant under $\mathrm{Br}_2(\mathbb{Q}) \to \mathrm{Br}_2(\mathbb{Q}_\ell)$. Taking $H = M_2(\mathbb{Q})$ in (3.1.1) and using the calculation (3.1.2), the global support invariant has the alternate characterization

$$\mathrm{hasse}\big(E^\sharp, Q_\alpha\big) = \mathrm{inv}(\alpha) \otimes \mathrm{hasse}\big(W_{M_2(\mathbb{Q})}, \mathrm{Nm}\big).$$

Loosely speaking, $\mathrm{inv}(\alpha)$ measures the disparity between the two quadratic spaces $(E^\sharp, Q_\alpha)$ and $(W_{M_2(\mathbb{Q})}, \mathrm{Nm})$. Define the *modified local support invariant*

$$\mathrm{inv}^*_\ell(\alpha) = \begin{cases} \mathrm{inv}_\ell(\alpha) & \text{if } \ell < \infty \\ -\mathrm{inv}_\ell(\alpha) & \text{if } \ell = \infty \end{cases}$$

and a finite set of places of $\mathbb{Q}$

$$\mathrm{Sppt}(\alpha) = \{\ell \le \infty : \mathrm{inv}^*_\ell(\alpha) = -1\}. \tag{3.3.4}$$

The product formula $\prod_{\ell \le \infty} \mathrm{inv}_\ell(\alpha) = 1$ implies that $\mathrm{Sppt}(\alpha)$ has odd cardinality. The term "support invariant" comes from the fact that the set $\mathrm{Sppt}(\alpha)$ determines the support of the stack $\mathcal{CM}_\Sigma(\alpha)$, as the following proposition makes clear.

Proposition 3.3.8. *Suppose* $\alpha \in (F^\sharp)^\times$.

(1) *If* $\#\mathrm{Sppt}(\alpha) > 1$ *then* $\mathcal{CM}_\Sigma(\alpha) = \emptyset$.
(2) *If* $\mathrm{Sppt}(\alpha) = \{\infty\}$ *then* $\mathcal{CM}_\Sigma(\alpha) = \emptyset$.

(3) *If* $\mathrm{Sppt}(\alpha) = \{p\}$ *with* $p < \infty$ *then every geometric point of* $\mathcal{CM}_\Sigma(\alpha)$ *has characteristic* p.

Proof. Suppose $(\mathbf{A}, j) \in \mathcal{CM}_\Sigma(\alpha)(k)$ is a geometric point. By Proposition 3.3.7 the field k has nonzero characteristic, and $\mathbf{A}$ is supersingular. Moreover, Propositions 3.3.7 and 3.1.3 tell us that there is an isomorphism of $\mathbb{Q}$-quadratic spaces $(E^\sharp, Q_\alpha) \cong (W_H, \mathrm{Nm})$, where H is the rational quaternion algebra of discriminant $p = \mathrm{char}(k)$. Comparing (3.1.2) with (3.3.3) shows that $\mathrm{inv}(\alpha) = H$, and hence $\mathrm{Sppt}(\alpha) = \{p\}$. All claims now follow easily. □

Proposition 3.3.9. *If* $\alpha \in (F^\sharp)^\times$ *then* $\mathcal{CM}_\Sigma(\alpha)$ *has dimension zero.*

Proof. Suppose $\mathfrak{q}$ is a prime of $\mathcal{O}_\Sigma$ and $z \in \mathcal{CM}_\Sigma(\alpha)(\mathbb{F}_\mathfrak{q}^{\mathrm{alg}})$. By [39, Proposition 2.9] and the proof of Proposition 3.3.5, the completed strictly Henselian local ring $\widehat{\mathcal{O}}^{\mathrm{sh}}_{\mathcal{CM}_\Sigma(\alpha),z}$ is a quotient of the complete discrete valuation ring $\widehat{\mathcal{O}}^{\mathrm{sh}}_{\Sigma,\mathfrak{q}}$. The stack $\mathcal{CM}_\Sigma(\alpha)$ has no points in characteristic 0, by Proposition 3.3.7, and so the quotient map $\widehat{\mathcal{O}}^{\mathrm{sh}}_{\Sigma,\mathfrak{q}} \to \widehat{\mathcal{O}}^{\mathrm{sh}}_{\mathcal{CM}_\Sigma(\alpha),z}$ is not an isomorphism. Therefore $\widehat{\mathcal{O}}^{\mathrm{sh}}_{\mathcal{CM}_\Sigma(\alpha),z}$ is Artinian, and the result follows easily. □

3.4. **Complex uniformization and Green functions.** We continue to take $\mathfrak{c} = \mathcal{O}_F$. Let $\mathbf{S} = (S, \kappa_S, \lambda_S)$ be an $\mathcal{O}_F$-polarized RM module (it is unique up to isomorphism, by Proposition 2.2.2), and let Λ_S be the F-symplectic form on $S_\mathbb{Q}$ determined by $\lambda_S = \mathrm{Tr}_{F/\mathbb{Q}} \circ \Lambda_S$. Let

$$G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{Sp}(S_\mathbb{Q}, \Lambda_S)$$

be the associated symplectic group, and set

$$\Gamma = \{\gamma \in G(\mathbb{Q}) : \gamma \cdot S = S\}.$$

A *cusp* of $\mathbf{S}$ is a maximal rational parabolic subgroup of $G_{/\mathbb{Q}^{\mathrm{alg}}}$. Every cusp of $\mathbf{S}$ is the stabilizer of an F-line in $S_\mathbb{Q}$, and the intersection of such a line with S is a projective rank one $\mathcal{O}_F$-direct summand $\mathfrak{a} \subset S$. This establishes a Γ-equivariant bijection between the set of cusps of $\mathbf{S}$ and the set of all such $\mathfrak{a} \subset S$. In particular, to any cusp (which we now fix) we may attach a short exact sequence of projective $\mathcal{O}_F$-modules

$$0 \to \mathfrak{a} \to S \to \mathfrak{b} \to 0.$$

Fix a splitting $S \cong \mathfrak{a} \oplus \mathfrak{b}$ and an F-basis $e_1, e_2 \in S_\mathbb{Q}$ with $e_1 \in \mathfrak{a}_\mathbb{Q}$, $e_2 \in \mathfrak{b}_\mathbb{Q}$, and $\Lambda_S(e_1, e_2) = 1$. These choices identify $S_\mathbb{Q} \cong F \times F$ (column vectors), and identify $\mathfrak{a}$ and $\mathfrak{b}$ with fractional $\mathcal{O}_F$-ideals in such a way that $\mathfrak{ab} = \mathfrak{D}_F^{-1}$. They also identify $G \cong \mathrm{Res}_{F/\mathbb{Q}}\mathrm{SL}_2$ in such a way that the chosen cusp is identified with the subgroup of upper triangular matrices (the "cusp at infinity"). As in the proof of Proposition 2.2.2, every $j \in V(\mathbf{S})$ is now of the form $js = J \cdot s^\sigma$ for some $J \in M_2(F)$, and $j \mapsto J$ defines an isomorphism of quadratic spaces

$$\big(V(\mathbf{S}), Q_\mathbf{S}\big) \cong \big(W_{M_2(\mathbb{Q})}, \det\big).$$

The action of $G(\mathbb{Q})$ on $V(\mathbf{S})$ defined by $g \bullet j = g \circ j \circ g^{-1}$ becomes the action $g \bullet J = g \cdot J \cdot (g^\sigma)^{-1}$ of $\mathrm{SL}_2(F)$ on $W_{M_2(\mathbb{Q})}$.

A *complex structure* on $S_\mathbb{R}$ is an $\mathtt{h} \in G(\mathbb{R})$ satisfying $\mathtt{h}^2 = -1$, and such that the symmetric bilinear form $\lambda_S(\mathtt{h}s_1, s_2)$ on $S_\mathbb{R}$ is positive definite. The set $\mathcal{D}$ of all complex structures carries a natural transitive $G(\mathbb{R})$ action $\gamma * \mathtt{h} = \gamma \mathtt{h} \gamma^{-1}$, and the stabilizer of any point is a maximal compact open subgroup. We can make $\mathcal{D}$ into a complex manifold by constructing explicit coordinates. Let $F_\mathbb{R}^{\gg 0} \subset F_\mathbb{R}$ be the subset of totally positive elements, and define a subset $\mathcal{H}_F \subset F_\mathbb{C}$ by

$$\mathcal{H}_F = F_\mathbb{R} + i \cdot F_\mathbb{R}^{\gg 0}.$$

A choice of isomorphism (which we do not make) $F_\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ identifies $\mathcal{H}_F$ with the product of two copies of the upper half complex plane. Fix a $\delta \in F$ such that $\delta^2 = d_F$. Given any $z = x + iy \in \mathcal{H}_F$ define

$$g_z = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \in G(\mathbb{R})$$

and

$$\mathtt{h}(z) = g_z \cdot \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \cdot g_z^{-1} \in \mathcal{D}.$$

The function $z \mapsto \mathtt{h}(z)$ establishes a bijection $\mathcal{H}_F \cong \mathcal{D}$.

We now construct complex uniformizations of $\mathcal{M}(\mathbb{C})$ and $\mathcal{T}(m)(\mathbb{C})$. Every $\mathtt{h} \in \mathcal{D}$ makes $S_\mathbb{R}$ into a $\mathbb{C}$-vector space in an obvious way ($i \cdot = \mathtt{h}$), and so defines a complex torus $A_\mathtt{h}(\mathbb{C}) = S_\mathbb{R}/S$ equipped with an action of $\mathcal{O}_F$ deduced from κ_S, and an $\mathcal{O}_F$-polarization deduced from λ_S. Thus to each point $\mathtt{h} \in \mathcal{D}$ we may associate an $\mathcal{O}_F$-polarized RM abelian surface $\mathbf{A}_\mathtt{h} = (A_\mathtt{h}, \kappa_{A_\mathtt{h}}, \lambda_{A_\mathtt{h}})$, and the construction $\mathtt{h} \mapsto \mathbf{A}_\mathtt{h}$ determines an isomorphism of complex orbifolds (for orbifolds see [11, Appendix B] and the references therein)

$$[\Gamma \backslash \mathcal{D}] \cong \mathcal{M}(\mathbb{C}).$$

For each $\mathtt{h} \in \mathcal{D}$ set

$$\mathcal{V}_\mathtt{h} = \{ j \in V(\mathbf{S})_\mathbb{R} : \mathtt{h} \circ j = -j \circ \mathtt{h} \}.$$

Given $j \in \mathcal{V}_\mathtt{h}$ and $x \in S$, both nonzero, the calculation

$$0 < \lambda_S(\mathtt{h}jx, jx) = -\lambda_S(j\mathtt{h}x, jx) = -Q_\mathbf{S}(j)\lambda_S(\mathtt{h}x, x)$$

shows that $Q_\mathbf{S}(j) < 0$. Thus the subspace $\mathcal{V}_\mathtt{h}$ is negative definite, and the same argument shows that its orthogonal complement

$$\mathcal{V}_\mathtt{h}^\perp = \{ j \in V(\mathbf{S})_\mathbb{R} : \mathtt{h} \circ j = j \circ \mathtt{h} \}$$

is positive definite. We know from Proposition 2.2.2 that $V(\mathbf{S})_\mathbb{R}$ has signature $(2,2)$, and therefore $\mathcal{V}_\mathtt{h}$ and $\mathcal{V}_\mathtt{h}^\perp$ have signatures $(0,2)$ and $(2,0)$, respectively. Note that $\mathcal{V}_\mathtt{h}^\perp$ is the space of special endomorphisms that are complex linear for the complex structure $\mathtt{h}$ on $S_\mathbb{R}$. For each $j \in V(\mathbf{S})_\mathbb{R}$ define

$$\mathcal{D}(j) = \{ \mathtt{h} \in \mathcal{D} : j \in \mathcal{V}_\mathtt{h}^\perp \},$$

so that for each $\mathtt{h} \in \mathcal{D}(j)$ the complex analytic map $j : S_{\mathbb{R}} \to S_{\mathbb{R}}$ determines a special endomorphism of $\mathbf{A}_{\mathtt{h}}$. As above there is an isomorphism of complex orbifolds

$$\Big[\Gamma\backslash \bigsqcup_{\substack{j\in L(\mathbf{S})\\ Q_{\mathbf{S}}(j)=m}} \mathcal{D}(j)\Big] \cong \mathcal{T}(m)(\mathbb{C}),$$

defined by sending a point $\mathtt{h} \in \mathcal{D}(j)$ to the pair $(\mathbf{A}_{\mathtt{h}}, j)$.

If $j_{\mathtt{h}}$ denotes the orthogonal projection of j to $\mathcal{V}_{\mathtt{h}}$, then

$$R(j, \mathtt{h}) = -Q_{\mathbf{S}}(j_{\mathtt{h}})$$

is a smooth nonnegative function on $\mathcal{D}$ whose zero locus is $\mathcal{D}(j)$. We can use the isomorphism $\mathcal{H}_F \cong \mathcal{D}$ to give an explicit formula for $R(j, \mathtt{h})$. The negative 2-plane $\mathcal{V}_z = \mathcal{V}_{\mathtt{h}(z)}$ in $W_{M_2(\mathbb{Q})} \otimes_{\mathbb{Q}} \mathbb{R}$ is spanned by the orthogonal vectors

$$g_z \bullet \begin{pmatrix} \delta & \\ & -\delta \end{pmatrix} = \frac{\delta}{\sqrt{yy^\sigma}} \begin{pmatrix} y & -xy^\sigma - x^\sigma y \\ & -y^\sigma \end{pmatrix}$$
$$g_z \bullet \begin{pmatrix} & \delta \\ \delta & \end{pmatrix} = \frac{\delta}{\sqrt{yy^\sigma}} \begin{pmatrix} x & yy^\sigma - xx^\sigma \\ 1 & -x^\sigma \end{pmatrix}.$$

If $j \in V(\mathbf{S})$ corresponds to

$$J = \begin{pmatrix} a & \delta b \\ \delta c & a^\sigma \end{pmatrix} \in W_{M_2(\mathbb{Q})}$$

then direct calculation shows that the function $R(j, \mathtt{h})$ on $\mathcal{D}$ is identified with the function

$$R(j, z) = \frac{|d_F czz^\sigma + \delta a^\sigma z - \delta a z^\sigma - d_F b|^2}{4 d_F yy^\sigma}$$

on $\mathcal{H}_F$. In particular, the holomorphic function

$$\Delta(j, z) = d_F czz^\sigma + \delta a^\sigma z - \delta a z^\sigma - d_F b$$

has divisor $\mathcal{D}(j) \subset \mathcal{D}$.

Next we construct a Green function for the divisor $\mathcal{T}(m)$, following ideas of Kudla [21] and Bruinier [3]. Let $\beta_1 : \mathbb{R}^{>0} \to \mathbb{R}$ be the exponential integral

$$\beta_1(r) = \int_1^\infty e^{-ru} u^{-1}\, du$$

of [26, (3.5.2)]. By [21, (11.22)] the function $\beta_1(r) + \log(r)$ extends continuously to $\mathbb{R}^{\geq 0}$, and $\beta_1(r) = O(e^{-r})$ as $r \to \infty$. For each $j \in V(\mathbf{S})$ and positive parameter $\mathbf{v} \in \mathbb{R}$, define a function

$$\mathrm{Gr}(j, \mathbf{v}, \mathtt{h}) = \beta_1(4\pi \mathbf{v} R(j, \mathtt{h}))$$

on $\mathcal{D} \smallsetminus \mathcal{D}(j)$. If $\Delta(j, \mathtt{h})$ is any holomorphic function on $\mathcal{D}$ with divisor $\mathcal{D}(j)$, the calculation of the preceding paragraph shows that

$$\mathrm{Gr}(j, \mathbf{v}, \mathtt{h}) + \log|\Delta(j, \mathtt{h})|^2$$

extends to a smooth function on all of $\mathcal{D}$. Thus $\mathrm{Gr}(j, \mathbf{v}, \mathtt{h})$ is a Green function for the divisor $\mathcal{D}(j)$, in the sense of [12, 43]. For any nonzero $m \in \mathbb{Z}$ the sum (see [3, Section 3] for the proof of convergence)

$$\mathrm{Gr}(m, \mathbf{v}, \mathtt{h}) = \sum_{\substack{j \in L(\mathbf{S}) \\ Q_{\mathbf{S}}(j)=m}} \mathrm{Gr}(j, \mathbf{v}, \mathtt{h})$$

is Γ-invariant, and defines a Green function for the orbifold divisor

$$\Big[\Gamma \backslash \bigsqcup_{\substack{j \in L(\mathbf{S}) \\ Q_{\mathbf{S}}(j)=m}} \mathcal{D}(j)\Big] \to [\Gamma \backslash \mathcal{D}].$$

In other words, $\mathrm{Gr}(m, \mathbf{v}, \mathtt{h})$ is a Green function for the Hirzebruch-Zagier divisor $\mathcal{T}(m)$ on $\mathcal{M}$. In particular, if $m < 0$ then $\mathrm{Gr}(m, \mathbf{v}, \mathtt{h})$ is a smooth function on $\mathcal{M}$. This can also be seen directly from the definition of $\mathrm{Gr}(j, \mathbf{v}, \mathtt{h})$: if $Q_{\mathbf{S}}(j) < 0$ then j cannot be orthogonal to any negative 2-plane, and so $R(j, \mathtt{h}) \neq 0$ for every $\mathtt{h} \in \mathcal{D}$.

Fix a CM type Σ of E. We will now evaluate $\mathrm{Gr}(m, \mathbf{v}, \mathtt{h})$ on the 0-cycle $\mathcal{CM}_\Sigma(\mathbb{C})$. Let X_Σ be the set of isomorphisms classes of $\mathcal{O}_F$-polarized CM modules of CM type Σ, as in Section 2.3. As $\mathbf{S}$ is unique up to isomorphism, to each $\mathbf{T} \in X_\Sigma$ we may attach a Γ-orbit of isomorphisms $\mathbf{T} \cong \mathbf{S}$ of $\mathcal{O}_F$-polarized RM modules. The CM type Σ determines an isomorphism $E_\mathbb{R} \cong \mathbb{C} \times \mathbb{C}$, and so makes $T_\mathbb{R} \cong S_\mathbb{R}$ into a complex vector space. Thus each $\mathbf{T} \in X_\Sigma$ determines a Γ-orbit of complex structures on $\mathbf{S}$, which we denote by $\mathcal{D}_\mathbf{T} \subset \mathcal{D}$. In this way we obtain an orbifold presentation

$$\bigsqcup_{\mathbf{T} \in X_\Sigma} \big[\Gamma \backslash \mathcal{D}_\mathbf{T}\big] \cong \mathcal{CM}_\Sigma(\mathbb{C}).$$

The action of Γ on $\mathcal{D}_\mathbf{T}$ is transitive, and the orbifold on the left is a finite set of points indexed by X_Σ, each with stabilizer $\mu(E)$.

For each $\mathbf{T} \in X_\Sigma$ we now fix an isomorphism $\mathbf{T} \cong \mathbf{S}$ of underlying $\mathcal{O}_F$-polarized RM modules. This singles out a representative $\mathtt{h}_\mathbf{T} \in \mathcal{D}_\mathbf{T}$. Suppose $j \in L(\mathbf{T})$ and set $\alpha = Q^\sharp_\mathbf{T}(j)$. Let $\epsilon^+, \epsilon^- \in F^\sharp_\mathbb{R}$ be the orthogonal idempotents inducing the splitting $F^\sharp_\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$, labeled so that $\epsilon^\pm$ corresponds to the archimedean place $\infty^\pm_\Sigma$ of Proposition 2.3.5. Scalar multiplication by complex numbers (in particular by $i = \mathtt{h}_\mathbf{T}$) commutes with the action of $E_\mathbb{R}$ on $T_\mathbb{R}$, and so the involution $j \mapsto \mathtt{h} \circ j \circ \mathtt{h}^{-1}$ commutes with the $E^\sharp$-module structure on $V(\mathbf{T})$. In particular, the decomposition

$$V(\mathbf{T})_\mathbb{R} = \epsilon^+ V(\mathbf{T})_\mathbb{R} \oplus \epsilon^- V(\mathbf{T})_\mathbb{R}$$

is stable under the involution $j \mapsto \mathtt{h} \circ j \circ \mathtt{h}^{-1}$. By Proposition 2.3.5 the summand $\epsilon^- V(\mathbf{T})_\mathbb{R}$ is a negative 2-plane, and it follows easily that $\mathcal{V}_{\mathtt{h}_\mathbf{T}} = \epsilon^- V(\mathbf{T})_\mathbb{R}$. From this we deduce $j_{\mathtt{h}_\mathbf{T}} = \epsilon^- j$, and

$$R(j, \mathtt{h}_\mathbf{T}) = -Q_\mathbf{T}(\epsilon^- j) = -\epsilon^- Q^\sharp_\mathbf{T}(j) = |\alpha|_{\infty^-_\Sigma}.$$

By Remark 3.2.6 there is a canonical bijection $X_\Sigma \to \mathcal{CM}_\Sigma(\mathbb{C})$, and we at last compute, at least formally, as $\mathcal{CM}_\Sigma(\mathbb{C})$ may contain points lying along the singularities of $\mathrm{Gr}(m, \mathbf{v}, \cdot)$,

$$\begin{aligned}\sum_{P\in\mathcal{CM}_\Sigma(\mathbb{C})} \frac{\mathrm{Gr}(m,\mathbf{v},P)}{\#\mathrm{Aut}(P)} &= \sum_{\mathbf{T}\in X_\Sigma} \frac{\mathrm{Gr}(m,\mathbf{v},\mathtt{h}_\mathbf{T})}{W_E} \\ &= \sum_{\mathbf{T}\in X_\Sigma} \sum_{\substack{j\in L(\mathbf{T})\\ Q_\mathbf{T}(j)=m}} \frac{\beta_1(4\pi \mathbf{v} R(j,\mathtt{h}_\mathbf{T}))}{W_E} \\ &= \sum_{\substack{\alpha\in F^\sharp\\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha)=m}} \beta_1(4\pi|\mathbf{v}\alpha|_{\infty_\Sigma^-}) \sum_{\mathbf{T}\in X_\Sigma} \sum_{\substack{j\in L(\mathbf{T})\\ Q^\sharp_\mathbf{T}(j)=\alpha}} \frac{1}{W_E}. \end{aligned} \tag{3.4.1}$$

This is our provisional formula for the Green function $\mathrm{Gr}(m, \mathbf{v}, \cdot)$ evaluated at $\mathcal{CM}_\Sigma(\mathbb{C})$, and should be viewed as an archimedean counterpart to the stack-theoretic decomposition (3.3.1). Note that as $Q^\sharp_\mathbf{T}$ is positive definite at ∞_Σ^+ and negative definite at ∞_Σ^- by Proposition 2.3.5, the only α that contribute to the sum are those of mixed sign: nonnegative at ∞_Σ^+ and nonpositive at ∞_Σ^-.

In all of the above calculations, we have viewed $\mathbb{C}$ as an $\mathcal{O}_\Sigma$-algebra using the inclusion $E_\Sigma \subset \mathbb{C}$. Now let $\iota : E_\Sigma \to \mathbb{C}$ be an arbitrary embedding, and extend ι to a field automorphism of $\mathbb{C}$. Let $\mathcal{CM}^\iota_\Sigma$ be the pullback of $\mathcal{CM}_\Sigma$ by ι, so that we have the cartesian diagram

$$\begin{array}{ccc} \mathcal{CM}^\iota_\Sigma & \longrightarrow & \mathcal{CM}_\Sigma \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_\Sigma). \end{array}$$

If we let $\mathbb{C}^\iota$ denote the complex numbers, viewed as an $\mathcal{O}_\Sigma$-algebra via $\iota : E_\Sigma \to \mathbb{C}$, then there is a tautological bijection $\mathcal{CM}^\iota_\Sigma(\mathbb{C}) \to \mathcal{CM}_\Sigma(\mathbb{C}^\iota)$. An object of the category $\mathcal{CM}_\Sigma(\mathbb{C}^\iota)$ is $\mathcal{O}_F$-polarized CM abelian surface $\mathbf{A}$ over $\mathbb{C}$ such that the characteristic polynomial of any $x \in \mathcal{O}_E$ acting on $\mathrm{Lie}(A)$ is equal to the image of $c_{\Sigma,x}(T)$ under $\iota : \mathcal{O}_\Sigma[T] \to \mathbb{C}[T]$. But of course this is the same as the image of $c_{\Sigma^\iota,x}(T)$ under the inclusion $\mathcal{O}_\Sigma[T] \to \mathbb{C}[T]$. In other words, the functor $\mathbf{A} \mapsto \mathbf{A}$ is an equivalence of categories $\mathcal{CM}_\Sigma(\mathbb{C}^\iota) \to \mathcal{CM}_{\Sigma^\iota}(\mathbb{C})$, and we have constructed a canonical equivalence

$$\mathcal{CM}^\iota_\Sigma(\mathbb{C}) \cong \mathcal{CM}_{\Sigma^\iota}(\mathbb{C}).$$

Define

$$\mathrm{Gr}(m, \mathbf{v}, \mathcal{CM}_\Sigma) = \sum_\iota \sum_{P\in\mathcal{CM}^\iota_\Sigma(\mathbb{C})} \frac{\mathrm{Gr}(m,\mathbf{v},P)}{\#\mathrm{Aut}(P)} \tag{3.4.2}$$

where the sum is over all embeddings $\iota : E_\Sigma \to \mathbb{C}$. Applying (3.4.1) with Σ replaced by Σ^ι and summing over all ι yields the following proposition.

Proposition 3.4.1. *For any nonzero $m \in \mathbb{Z}$ and any $\mathbf{v} \in \mathbb{R}^{>0}$, we have*

$$\mathrm{Gr}(m, \mathbf{v}, \mathcal{CM}_\Sigma) = \sum_{\iota} \sum_{\substack{\alpha \in F^\sharp \\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha) = m}} \beta_1(4\pi |\mathbf{v}\alpha|_{\infty^-_{\Sigma^\iota}}) \sum_{\mathbf{T} \in X_{\Sigma^\iota}} \sum_{\substack{j \in L(\mathbf{T}) \\ Q^\sharp_{\mathbf{T}}(j) = \alpha}} \frac{1}{W_E}.$$

We allow the possibility that both sides are infinite; this can only happen if $F^\sharp$ is not a field, so that the right hand side includes a term with $|\alpha|_{\infty^-_{\Sigma^\iota}} = 0$.

Proof. This is clear from (3.4.1) and the discussion above. □

4. Eisenstein series

In this section, we review some general facts on Eisenstein series, and construct the Eisenstein series $E(\tau, s, \mathbf{T})$ of the introduction.

For $x \in \mathbb{R}$ set $e(x) = e^{2\pi i x}$. Let $\psi : \mathbb{A} \to \mathbb{C}^\times$ be the unique unramified additive character that is trivial on $\mathbb{Q}$ and satisfies $\psi_\infty(x) = e(x)$. Define an additive character $\psi_{F^\sharp} : \mathbb{A}_{F^\sharp} \to \mathbb{C}^\times$ by

$$\psi_{F^\sharp} = \psi \circ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}.$$

Let $\chi^\sharp : \mathbb{A}_{F^\sharp}^\times \to \mathbb{C}^\times$ be the quadratic Hecke character associated to $E^\sharp/F^\sharp$. Define an algebraic group over $\mathbb{Q}$ by $G = \mathrm{Res}_{F^\sharp/\mathbb{Q}} \mathrm{SL}_2$.

4.1. **General constructions.** Let $\mathcal{H}_{F^\sharp}$ be the $F^\sharp$ upper half-plane of the introduction. For $\tau = \mathbf{u} + i\mathbf{v} \in \mathcal{H}_{F^\sharp}$ set

$$g_\tau = \begin{pmatrix} 1 & \mathbf{u} \\ & 1 \end{pmatrix} \begin{pmatrix} \mathbf{v}^{1/2} & \\ & \mathbf{v}^{-1/2} \end{pmatrix} \in \mathrm{SL}_2(F^\sharp_\mathbb{R})$$

viewed as an element of $\mathrm{SL}(\mathbb{A}_{F^\sharp})$ with trivial nonarchimedean components. For a finite prime $\mathfrak{p}$ of $F^\sharp$ let $K_\mathfrak{p} = \mathrm{SL}_2(\mathcal{O}_{F^\sharp,\mathfrak{p}})$ be the usual maximal compact open subgroup of $\mathrm{SL}_2(F^\sharp_\mathfrak{p})$. If v is an infinite place of $F^\sharp$, let

$$K_v = \{k_\theta : \theta \in \mathbb{R}\} \cong \mathrm{SO}_2(\mathbb{R})$$

be the usual maximal compact subgroup of $\mathrm{SL}_2(F^\sharp_v)$, where

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Let $K = \prod_\mathfrak{p} K_\mathfrak{p}$ (including the infinite primes).

Let

$$I(s, \chi^\sharp) = \bigotimes_v I(s, \chi^\sharp_v) = \mathrm{Ind}_{B(\mathbb{A}_{F^\sharp})}^{\mathrm{SL}_2(\mathbb{A}_{F^\sharp})}(\chi^\sharp \cdot |\ |^s)$$

be the representation of $\mathrm{SL}_2(\mathbb{A}_{F^\sharp})$ induced from the character $\chi^\sharp \cdot |\ |^s$ on $B(\mathbb{A}_{F^\sharp})$. Here $B \subset \mathrm{SL}_2$ is the subgroup of upper triangular matrices. If R is any $F^\sharp$-algebra we define subgroups of $B(R)$ by

$$N(R) = \{n(b) : b \in R\} \qquad M(R) = \{m(a) : a \in R^\times\}$$

where

$$n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

A section (element) in $I(s, \chi^\sharp)$ is a smooth K-finite function Φ on $\mathrm{SL}_2(\mathbb{A}_F^\sharp)$ such that

$$\Phi(n(b)m(a)g, s) = \chi^\sharp(a)|a|^{s+1}\Phi(g, s)$$

for all $a \in \mathbb{A}_{F^\sharp}^\times$, $b \in \mathbb{A}_{F^\sharp}$, and $g \in \mathrm{SL}_2(\mathbb{A}_{F^\sharp})$. A section $\Phi \in I(s, \chi^\sharp)$ is called *standard* if $\Phi|_K$ is independent of s. A section Φ is called *factorizable* if $\Phi = \otimes \Phi_v$ with $\Phi_v \in I(s, \chi^\sharp_v)$. For our purpose, it is sometimes convenient to consider $I(s,\chi^\sharp)$ as a representation of the group $G(\mathbb{A})$. In this view, there is a factorization

$$I(s,\chi^\sharp) = \bigotimes_p I(s,\chi_p^\sharp) \qquad I(s,\chi_p^\sharp) = \bigotimes_{v|p} I(s,\chi_v^\sharp).$$

A section $\Phi \in I(s,\chi^\sharp)$ is called $\mathbb{Q}$-factorizable if $\Phi = \otimes\Phi_p$ with $\Phi_p \in I(s,\chi_p^\sharp)$.

Associated to a standard section Φ is an Eisenstein series on $\mathrm{SL}_2(\mathbb{A}_{F^\sharp})$

$$E(g,s,\Phi) = \sum_{\gamma\in B(F^\sharp)\backslash \mathrm{SL}_2(F^\sharp)} \Phi(\gamma g, s). \tag{4.1.1}$$

According to the general theory of Langlands on Eisenstein series [33], the summation defining $E(g,s,\Phi)$ is absolutely convergent when $\mathrm{Re}(s)$ is sufficiently large, and has meromorphic continuation to the whole complex plane with finitely many poles. The meromorphic continuation is holomorphic along the unitary axis $\mathrm{Re}(s)=0$ and satisfies a functional equation in $s\mapsto -s$. Furthermore, there is a Fourier expansion

$$E(g,s,\Phi) = \sum_{\alpha\in F^\sharp} E_\alpha(g,s,\Phi)$$

where

$$E_\alpha(g,s,\Phi) = \int_{F^\sharp\backslash\mathbb{A}_{F^\sharp}} E(n(b)g,s,\Phi)\cdot \psi_{F^\sharp}(-b\alpha)\, db.$$

Here db is the Haar measure on $F^\sharp\backslash\mathbb{A}_{F^\sharp}$ self-dual with respect to $\psi_{F^\sharp}$. If $\Phi = \otimes\Phi_p$ is $\mathbb{Q}$-factorizable and $\alpha\in (F^\sharp)^\times$, there is a factorization

$$E_\alpha(g,s,\Phi) = \prod_p W_{\alpha,p}(g_p,s,\Phi_p) \tag{4.1.2}$$

in which

$$W_{\alpha,p}(g_p,s,\Phi_p) = \int_{F_p^\sharp} \Phi_p(w^{-1}n(b)g_p,s)\cdot\psi_{F_p^\sharp}(-\alpha b)\, db,$$

and $w = \left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)$. If $\Phi=\otimes\Phi_v$ is factorizable over the places of $F^\sharp$, then

$$W_{\alpha,p}(g_p,s,\Phi_p) = \prod_{v|p} W_{\alpha,v}(g_v,s,\Phi_v)$$

and

$$E_\alpha(g,s,\Phi) = \prod_v W_{\alpha,v}(g_v,s,\Phi_v).$$

Here $W_{\alpha,v}(\cdot)$ is defined the same way as $W_{\alpha,p}(\cdot)$ but with $F_p^\sharp$ replaced by $F_v^\sharp$.

When v is an infinite prime of $F^\sharp$ the compact group $K_v\cong \mathrm{SO}_2(\mathbb{R})$ is abelian with characters $k_\theta\mapsto e^{i\ell\theta}$ indexed by $\ell\in\mathbb{Z}$. Using the decomposition

$$\mathrm{SL}_2(F_v^\sharp) = B(F_v^\sharp)\cdot K_v$$

and the fact that the character $\chi^\sharp$ is odd, it follows that

$$I(s,\chi_v^\sharp) = \bigoplus_{\ell \text{ odd}} \mathbb{C}\cdot\Phi_v^\ell, \tag{4.1.3}$$

where $\Phi_v^\ell \in I(s,\chi_v^\sharp)$ is the unique standard section whose restriction to K_v satisfies

$$\Phi_v^\ell(k_\theta) = e^{i\ell\theta}.$$

Write $\Phi_\infty^\ell = \prod_{v|\infty}\Phi_v^\ell \in I(s,\chi_\infty^\sharp)$. For a standard section

$$\Phi_f \in \bigotimes_{v\nmid\infty} I(s,\chi_v^\sharp)$$

the function ($\tau = \mathbf{u}+i\mathbf{v}\in\mathcal{H}_{F^\sharp}$)

$$E(\tau,s,\Phi_f\otimes\Phi_\infty^\ell) = \mathrm{Norm}_{F^\sharp/\mathbb{Q}}(\mathbf{v})^{-\ell/2}\cdot E(g_\tau,s,\Phi_f\otimes\Phi_\infty^\ell)$$

is a non-holomorphic Hilbert modular form of parallel weight ℓ. Given a prime $p<\infty$ of $\mathbb{Q}$ abbreviate

$$\begin{aligned} W_{\alpha,p}(s,\Phi_p) &= W_{\alpha,p}(1,s,\Phi_p) \\ W_{\alpha,\infty}(\tau,s,\Phi_\infty^\ell) &= \mathrm{Norm}_{F^\sharp/\mathbb{Q}}(\mathbf{v})^{-\ell/2}\cdot W_{\alpha,\infty}(g_\tau,s,\Phi_\infty^\ell). \end{aligned}$$

4.2. **The Weil representation.** There is a systematic way to construct standard section in the induced representation using quadratic spaces and Weil representations. We review some basic facts about the construction in this section, and compute the associated local Whittaker functions in Section 4.6. We will use slightly different notation in this subsection and Section 4.6. Let $\mathcal{F}$ be a local field of characteristic not equal to 2, and let ψ be a non-trivial additive character of $\mathcal{F}$. Let $V=(V,Q)$ be a non-degenerate quadratic space over $\mathcal{F}$ of even dimension $2m$. The reductive dual pair $(\mathrm{SL}_2(\mathcal{F}),O(V))$ gives a Weil representation $\omega_{V,\psi}$ of $\mathrm{SL}_2(\mathcal{F})$ on $S(V)$, the space of Schwartz functions on V, which is determined by the formulas

$$\begin{aligned} \omega_{V,\psi}(n(b))\phi(x) &= \psi(bQ(x))\phi(x) \\ \omega_{V,\psi}(m(a))\phi(x) &= \chi_V(a)|a|^m\phi(xa) \\ \omega_{V,\psi}(w^{-1})\phi(x) &= \gamma(V)\int_V \phi(y)\psi(-[x,y])\,d_Vy \end{aligned} \tag{4.2.1}$$

for $\phi\in S(V)$, $a\in\mathcal{F}^\times$, and $b\in\mathcal{F}$. Here $\chi_V(a) = ((-1)^m\det V,a)_{\mathcal{F}}$ is the quadratic character of $\mathcal{F}^\times$ associated to V, $[x,y]$ is the bilinear form on V defined by $[x,y]=Q(x+y)-Q(x)-Q(y)$, and

$$w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right).$$

Finally $\gamma(V)=\gamma(V,\psi)$ is the local splitting index defined in [19, Theorem 3.1]. See also [20, I.3], where $\gamma(V,\psi)$ is denoted $\gamma(\psi\circ Q)$. We refer to [19], [20], and [37] for basics on Weil representations. Define also

$$n_-(b) = \left(\begin{smallmatrix} 1 & 0 \\ b & 1\end{smallmatrix}\right).$$

Lemma 4.2.1. *Let V_1 and V_2 be two quadratic spaces over $\mathcal{F}$ of the same dimension, same quadratic character, but different Hasse invariants. Then*

$$\gamma(V_1) = -\gamma(V_2).$$

Proof. This is a direct calculation either from from the definitions, or from [20, Lemma I.4.2]. □

For $s_0 = m - 1$ the map

$$\lambda = \lambda_{V,\psi} : S(V) \to I(s_0, \chi_V)$$

defined by $\lambda(\phi)(g) = \omega_{V,\psi}(g)\phi(0)$ is $\mathrm{SL}_2(\mathcal{F})$-equivariant. Given a Schwartz function $\phi \in S(V)$, let $\Phi(g, s) \in I(s, \chi_V)$ be the unique standard section satisfying

$$\Phi(g, s_0) = \lambda(\phi).$$

Concretely, if we factor $g = n(b)m(a)k$ with $b \in \mathcal{F}$, $a \in \mathcal{F}^\times$, and $k \in \mathrm{SL}_2(\mathcal{O}_\mathcal{F})$ (when $\mathcal{F}$ is p-adic), $k \in \mathrm{SO}_2(\mathbb{R})$ (when $\mathcal{F} = \mathbb{R}$), or $k \in U(2)$ (when $\mathcal{F} = \mathbb{C}$), then

$$\Phi(g, s_0 + s) = \lambda(\phi)|a|^s.$$

Define the Whittaker function

$$W_\alpha(g, s, \Phi) = \int_\mathcal{F} \Phi(w^{-1}n(b)g, s) \cdot \psi(-\alpha b)\, db.$$

Here db is the Haar measure on $\mathcal{F}$ self-dual relative to ψ. The Whittaker function depends on ψ, and we will write $W^\psi_\alpha(\ldots)$ when we wish to emphasize this dependence.

For $a \in \mathcal{F}^\times$ we denote by $V^a = (V, aQ)$ the vector space V with the new quadratic form $x \mapsto aQ(x)$, and denote by $a\psi$ the additive character $a\psi(x) = \psi(ax)$. The following follows from [20, Corollary 6.1] and can also be checked easily from the explicit formula of the Weil representation.

Lemma 4.2.2. *Fix $a \in \mathcal{F}^\times$.*

1. *One has $\omega_{V^a,\psi} = \omega_{V,a\psi}$.*
2. *If $\phi \in S(V)$ and $\phi^a \in S(V^a)$ is the image of ϕ under the identification $S(V) = S(V^a)$, then*

$$\lambda_{V^a,\psi}(\phi^a) = \lambda_{V,a\psi}(\phi) \in I(s_0, \chi_V).$$

Moreover, the associated Whittaker functions are related by

$$W^{a\psi}_\alpha(g, s, \lambda_{V,a\psi}(\phi)) = |a|^{\frac{1}{2}} W^\psi_{\alpha a}(g, s, \lambda_{V^a,\psi}(\phi^a)).$$

We end this subsection with a well-known fact (see for example [42, Lemma 1.2]). Note that in [51] the factor of 2 is mistakenly omitted in definition of the function ϕ of the following lemma. Recall that $\psi_\infty : \mathbb{R} \to \mathbb{C}^\times$ is the additive character $\psi_\infty(x) = e(x)$.

Lemma 4.2.3. *Let (V,Q) be a quadratic space over $\mathcal{F}=\mathbb{R}$ of signature (p,q), and let $\chi_V=((-1)^{\frac{(p-q)(p-q-1)}{2}},\cdot)_{\mathbb{R}}$ be the associated quadratic character of $\mathbb{R}^\times$. Fix an orthogonal decomposition $V=V_+\oplus V_-$ where $V_\pm$ are positive (negative) definite subspace of V, and write $x=x_++x_-$ according to the decomposition. Let*

$$\phi(x)=e^{-2\pi(Q(x_+)-Q(x_-))}\in S(V).$$

Then ϕ is an eigenfunction of $(\mathrm{SO}_2(\mathbb{R}),\omega_{V,\psi_\infty})$ with eigencharacter $k_\theta\mapsto e^{\frac{p-q}{2}i\theta}$. In particular, $\lambda(\phi)=\Phi_{\mathbb{R}}^{\frac{p-q}{2}}\in I(s,\chi_V)$ is the normalized weight $\frac{p-q}{2}$ vector in the decomposition (4.1.3).

4.3. **Coherent and incoherent Eisenstein series.** In this section we review some basic facts about coherent and incoherent Eisenstein series. The reader is referred to [21], [23] and [25], and the references therein for more details.

Given a place v of $F^\sharp$ let $\mathcal{C}_v$ be a binary $F^\sharp_v$-quadratic space whose character

$$\chi_{\mathcal{C}_v}(x)=(-\det(\mathcal{C}_v),x)_v$$

is equal to $\chi^\sharp_v$. Let $\mathrm{hasse}(\mathcal{C}_v)$ be the Hasse invariant of $\mathcal{C}_v$. The reductive dual pair $\big(\mathrm{SL}_2(F^\sharp_v),\mathrm{O}(\mathcal{C}_v)\big)$ determines a Weil representation $\omega_v=\omega_{\mathcal{C}_v,\psi^\sharp_v}$ of $\mathrm{SL}_2(F^\sharp_v)$ on the space of Schwartz functions $S(\mathcal{C}_v)$. Moreover, the Weil representation provides an $\mathrm{SL}_2(F^\sharp_v)$-equivariant map

$$\lambda_v:S(\mathcal{C}_v)\to I(0,\chi^\sharp_v)$$

defined by

$$\lambda_v(\phi)(g)=\big(\omega_v(g)\phi\big)(0).$$

Let $R(\mathcal{C}_v)$ be the image of λ_v. It is a beautiful fact [21] that

$$I(0,\chi^\sharp_v)=\bigoplus_{\mathcal{C}_v}R(\mathcal{C}_v) \tag{4.3.1}$$

where the direct sum is over all binary $F^\sharp_v$-quadratic spaces $\mathcal{C}_v$ of character $\chi^\sharp_v$.

Suppose that $\mathcal{C}=\prod_v\mathcal{C}_v$ is a free $\mathbb{A}_{F^\sharp}$-module of rank two equipped with an $\mathbb{A}_{F^\sharp}$-quadratic form whose character

$$\chi_{\mathcal{C}}(x)=\prod_v(-\det(\mathcal{C}),x)_v$$

is equal to $\chi^\sharp$. If

$$\prod_v\mathrm{hasse}(\mathcal{C}_v)=1$$

then there is a global $F^\sharp$-quadratic space V such that $\mathcal{C}\cong V\otimes_{F^\sharp}\mathbb{A}_{F^\sharp}$, and when this is the case we say that $\mathcal{C}$ is *coherent*. If

$$\prod_v\mathrm{hasse}(\mathcal{C}_v)=-1$$

then no such V exists, and we say that $\mathcal{C}$ is *incoherent*. We will usually think of $\mathcal{C}$ as a collection of local quadratic spaces $\{\mathcal{C}_v\}$ rather than as a quadratic space over the adele ring $\mathbb{A}_{F^\sharp}$. It follows from (4.3.1) that there is a global decomposition

$$I(0,\chi^\sharp) = \left(\bigoplus_{\mathcal{C} \text{ coherent}} R(\mathcal{C}) \right) \oplus \left(\bigoplus_{\mathcal{C} \text{ incoherent}} R(\mathcal{C}) \right),$$

where

$$R(\mathcal{C}) = \bigotimes_v R(\mathcal{C}_v).$$

Remark 4.3.1. Pick any $\widehat{\mathcal{O}}_{F^\sharp}$-lattice $L \subset \widehat{\mathcal{C}} = \prod_{v \nmid \infty} \mathcal{C}_v$, and for a finite place v of $F^\sharp$ let L_v be the component of L at v. The tensor product defining $R(\mathcal{C})$ is understood to mean the restricted tensor product of the $R(\mathcal{C}_v)$ with respect to the vectors $\lambda_v(\mathrm{char}(L_v))$, where $\mathrm{char}(L_v) \in S(\mathcal{C}_v)$ is the characteristic function of L_v.

Associated to any $\Phi(g) \in I(0,\chi^\sharp)$ there is a unique standard section

$$\Phi(g,s) \in I(s,\chi^\sharp)$$

such that $\Phi(g,0) = \Phi(g)$. The formation of Eisenstein series (4.1.1) gives an $\mathrm{SL}_2(\mathbb{A}_{F^\sharp})$-equivariant map

$$\mathrm{Eis} : I(0,\chi^\sharp) \to \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \tag{4.3.2}$$

defined by

$$\Phi \mapsto E(g,0,\Phi),$$

where $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is the space of automorphic forms on G. Kudla showed in [21] that

$$\ker(\mathrm{Eis}) = \bigoplus_{\mathcal{C} \text{ incoherent}} R(\mathcal{C}).$$

Given a coherent $\mathbb{A}_{F^\sharp}$-quadratic space $\mathcal{C}$ as above and a

$$\phi \in S(\mathcal{C}) = \bigotimes_v S(\mathcal{C}_v)$$

(the tensor product on the right is the restricted tensor product with respect to the characteristic functions of lattices $L_v \subset \mathcal{C}_v$ as in Remark 4.3.1) the Eisenstein series $E(g,0,\lambda(\phi))$ is related to a theta integral by the Siegel-Weil formula. Given an incoherent collection $\mathcal{C}$ and a $\phi \in S(\mathcal{C})$ the Eisenstein series $E(g,s,\lambda(\phi))$ vanishes at $s=0$ by (4.3.2). One of the primary goals of Kudla's program on arithmetic Siegel-Weil formulas [23] is to understand the derivative of $E(g,s,\lambda(\phi))$ at $s=0$.

Given an incoherent collection $\mathcal{C}$ of local binary quadratic spaces as above, and an $\alpha \in (F^\sharp)^\times$, define $\mathrm{Diff}(\alpha,\mathcal{C})$ to be the set of all places v of $F^\sharp$ for which the quadratic space $\mathcal{C}_v$ does not represent α. Equivalently

$$\mathrm{Diff}(\alpha,\mathcal{C}) = \{\text{places } v \text{ of } F^\sharp : \chi_v^\sharp(\alpha) \neq \mathrm{hasse}(\mathcal{C}_v)\}.$$

From the second description it is clear that $\mathrm{Diff}(\alpha, \mathcal{C})$ is a finite set of odd cardinality. When $\phi = \otimes\phi_v \in S(\mathcal{C})$ is factorizable,

$$v \in \mathrm{Diff}(\alpha, \mathcal{C}) \implies W_{\alpha,v}(g, 0, \phi_v) = 0. \tag{4.3.3}$$

See [21]. Similarly, if $\phi = \otimes\phi_p \in S(\mathcal{C})$ is $\mathbb{Q}$-factorizable then

$$W_{\alpha,p}(g, s, \phi_p) = 0 \tag{4.3.4}$$

for every place $p \leq \infty$ of $\mathbb{Q}$ such that $\mathrm{Diff}(\alpha, \mathcal{C})$ contains a place above p.

4.4. **CM abelian surfaces and coherent Eisenstein series.** Let $\mathfrak{c} \supset \mathcal{O}_F$ be a fractional $\mathcal{O}_F$-ideal, and fix a prime p. Fix a CM type Σ of E, let $\mathfrak{q}$ be a prime of $\mathcal{O}_\Sigma$, and let $\mathbf{A}$ be a $\mathfrak{c}$-polarized supersingular CM abelian surface over $\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}$. As in Section 3.2, the space of special endomorphisms $L(\mathbf{A})$ is a $\mathbb{Z}$-lattice in the free rank two $F^\sharp$-module $V(\mathbf{A})$, which is equipped with the totally positive definite $F^\sharp$-quadratic form $Q^\sharp_{\mathbf{A}}$ of character $\chi_{V(\mathbf{A})} = \chi^\sharp$. Let

$$\phi_{\mathbf{A}} = \mathrm{char}(\widehat{L}(\mathbf{A}))$$

denote the characteristic function of $\widehat{L}(\mathbf{A}) \subset \widehat{V}(\mathbf{A})$. Lemma 4.2.3 implies that the Eisenstein series

$$E(\tau, s, \mathbf{A}) = E(\tau, s, \widehat{\lambda}(\phi_{\mathbf{A}}) \otimes \Phi^1_\infty)$$

associated to the coherent collection $V(\mathbf{A}) \otimes_{\mathbb{Q}} \mathbb{A}$ is a Hilbert modular Eisenstein series of weight 1. Moreover, $E(\tau, 0, \mathbf{A})$ is holomorphic. We write $W_{\alpha,p}(g, s, \mathbf{A})$ for the associated local Whittaker functions, and

$$W_{\alpha,p}(s, \mathbf{A}) = W_{\alpha,p}(1, s, \mathbf{A}).$$

Proposition 4.4.1. *Recall the group $C_0(E)$ defined in Section 2.4. For any totally positive $\alpha \in (F^\sharp)^\times$*

$$\sum_{\mathbf{Z} \in C_0(E)} \sum_{\substack{j \in L(\mathbf{A} \otimes \mathbf{Z}) \\ Q^\sharp_{\mathbf{A}\otimes\mathbf{Z}}(j) = \alpha}} \frac{1}{\#\mathrm{Aut}(\mathbf{A} \otimes \mathbf{Z})} \cdot q^\alpha = \frac{\#C_0(E)}{2W_E} \cdot E_\alpha(\tau, 0, \mathbf{A}).$$

Proof. Recall the homomorphism $\nu_E : T_E \to S_E$ of algebraic groups of Section 2.4. Abbreviate $[S_E] = [S_E(\mathbb{Q})\backslash S_E(\mathbb{A})]$ and fix a Haar measure on $[S_E]$. For any function f on

$$[S_E]/S_E(\mathbb{R})\nu_E(U_E) \cong S_E(\mathbb{Q})\backslash S_E(\mathbb{A}_f)/\nu_E(U_E),$$

the isomorphisms

$$C_0(E) \cong T_E(\mathbb{Q})\backslash T_E(\mathbb{A}_f)/U_E \xrightarrow{\nu_E} S_E(\mathbb{Q})\backslash S_E(\mathbb{A}_f)/\nu_E(U_E)$$

imply that

$$\int_{[S_E]} f(h)dh = \frac{\mathrm{Vol}([S_E])}{\#C_0(E)} \sum_{t \in C_0(E)} f(\nu_E(t)). \tag{4.4.1}$$

For an infinite prime v of $F^\sharp$, define an $S_E(F^\sharp_v)$-invariant function on $V_v(\mathbf{A})$ by

$$\phi_v(x) = e^{-2\pi Q^\sharp_{\mathbf{A}}(x)},$$

and recall from Lemma 4.2.3 that $\lambda_v(\phi_v) = \Phi^1_v$. Let $\phi_\infty = \otimes_{v|\infty}\phi_v$. For $\tau = \mathbf{u} + i\mathbf{v} \in \mathcal{H}_{F^\sharp}$ and $h \in S_E(\mathbb{A})$ let

$$\theta(\tau, h, \mathbf{A}) = \mathrm{Norm}_{F^\sharp/\mathbb{Q}}(\mathbf{v})^{-\frac{1}{2}} \sum_{x \in V(\mathbf{A})} \big(\omega_{V(\mathbf{A})}(g_\tau)\phi_\infty\big)(h_\infty^{-1}x) \cdot \phi_{\mathbf{A}}(h_f^{-1}x)$$

be the theta kernel on $\mathcal{H}_{F^\sharp} \times S_E(\mathbb{A})$. Simple calculation using (4.2.1) shows

$$\theta(\tau, h, \mathbf{A}) = \sum_{x \in V(\mathbf{A})} \phi_{\mathbf{A}}(h_f^{-1}x) \cdot q^{Q^\sharp_{\mathbf{A}}(x)}.$$

Let

$$I(\tau, \mathbf{A}) = \int_{[S_E]} \theta(\tau, h, \mathbf{A})\, dh$$

be the theta integral. Then $I(\tau, \mathbf{A})$ is a Hilbert modular form of weight 1, and the Siegel-Weil formula of [22, Theorem 4.1] asserts that

$$I(\tau, \mathbf{A}) = \frac{\mathrm{Vol}([S_E])}{2} E(\tau, 0, \mathbf{A}). \tag{4.4.2}$$

Note that $\mathrm{Vol}([S_E]) = 2$, in the normalization of [22, Theorem 4.1].

Next, using (4.4.1), the α^{th} Fourier coefficient of $I(\tau, \mathbf{A})$ is given by

$$\begin{aligned} I_\alpha(\tau, \mathbf{A}) \cdot q^{-\alpha} &= \int_{[S_E]} \sum_{\substack{j \in V(\mathbf{A}) \\ Q^\sharp_{\mathbf{A}}(j) = \alpha}} \phi_{\mathbf{A}}(h_f^{-1}j)\, dh \\ &= \frac{\mathrm{Vol}([S_E])}{\#C_0(E)} \sum_{t \in C_0(E)} \sum_{\substack{j \in V(\mathbf{A}) \\ Q^\sharp_{\mathbf{A}}(j) = \alpha}} \phi_{\mathbf{A}}(\nu_E(t)^{-1}j) \\ &= \frac{\mathrm{Vol}([S_E])}{\#C_0(E)} \sum_{\mathbf{Z} \in C_0(E)} \sum_{\substack{j \in V(\mathbf{A}\otimes\mathbf{Z}) \\ Q^\sharp_{\mathbf{A}\otimes\mathbf{Z}}(j) = \alpha}} \phi_{\mathbf{A}\otimes\mathbf{Z}}(j) \end{aligned}$$

where the final equality follows from Proposition 3.2.5. Combining this with Lemma 3.2.3 shows that

$$I_\alpha(\tau, \mathbf{A}) \cdot q^{-\alpha} = \frac{\mathrm{Vol}([S_E]) \cdot W_E}{\#C_0(E)} \sum_{\mathbf{Z} \in C_0(E)} \sum_{\substack{j \in L(\mathbf{A}\otimes\mathbf{Z}) \\ Q^\sharp_{\mathbf{A}\otimes\mathbf{Z}}(j) = \alpha}} \frac{1}{\#\mathrm{Aut}(\mathbf{A}\otimes\mathbf{Z})}. \tag{4.4.3}$$

Combining (4.4.3) with (4.4.2) completes the proof. $\square$

4.5. **CM modules and (in)coherent Eisenstein series.** Fix a fractional $\mathcal{O}_F$-ideal $\mathfrak{c} \supset \mathcal{O}_F$, and let $\mathbf{T}$ be a $\mathfrak{c}$-polarized CM module as in Section 2.3. Let $L(\mathbf{T})$ be the space of special endomorphisms and recall that the rank one $E^\sharp$-module $V(\mathbf{T}) = L(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is equipped with the $F^\sharp$-quadratic form $Q^\sharp_{\mathbf{T}}$. By Proposition 2.3.5, the $F^\sharp$-quadratic space $(V(\mathbf{T}), Q^\sharp_{\mathbf{T}})$ has signature $(2,0)$ at one infinite place of $F^\sharp$, and has signature $(0,2)$ at the other infinite place. Let $\mathcal{C}(\mathbf{T}) = \prod \mathcal{C}_v(\mathbf{T})$ be the incoherent binary quadratic space over $\mathbb{A}_{F^\sharp}$ obtained from $V(\mathbf{T})$ by replacing the quadratic space $V_v(\mathbf{T})$ at the infinite place v at which $V_v(\mathbf{T})$ has signature $(0,2)$ by a quadratic space of signature $(2,0)$. In other words, for every place v of $F^\sharp$ we let $\mathcal{C}_v(\mathbf{T}) = V_v(\mathbf{T})$ as an $F^\sharp_v$-module; if $v \nmid \infty$ we give $\mathcal{C}_v$ the quadratic form $Q^\sharp_{\mathbf{T}}$, and if $v \mid \infty$ we give $\mathcal{C}_v$ the positive definite quadratic form. If we set

$$\widehat{\mathcal{C}}(\mathbf{T}) = \prod_{v \nmid \infty} \mathcal{C}_v(\mathbf{T})$$

then $\widehat{L}(\mathbf{T})$ can be viewed as a $\widehat{\mathbb{Z}}$-lattice in the $\widehat{F}^\sharp$-module $\widehat{\mathcal{C}}(\mathbf{T})$. Define an incoherent Eisenstein series associated to the incoherent collection $\mathcal{C}(\mathbf{T})$ by

$$E(\tau, s, \mathbf{T}) = E(\tau, s, \widehat{\lambda}(\phi_{\mathbf{T}}) \otimes \Phi^1_\infty)$$

where

$$\phi_{\mathbf{T}} = \mathrm{char}(\widehat{L}(\mathbf{T})) \in S(\widehat{\mathcal{C}}(\mathbf{T}))$$

is the characteristic function of $\widehat{L}(\mathbf{T})$. Lemma 4.2.3 implies that $E(\tau, s, \mathbf{T})$ is a Hilbert modular form of weight 1, and the incoherence of $\mathcal{C}(\mathbf{T})$ implies that $E(\tau, 0, \mathbf{T}) = 0$. We write $W_{\alpha,p}(g, s, \mathbf{T})$ for the associated local Whittaker functions, and

$$W_{\alpha,p}(s, \mathbf{T}) = W_{\alpha,p}(1, s, \mathbf{T}).$$

Remark 4.5.1. The section $\widehat{\lambda}(\phi_{\mathbf{T}}) \in I(s, \chi^\sharp_f)$ is $\mathbb{Q}$-factorizable, but as the $\widehat{\mathbb{Z}}$-lattice $\widehat{L}(\mathbf{T}) \subset \widehat{\mathcal{C}}(\mathbf{T})$ need not be stable under the action of $\widehat{\mathcal{O}}_{F^\sharp}$, the section $\widehat{\lambda}(\phi_{\mathbf{T}})$ need not admit a factorization over the finite places of $F^\sharp$.

For $\alpha \in (F^\sharp)^\times$ abbreviate $\mathrm{Diff}(\alpha, \mathbf{T}) = \mathrm{Diff}(\alpha, \mathcal{C}(\mathbf{T}))$. If α is totally positive then $v \in \mathrm{Diff}(\alpha, \mathbf{T})$ if and only if v is finite and $V_v(\mathbf{T})$ does not represent α.

Lemma 4.5.2. *Suppose $\alpha \in (F^\sharp)^\times$. Every place $v \in \mathrm{Diff}(\alpha, \mathbf{T})$ is nonsplit in $E^\sharp$. Furthermore, if $p \in \mathrm{Sppt}(\alpha)$ then $\mathrm{Diff}(\alpha, \mathbf{T})$ contains a unique prime above p.*

Proof. The first claim is clear: if $v \in \mathrm{Diff}(\alpha, \mathbf{T})$ then $(V_v(\mathbf{T}), Q^\sharp_{\mathbf{T}})$ does not represent α. By (2.3.4) this requires that v is nonsplit in $E^\sharp$. Now suppose $p \in \mathrm{Sppt}(\alpha)$. We first prove that there is at least one prime of $F^\sharp$ above p which belongs to $\mathrm{Diff}(\alpha, \mathbf{T})$. Suppose not. Then the $F^\sharp_p$-quadratic space

$V_p(\mathbf{T})$ represents α, and it follows from (2.3.4) that there is an isomorphism of $F^\sharp_p$-quadratic spaces

$$(V_p(\mathbf{T}), Q^\sharp_{\mathbf{T}}) \cong (E^\sharp_p, \alpha x x^\dagger).$$

This implies that there is an isomorphism of $\mathbb{Q}_p$-quadratic spaces

$$(V_p(\mathbf{T}), Q_{\mathbf{T}}) \cong (E^\sharp_p, \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha x x^\dagger)).$$

Comparing Proposition 2.2.2 with (3.3.3) shows that $\mathrm{inv}_p(\alpha) = 1$, contradicting $p \in \mathrm{Sppt}(\alpha)$.

Now we prove that $\mathrm{Diff}(\alpha, \mathbf{T})$ contains exactly one prime above p. We may assume that $p\mathcal{O}_{F^\sharp} = \mathfrak{p}_1\mathfrak{p}_2$ is split in $F^\sharp$, and so

$$F^\sharp_p \cong F^\sharp_{\mathfrak{p}_1} \times F^\sharp_{\mathfrak{p}_2} \cong \mathbb{Q}_p \times \mathbb{Q}_p.$$

The idempotents on the right induce a decomposition of $E^\sharp_p$ as a product of $\mathbb{Q}_p$-algebras $E^\sharp_p \cong E_1 \times E_2$. For $\beta = (\beta_1, \beta_2) \in F^\sharp_p$ with $\beta_i \neq 0$, define an $F^\sharp_p$-quadratic form

$$Q^\sharp_\beta(z) = \beta z z^\dagger = (\beta_1 z_1 z_1^\dagger, \beta_2 z_2 z_2^\dagger)$$

on $E^\sharp_p$, and a $\mathbb{Q}_p$-quadratic form on the same space $Q_\beta(z) = \beta_1 z_1 z_1^\dagger + \beta_2 z_2 z_2^\dagger$. If d_i is the discriminant of E_i then a simple Hilbert symbol calculation shows

$$\mathrm{hasse}(E^\sharp_p, Q_\beta) = (\beta_1, d_1)_p \cdot (\beta_2, d_2)_p \cdot (-d_1, -d_2)_p.$$

Recalling (3.3.3), (2.3.4), and Proposition 2.2.2

$$-\mathrm{hasse}(E^\sharp_p, Q_\alpha) = (-d_F, -1)_p = \mathrm{hasse}(E^\sharp_p, Q_{\beta(\mathbf{T})}).$$

Writing $\beta(\mathbf{T}) = (\beta_1, \beta_2)$, we find

$$(\beta_1, d_1)_p \cdot (\beta_2, d_2)_p \cdot (-d_1, -d_2)_p \neq (\alpha_1, d_1)_p \cdot (\alpha_2, d_2)_p \cdot (-d_1, -d_2)_p.$$

Thus

$$(\beta_1, d_1)_p \neq (\alpha_1, d_1)_p \quad \text{or} \quad (\beta_2, d_2)_p \neq (\alpha_2, d_2)_p$$

but not both. It follows that α is represented by $(E^\sharp_p, \beta(\mathbf{T})$ locally at exactly one of $\mathfrak{p}_1$ and $\mathfrak{p}_2$, and so either $\mathfrak{p}_1 \in \mathrm{Diff}(\alpha, \mathbf{T})$ or $\mathfrak{p}_2 \in \mathrm{Diff}(\alpha, \mathbf{T})$, but not both. □

Corollary 4.5.3. *If $\alpha \in (F^\sharp)^\times$ is totally positive and $\#\mathrm{Sppt}(\alpha) > 1$, then*

$$E'_\alpha(g, 0, \mathbf{T}) = 0.$$

Proof. By Lemma 4.5.2, for each $p \in \mathrm{Sppt}(\alpha)$ there is a prime above p in $\mathrm{Diff}(\alpha, \mathbf{T})$. Therefore (4.3.4) and the factorization (4.1.2) imply that $E_\alpha(g, s, \mathbf{T})$ vanishes to order at least two. □

Associated to a polarized CM module $\mathbf{T}$ of CM type Σ, there is also a coherent Eisenstein series $E(\tau, s, V(\mathbf{T}))$ that will be needed in the proof of Lemma 5.3.2. Let $\infty^\pm = \infty^\pm_\Sigma$ be two archimedean places of $F^\sharp$, as in Proposition 2.3.5, so that $(V(\mathbf{T}), Q^\sharp_{\mathbf{T}})$ has signature $(2, 0)$ at ∞^+ and signature $(0, 2)$ at ∞^-. For $\alpha \in F^\sharp$, let $\alpha_\pm \in \mathbb{R}$ be the image of α in the completion of $F^\sharp$ at $\infty^\pm$. If $\tau = \mathbf{u} + i\mathbf{v} \in \mathcal{H}_{F^\sharp}$, let $\tau_\pm = \mathbf{u}_\pm + i\mathbf{v}_\pm \in \mathcal{H}$ be the image of τ under the map $\mathcal{H}_{F^\sharp} \to \mathcal{H}$ determined by $\infty^\pm$.

Define a Schwartz function $\phi_{\mathbf{T},\mathbb{A}} = \phi_{\mathbf{T}} \otimes \phi_{\mathbf{T},\infty^+} \otimes \phi_{\mathbf{T},\infty^-}$ on the adelization of the $F^\sharp$-module $V(\mathbf{T})$ as follows. Let $\phi_{\mathbf{T}}$ be the characteristic function of $\widehat{L}(\mathbf{T})$, and set

$$\phi_{\mathbf{T},\infty^\pm}(x) = e^{\mp 2\pi Q^\sharp_{\mathbf{T}}(x)}$$

for $x \in V_{\infty^\pm}(\mathbf{T})$. By Lemma 4.2.3, the Eisenstein series

$$\begin{aligned} E(\tau, s, V(\mathbf{T})) &= (\mathbf{v}_+^{-1}\mathbf{v}_-)^{\frac{1}{2}} E(g_\tau, s, \lambda(\phi_{\mathbf{T},\mathbb{A}})) \\ &= (\mathbf{v}_+^{-1}\mathbf{v}_-)^{\frac{1}{2}} E(g_\tau, s, \lambda(\phi_{\mathbf{T}}) \otimes \Phi^1_{\infty^+} \otimes \Phi^{-1}_{\infty^-}) \end{aligned}$$

has weight 1 at ∞^+ and weight -1 and ∞^-. We denote by $W_{\alpha,p}(g, s, V(\mathbf{T}))$ the associated local Whittaker function, and

$$W_{\alpha,p}(s, V(\mathbf{T})) = W_{\alpha,p}(1, s, V(\mathbf{T})).$$

We now have the following variant of Proposition 4.4.1.

Proposition 4.5.4. *Fix $\mathbf{T} \in X_\Sigma$ and $\alpha \in F^\sharp$. If $\alpha_+ > 0$ and $\alpha_- < 0$, then*

$$\sum_{\mathbf{Z} \in C_0(E)} \sum_{\substack{j \in L(\mathbf{T}\otimes\mathbf{Z}) \\ Q^\sharp_{\mathbf{T}\otimes\mathbf{Z}}(j) = \alpha}} \frac{1}{W_E} \cdot q^\alpha = \frac{\#C_0(E)}{2W_E} \mathbf{v}_-^{-1} e^{-4\pi\alpha_-\mathbf{v}_-} E_\alpha(\tau, 0, V(\mathbf{T})).$$

Proof. The proof is similar to that of Proposition 4.4.1, and we only give a sketch. Let

$$\theta(\tau, h, V(\mathbf{T})) = (\mathbf{v}_+^{-1}\mathbf{v}_-)^{\frac{1}{2}} \sum_{x \in V(\mathbf{T})} \omega_{V(\mathbf{T})}(g_\tau)\phi_{\mathbf{T},\mathbb{A}}(h^{-1}x)$$

be the theta function associated to $\phi_{\mathbf{T},\mathbb{A}}$, where $h \in \mathrm{SO}(V(\mathbf{T}))(\mathbb{A})$. A simple calculation gives

$$\theta(\tau, h, V(\mathbf{T})) = \mathbf{v}_- \sum_{\alpha \in F^\sharp} e^{4\pi\alpha_-\mathbf{v}_-} q^\alpha \sum_{\substack{x \in V(\mathbf{T}) \\ Q^\sharp_{\mathbf{T}}(x) = \alpha}} \phi_{\mathbf{T}}(\widehat{h}^{-1}x)$$

where $q^\alpha = e(\alpha_+\tau_+)e(\alpha_-\tau_-)$, and $\widehat{h}$ is the finite part of h. Let

$$I(\tau, V(\mathbf{T})) = \int_{[S_E]} \theta(\tau, h, V(\mathbf{T}))dh$$

be the associated theta integral. Unfolding, as in the proof of Proposition 4.4.1, implies that the α^{th} Fourier coefficient of $I(\tau, V(\mathbf{T}))$ is given by

$$I_\alpha(\tau, V(\mathbf{T})) = \frac{\mathrm{Vol}([S_E])}{\#C_0(E)} \mathbf{v}_- e^{4\pi\mathbf{v}_-\alpha_-} \sum_{\mathbf{Z} \in C_0(E)} \sum_{\substack{j \in L(\mathbf{T}\otimes\mathbf{Z}) \\ Q^\sharp_{\mathbf{T}\otimes\mathbf{Z}}(j) = \alpha}} 1.$$

On the other hand, the Siegel-Weil formula [22, Theorem 4.1] gives

$$I(\tau, V(\mathbf{T})) = \frac{\mathrm{Vol}([S_E])}{2} E(\tau, 0, V(\mathbf{T})).$$

Now the proposition is clear.

□

In Section 5, we will see a close relation between the incoherent Eisenstein series defined in this subsection and the coherent Eisenstein series defined in this and the previous subsections. For example, suppose Σ is a CM type of E and $\mathfrak{q}$ is a prime of $\mathcal{O}_\Sigma$. If $\mathbf{A}$ is a supersingular $\mathcal{O}_F$-polarized CM abelian surface over an algebraic closure of $\mathcal{O}_\Sigma/\mathfrak{q}$, then $\mathbf{A}$ admits a unique lift to characteristic 0. This lift corresponds, by Remark 3.2.6, to an $\mathcal{O}_F$-polarized CM module $\mathbf{T}$, and the coherent and incoherent quadratic spaces $V(\mathbf{A})$ and $\mathcal{C}(\mathbf{T})$ differ at a unique place of $F^\sharp$. This implies a close relation between the Whittaker functions $W_\alpha(g, s, \mathbf{A})$ and $W_\alpha(g, s, \mathbf{T})$.

4.6. **Local Whittaker functions.** In this subsection we compute some local Whittaker functions in specific cases. Some of the formulae have appeared in [51], among other places.

Fix a prime p, and let $\mathcal{F}$ be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{E}$ be a either a quadratic field extension of $\mathcal{F}$, or $\mathcal{E} \cong \mathcal{F} \times \mathcal{F}$. In the applications we will take $\mathcal{F} = F^\sharp_v$ and $\mathcal{E} = E^\sharp_v$ for a place v of $F^\sharp$. Let $\mathfrak{D}_{\mathcal{E}/\mathcal{F}}$ and $d_{\mathcal{E}/\mathcal{F}} = \mathrm{Nm}(\mathfrak{D}_{\mathcal{E}/\mathcal{F}})$ be the relative different and discriminant. Denote by $\chi : \mathcal{F}^\times \to \mathbb{C}^\times$ the quadratic character associated to $\mathcal{E}/\mathcal{F}$, and by $x \mapsto x^\dagger$ the nontrivial automorphism of $\mathcal{E}/\mathcal{F}$. Let $\pi \in \mathcal{F}$ be a uniformizing parameter and abbreviate

$$f = \mathrm{ord}_\pi(d_{\mathcal{E}/\mathcal{F}}) \quad \text{and} \quad q = \#\mathcal{O}_\mathcal{F}/\pi\mathcal{O}_\mathcal{F}.$$

Fix a nonzero $\beta \in \mathcal{O}_\mathcal{F}$ and let $V_\beta = \mathcal{E}$ with the $\mathcal{F}$-quadratic form $Q_\beta(x) = \beta x x^\dagger$. Let ψ be an unramified additive character of $\mathcal{F}$, and let $\omega = \omega_{V_\beta,\psi}$ be the associated Weil representation of $\mathrm{SL}_2(\mathcal{F})$ on the space of Schwartz functions $S(V_\beta)$. For $\mu \in \mathfrak{D}^{-1}_{\mathcal{E}/\mathcal{F}}$ write

$$\Phi^\mu_\beta = \lambda_{V_\beta,\psi}(\phi^\mu) \in I(s,\chi)$$

for the standard section associated to the Schwartz function

$$\phi^\mu = \mathrm{char}(\mu + \mathcal{O}_\mathcal{E})$$

Define the normalized Whittaker function

$$W^*_\alpha(g, s, \Phi) = |d_{\mathcal{E}/\mathcal{F}}|^{-\frac{1}{2}} L(s+1,\chi) W_\alpha(g, s, \Phi),$$

and abbreviate

$$W^*_\alpha(s, \Phi) = W^*_\alpha(1, s, \Phi).$$

Lemma 4.6.1.

(1) *Let $d_\beta z$ be the self-dual Haar measure on $\mathcal{E}$ with respect to the $\mathcal{F}$-bilinear form $[x, y] = \psi(\mathrm{Tr}_{\mathcal{E}/\mathcal{F}}(\beta x \overline{y}))$, and let dz be the standard Haar measure on $\mathcal{E}$ with $\mathrm{Vol}(\mathcal{O}_\mathcal{E}, dz) = 1$. Then $d_\beta z = |\beta^2 d_{\mathcal{E}/\mathcal{F}}|^{\frac{1}{2}}$.*

(2) *If $\mathcal{E}/\mathcal{F}$ is a field extension then*

$$\int_\mathcal{E} \phi(z\bar{z})\, dz = C(\mathcal{E}/\mathcal{F}) \int_\mathcal{F} \phi(x)(1+\chi(x))\, dx$$

for every $\phi \in S(\mathcal{F})$. Here dx is the standard Haar measure on $\mathcal{F}$ with $\mathrm{Vol}(\mathcal{O}_\mathcal{F}, dx) = 1$*, and*

$$C(\mathcal{E}/\mathcal{F}) = \begin{cases} 1 & \text{if } \mathcal{E}/\mathcal{F} \text{ is ramified} \\ \frac{1}{2}(1+q^{-1}) & \text{if } \mathcal{E}/\mathcal{F} \text{ is inert.} \end{cases}$$

Proof. Claim (1) is trivial. We now verify (2). As $\mathcal{E}$ is a field, for any $f \in S(\mathcal{E})$

$$\int_\mathcal{E} f(z)\,dz = \int_{\mathcal{E}^\times} |z| f(z) \frac{dz}{|z|} = \int_{\mathcal{E}^\times/\mathcal{E}^1} \int_{\mathcal{E}^1} |t| f(gt)\,dg\,dt.$$

Here $\mathcal{E}^1$ is the group of norm one elements in $\mathcal{E}$, dg is the Haar measure on $\mathcal{E}^1$ with $(\mathrm{Vol}(\mathcal{E}^1)) = 1$, and dt is the quotient measure. Now $z \mapsto zz^\dagger$ can be used to identify $\mathcal{E}^\times/\mathcal{E}^1$ with its image in $\mathcal{F}^\times$, whose characteristic function can be given as $\frac{1}{2}(1+\chi(x))$. Using the standard Haar measure $\frac{dx}{|x|}$, we have

$$\int_\mathcal{E} f(z)\,dz = C_1(\mathcal{E}/\mathcal{F}) \int_{F^\times} \tilde{f}(x) \frac{dx}{|x|}$$

for some constant $C_1(\mathcal{E}/\mathcal{F})$ (the transfer constant from dt on $\mathcal{E}^\times/\mathcal{E}^1$ to $\frac{dx}{|x|}$), where

$$\tilde{f}(x) = \frac{1+\chi(x)}{2} \int_{\mathcal{E}^1} |z| f(gz)\,dg.$$

Here $z \in \mathcal{E}^\times$ is any element with $zz^\dagger = x$; if no such z exists, then $\chi(x) = -1$ and we take $\tilde{f}(x) = 0$. If we let $f(z) = \phi(zz^\dagger)$ for $\phi \in S(\mathcal{F})$, then

$$\tilde{f}(x) = \frac{1+\chi(x)}{2} |x| \int_{\mathcal{E}^1} \phi(x)\,dg = \frac{1+\chi(x)}{2} |x| \phi(x),$$

and so

$$\int_\mathcal{E} \phi(zz^\dagger) dz = C(\mathcal{E}/\mathcal{F}) \int_\mathcal{F} \phi(x)(1+\chi(x))\,dx$$

where $2 \cdot C(\mathcal{E}/\mathcal{F}) = C_1(\mathcal{E}/\mathcal{F})$. Taking $\phi = \mathrm{char}(\mathcal{O}_\mathcal{F})$ yields the desired formula for $C(\mathcal{E}/\mathcal{F})$. □

For any ideal $\mathfrak{a} \subset \mathcal{O}_\mathcal{F}$ define a compact open subgroup

$$K_0(\mathfrak{a}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_\mathcal{F}) : c \in \mathfrak{a} \right\}.$$

The proof of the following proposition is the same as [51, Propostions 2.1 and 2.2], and is left to the reader. When $\mathfrak{a} \neq \mathcal{O}_\mathcal{F}$ is contained in the conductor of χ, *i.e.* $\chi(1+\mathfrak{a}) = 1$, we extend χ to a character of $K_0(\mathfrak{a})$ via

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \chi(d).$$

Proposition 4.6.2. *Assume that $\mathcal{E}/\mathcal{F}$ is unramified, and fix an $\alpha \in \mathcal{F}^\times$. If $\alpha \notin \mathcal{O}_\mathcal{F}$ then $W_\alpha(s, \Phi^0_\beta) = 0$. Assume that $\alpha \in \mathcal{O}_\mathcal{F}$.*

(1) *If* $\mathrm{ord}_\pi(\alpha) < \mathrm{ord}_\pi(\beta)$ *then*
$$\frac{W^*_\alpha(s,\Phi^0_\beta)}{\gamma(V_\beta)} = |\beta| L(s+1,\chi)(1-q^{-s}) \sum_{n=0}^{\mathrm{ord}_\pi(\alpha)} q^{n(1-s)}.$$
In particular, $W^*_\alpha(0,\Phi^0_\beta) = 0$, *and*
$$\frac{W^{*,\prime}_\alpha(0,\Phi^0_\beta)}{\gamma(V_\beta)} = |\beta| \log(q)\, L(1,\chi) \sum_{n=0}^{\mathrm{ord}_\pi(\alpha)} q^n.$$

(2) *If* $\mathrm{ord}_\pi(\alpha) \geq \mathrm{ord}_\pi(\beta)$ *then*
$$\frac{W^*_\alpha(s,\Phi^0_\beta)}{\gamma(V_\beta)} = |\beta| L(s+1,\chi)(1-q^{-s}) \sum_{n=0}^{\mathrm{ord}_\pi(\beta)-1} q^{n(1-s)} + |\beta| \sum_{n=0}^{\mathrm{ord}_\pi(\alpha/\beta)} \chi(\pi^n) q^{-ns}.$$
In particular,
$$\frac{W^*_\alpha(0,\Phi^0_\beta)}{\gamma(V_\beta)} = |\beta| \sum_{n=0}^{\mathrm{ord}_\pi(\alpha/\beta)} \chi(\pi)^n = |\beta| \begin{cases} \mathrm{ord}_\pi(\alpha/\beta)+1 & \text{if } \chi(\pi)=1 \\ \frac{1+(-1)^{\mathrm{ord}_\pi(\alpha/\beta)}}{2} & \text{if } \chi(\pi)=-1. \end{cases}$$
So $W^*_\alpha(0,\Phi^0_\beta) = 0$ *if and only if* $\chi(\pi) = -1$ *and* $\mathrm{ord}_\pi(\alpha/\beta)$ *is odd. When this is the case,*
$$\frac{W^{*,\prime}_\alpha(0,\Phi^0_\beta)}{\gamma(V_\beta)} = |\beta| \log(q) \left[\frac{\mathrm{ord}_\pi(\alpha/\beta)+1}{2} + \frac{1-q^{-\mathrm{ord}_\pi(\beta)}}{q(1-q^{-2})} \right].$$

Proposition 4.6.3. *Assume that* $\mathcal{E}/\mathcal{F}$ *is ramified, and fix an* $\alpha \in \mathcal{F}^\times$. *If* $\alpha \notin \mathcal{O}_\mathcal{F}$ *then* $W^*_\alpha(s,\Phi^0_\beta) = 0$. *Assume* $\alpha \in \mathcal{O}_\mathcal{F}$.

(1) *If* $0 \leq \mathrm{ord}_\pi(\alpha) < \mathrm{ord}_\pi(\beta)$ *then*
$$\frac{W^*_\alpha(s,\Phi^0_\beta)}{\gamma(V_\beta)} = |\beta|(1-q^{-s}) \sum_{n=0}^{\mathrm{ord}_\pi(\alpha)} q^{n(1-s)}.$$
In particular, $W^*_\alpha(0,\Phi^0_\beta) = 0$, *and*
$$\frac{W^{*,\prime}_\alpha(0,\Phi^0_\beta)}{\gamma(V_\beta)} = |\beta| \log(q) \sum_{n=0}^{\mathrm{ord}_\pi(\alpha)} q^n$$

(2) *If* $\mathrm{ord}_\pi(\alpha) \geq \mathrm{ord}_\pi(\beta)$ *then*
$$\frac{W^*_\alpha(s,\Phi^0_\beta)}{\gamma(V_\beta)} = |\beta|^s + \chi(\alpha/\beta)|\alpha d_{\mathcal{E}/FF}|^s + |\beta|(1-q^{-s}) \sum_{n=0}^{\mathrm{ord}_\pi(\beta)-1} q^{n(1-s)}.$$

In particular,

$$\frac{W^*_\alpha(0,\Phi^0_\beta)}{\gamma(V_\beta)} = 1+\chi(\alpha/\beta).$$

If $\chi(\alpha/\beta)=-1$ *then*

$$\frac{W^{*,\prime}_\alpha(0,\Phi^0_\beta)}{\gamma(V_\beta)} = \log(q)\left[\frac{1-q^{-\mathrm{ord}_\pi(\beta)}}{q(1-q^{-1})}+\mathrm{ord}_\pi(\alpha/\beta)+f\right].$$

Proof. The case $\beta\in\mathcal{O}_\mathcal{F}^\times$ is [51, Proposition 2.3]. We may choose the uniformizer π of $\mathcal{F}$ so that $\chi(\pi)=1$ and a uniformizer $\pi_\mathcal{E}$ of $\mathcal{E}$ such that $\mathrm{Nm}_{\mathcal{E}/\mathcal{F}}(\pi_\mathcal{E})=\pi$. Abbreviate $X=q^{-s}$, $c=\mathrm{ord}_\pi(\beta)$, and $N=\mathrm{ord}_\pi(\alpha)$, and recall $f=\mathrm{ord}_\pi(d_{\mathcal{E}/\mathcal{F}})$. As in [51, Proposition 2.3], the Schwartz function ϕ^0 is an eigenfunction of $K_0(\beta d_{\mathcal{E}/\mathcal{F}})$ with eigencharacter χ (with respect to the Weil representation). So for every $k\in K_0(\beta d_{\mathcal{E}/\mathcal{F}})$ and $g\in\mathrm{SL}_2(\mathcal{F})$ one has

$$\Phi^0_\beta(gk)=\chi(k)\Phi^0_\beta(g).$$

Furthermore, one checks $\Phi^0_\beta(1)=1$ and

$$\Phi^0_\beta(w^{-1})=\gamma(V_\beta)|\beta^2 d_{\mathcal{E}/\mathcal{F}}|^{\frac12}=\chi(\beta)\gamma(V_1)|\beta^2 d_{\mathcal{E}/\mathcal{F}}|^{\frac12}.$$

We now verify that for $b\in\mathcal{O}_\mathcal{F}$ one has

$$\Phi^0_\beta(n_-(b))=\Phi^0_\beta(w^{-1})\chi(b)|b|^{-1}\mathrm{char}(\mathcal{O}_\mathcal{F})(\beta/b).$$

Using $n_-(b)=wn(-b)w^{-1}$, unwinding the definition of the Weil representation shows

$$\Phi^0_\beta(n_-(b))=\int_{\mathcal{O}_\mathcal{E}}\psi\left(-\frac{bz\overline{z}}{\beta\pi^f}\right)dz.$$

Lemma 4.6.1 shows that if $0\le\mathrm{ord}_\pi(b)<\mathrm{ord}_\pi(\beta)+f$ then

$$\begin{aligned}\Phi^0_\beta(n_-(b)) &= \int_{\mathcal{O}_\mathcal{F}}\psi\left(-\frac{bx}{\beta\pi^f}\right)\cdot(1+\chi(x))\,dx\\ &= \int_{\mathcal{O}_\mathcal{F}}\psi\left(-\frac{bx}{\beta\pi^f}\right)\chi(x)\,dx\\ &= \sum_{n=0}^\infty q^{-n}\int_{\mathcal{O}_\mathcal{F}^\times}\psi\left(-\frac{b\pi^{n-f}x}{\beta}\right)\psi(x)\,dx\\ &= |\beta/b|q^{-\frac f2}\chi\left(b/\beta\right)\gamma(-\pi^{-f}\psi,\chi)\cdot\mathrm{char}(\mathcal{O}_\mathcal{F})\left(\beta/b\right).\end{aligned}$$

Here

$$\gamma(-\pi^{-f}\psi,\chi)=q^{\frac f2}\int_{\mathcal{O}_\mathcal{F}^\times}\psi(-\pi^{-f}x)\chi(x)\,dx.$$

Take $\beta=b=1$ for a moment. From $n_-(1)=n(1)w^{-1}n(1)$ it follows that

$$\Phi^0_1(n_-(1))=\Phi^0_1(w^{-1})=\gamma(V_1)q^{-\frac f2}.$$

Comparing these two equalities shows $\gamma(-\pi^{-f}\psi,\chi)=\gamma(V_1)$. Now the formula for $\Phi^0_\beta(n_-(b))$ is clear.

Now we are ready to prove the formulae for $W_\alpha(s,\Phi^0_\beta)$. Using

$$w^{-1}n(b)=n(-b^{-1})m(b^{-1})n_-(b^{-1})$$

and the formulae just proved, we have

$$\begin{aligned}W_\alpha(s,\Phi^0_\beta)=&\int_{\mathcal{O}_\mathcal{F}}\Phi^0_\beta(w^{-1})\psi(-b\alpha)\,db\\&+\sum_{n=1}^{c+f-1}\int_{\pi^{-n}\mathcal{O}_\mathcal{F}^\times}\chi(b)^{-1}|b|^{-s-1}\Phi^0_\beta(n_-(b^{-1}))\psi(-b\alpha)\,db\\&+\sum_{n=c+f}^{\infty}\int_{\pi^{-n}\mathcal{O}_\mathcal{F}^\times}\chi(b)^{-1}|b|^{-s-1}\psi(-b\alpha)\,db\\=&\,\Phi^0_\beta(w^{-1})\mathrm{char}(\mathcal{O}_\mathcal{F})(\alpha)+\sum_{n\geq c+f}X^n\chi(\alpha)q^{-\frac{f}{2}}\gamma(V_1)\mathrm{char}(\pi^{n-f}\mathcal{O}_\mathcal{F}^\times)(\alpha)\\&+\Phi^0_\beta(w^{-1})\sum_{0<n\leq c}(qX)^n\big[\mathrm{char}(\pi^n\mathcal{O}_\mathcal{F})(\alpha)-q^{-1}\mathrm{char}(\pi^{n-1}\mathcal{O}_\mathcal{F})(\alpha)\big].\end{aligned}$$

Therefore $W_\alpha(s,\Phi^0_\beta)=0$ unless $\alpha\in\mathcal{O}_\mathcal{F}$. When $\alpha\in\mathcal{O}_\mathcal{F}$, one has

$$\frac{W_\alpha(s,\Phi^0_\beta)}{\Phi^0_\beta(w^{-1})}=\cdot\begin{cases}(1-X)\sum_{n=0}^N(qX)^n & \text{if } N<c\\ q^c\left[X^c+\chi(\beta\alpha)X^{f+N}\right]+(1-X)\sum_{n=0}^{c-1}(qX)^n & \text{if } N\geq c,\end{cases}$$

and the proposition is clear. ∎

Proposition 4.6.4. *Assume that $\mathcal{E}/\mathcal{F}$ is ramified, $\alpha\in\mathcal{F}^\times$, $p\neq 2$, and $\mu\in\mathfrak{D}^{-1}_{\mathcal{E}/\mathcal{F}}\smallsetminus\mathcal{O}_\mathcal{E}$. Then $W^*_\alpha(s,\Phi^\mu_\beta)=0$ unless $c(\alpha,\mu)=\mathrm{ord}_\mathcal{F}(\alpha-Q_\beta(\mu))\geq 0$. Assume $c(\alpha,\mu)\geq 0$.*

(1) *If $0\leq c(\alpha,\mu)<\mathrm{ord}_\pi(\beta)$, then*

$$\frac{W^*_\alpha(s,\Phi^\mu_\beta)}{\gamma(V_\beta)}=|\beta|(1-q^{-s})\sum_{n=0}^{c(\alpha,\mu)}q^{n(1-s)}.$$

In particular, $W^*_\alpha(0,\Phi^\mu_\beta)=0$, and

$$\frac{W^{*,\prime}_\alpha(0,\Phi^\mu_\beta)}{\gamma(V_\beta)}=|\beta|\log(q)\sum_{n=0}^{c(\alpha,\mu)}q^n.$$

(2) *If $c(\alpha,\mu)\geq\mathrm{ord}_\pi(\beta)$, then*

$$\frac{W^*_\alpha(s,\Phi^\mu_\beta)}{\gamma(V_\beta)}=q^{-s\cdot\mathrm{ord}_\pi(\beta)}+|\beta|(1-q^{-s})\sum_{n=0}^{\mathrm{ord}_\pi(\beta)-1}q^{n(1-s)}.$$

In particular $W^*_\alpha(0,\Phi^\mu_\beta)=1$.

Proof. The proof is similar to Proposition 4.6.3. We again abbreviate $X = q^{-s}$, $c = \mathrm{ord}_\pi(\beta)$ and $N = \mathrm{ord}_\pi(\alpha)$. We don't assume that $p \neq 2$ until the actual calculation of the Whittaker functions. First it is easy to check the following explicit formulae

$$\begin{aligned}
\omega(n(b))\phi^\mu &= \psi(bQ_\beta(\mu))\phi^\mu && \text{for } b \in \mathcal{O}_\mathcal{F}, \\
\omega(m(a))\phi^\mu &= \chi(a)\phi^{a^{-1}\mu} && \text{for } a \in \mathcal{O}_\mathcal{F}^\times, \\
\omega(n_-(b))\phi^\mu &= \phi^\mu && \text{for } b \in \beta\mathfrak{D}_{\mathcal{E}/\mathcal{F}}, \\
\omega(w^{-1})\phi(x) &= \Phi^0_\beta(w^{-1})\psi(-\mathrm{Tr}_{\mathcal{E}/\mathcal{F}}(\beta\mu^\dagger x)).
\end{aligned}$$

From this, one sees immediately that Φ^μ_β has the following properties.

(1) For every $b \in \mathcal{O}_\mathcal{F}$, we have

$$\Phi^\mu_\beta(gn(b)) = \psi(bQ_\beta(\mu)) \cdot \Phi^\mu_\beta(g).$$

(2) For every $b \in \beta\mathfrak{D}_{\mathcal{E}/\mathcal{F}}$, we have

$$\Phi^\mu_\beta(gn_-(b)) = \Phi^\mu_\beta(g).$$

(3) We have $\Phi^\mu_\beta(1) = 0$ and $\Phi^\mu_\beta(w^{-1}) = \Phi^0_\beta(w^{-1})$.

Finally, for every $b \in \mathcal{O}_\mathcal{F}$,

$$\Phi^\mu_\beta(n_-(b)) = \begin{cases} \Phi^0_\beta(n_-(b))\psi(b^{-1}Q_\beta(\mu)) & \text{if } \mathrm{ord}_\mathcal{E}(b) \leq \mathrm{ord}_\mathcal{E}(\mu\beta\mathfrak{D}_{\mathcal{E}/\mathcal{F}}) \\ 0 & \text{if } \mathrm{ord}_\mathcal{E}(b) \geq \mathrm{ord}_\mathcal{E}(\beta\mathfrak{D}_{\mathcal{E}/\mathcal{F}}). \end{cases}$$

Notice that if $p \neq 2$ (which is our assumption in the proposition) this covers all possibilities. We now verify the last formula. The case $b \in \beta d_{\mathcal{E}/\mathcal{F}}$ follows from (2) and (3) above. Assume $b \notin \beta d_{\mathcal{E}/\mathcal{F}}$ (i.e. $\mathrm{ord}_\pi(b) < c + f$). Using $n_-(b) = wn(-b)w^{-1}$ we find

$$\Phi^\mu_\beta(n_-(b)) = \int_{\mathcal{O}_\mathcal{E}} \psi\left(-\frac{bzz^\dagger}{\beta\pi^f} - \mathrm{Tr}_{\mathcal{E}/\mathcal{F}}(\mu^\dagger z\pi_\mathcal{E}^{-f})\right) dz.$$

If $\mathrm{ord}_\mathcal{E}(b) \leq \mathrm{ord}_\mathcal{E}(\mu\beta\mathfrak{D}_{\mathcal{E}/\mathcal{F}})$ then $\beta\mu\pi_\mathcal{E}^f b^{-1} \in \mathcal{O}_\mathcal{E}$. The substitution $z \mapsto z - \beta\mu\pi_\mathcal{E}^f b^{-1}$ gives

$$\begin{aligned}
\Phi^\mu_\beta(n_-(b)) &= \psi(b^{-1}\beta\mu\mu^\dagger) \int_{\mathcal{O}_\mathcal{E}} \psi\left(-\frac{bzz^\dagger}{\beta\pi^f}\right) dz \\
&= \psi(b^{-1}Q_\beta(\mu)) \cdot \Phi^0_\beta(n_-(b))
\end{aligned}$$

as claimed.

Now we assume $p \neq 2$ and prove the formula for $W^*_\alpha(s, \Phi^\mu_\beta)$. The assumption $p \neq 2$ implies $f = 1$. Recall

$$w^{-1}n(b) = n(-b^{-1})m(b^{-1})n_-(b^{-1}).$$

It follows that

$$\Phi^\mu_\beta(w^{-1}n(b)) = \begin{cases} \psi(bQ_\beta(\mu)) \cdot \Phi^0_\beta(w^{-1}) & \text{if } b \in \mathcal{O}_\mathcal{F} \\ \Phi^0_\beta(w^{-1})|b|^{-s} & \text{if } \mathrm{ord}_\pi(b) \geq -c \\ 0 & \text{if } \mathrm{ord}_\pi(b) < -c, \end{cases}$$

and thus, setting $\alpha(\mu) = \alpha - Q_\beta(\mu)$,

$$\begin{aligned} W_\alpha(s, \Phi^\mu_\beta) &= \int_F \Phi^\mu_\beta(w^{-1}n(b))\psi(-\alpha b)\, db \\ &= \Phi^0_\beta(w^{-1})\Big[\mathrm{char}(\mathcal{O}_\mathcal{F})(\alpha(\mu)) + \sum_{0<n\leq c} \int_{\pi^{-n}\mathcal{O}_\mathcal{F}^\times} |b|^{-s+1}\psi(-b\alpha(\mu))\, \frac{db}{|b|}\Big] \\ &= \Phi^0_\beta(w^{-1})\Big[\mathrm{char}(\mathcal{O}_\mathcal{F})(\alpha(\mu)) \\ &\quad + \sum_{0<n\leq c} (qX)^n \big[\mathrm{char}(\pi^n\mathcal{O}_\mathcal{F})(\alpha(\mu)) - q^{-1}\mathrm{char}(\pi^{n-1}\mathcal{O}_\mathcal{F})(\alpha(\mu))\big]\Big]. \end{aligned}$$

Since $\mu \notin \mathcal{O}_\mathcal{E}$, $\mathrm{ord}_\pi(Q_\beta(\mu)) < c$. If $c(\alpha, \mu) = \mathrm{ord}_\pi(\alpha(\mu)) < c$ then

$$W_\alpha(s, \Phi^\mu_\beta) = \Phi^0_\beta(w^{-1})(1 - X) \sum_{0\leq n\leq c(\alpha,\mu)} (qX)^n.$$

If $c(\alpha, \mu) \geq c$ then

$$\frac{W_\alpha(s, \Phi^\mu_\beta)}{\Phi^0_\beta(w^{-1})} = (1 - X) \sum_{0\leq n<c} (qX)^n + (qX)^c.$$

Notice that $c(\alpha, \mu) \geq c$ implies $\chi(\alpha\beta^{-1}) = 1$. Indeed, write $\mu = \mu_0\pi_\mathcal{E}^{-1}$ with $\mu_0 \in \mathcal{O}_\mathcal{E}^\times$. Then $c(\alpha, \mu) \geq c$ implies

$$\mu_0\mu_0^\dagger \equiv \alpha\beta^{-1} \mod \pi\mathcal{O}_\mathcal{F},$$

and $\chi(\alpha\beta^{-1}) = 1$ follows. □

5. The main results

This section contains our main results. In Section 5.1 we state a result expressing the degree of the 0-dimension stack $\mathcal{CM}_\Sigma(\alpha)$ in terms of the α^{th}-Fourier coefficient of the central derivative of an incoherent Hilbert modular Eisenstein series. The proof of this result is contained in Section 5.2, with the exception of certain local calculations (which are, in fact, the techincal core of the proof) postponed until Section 6. In Section 5.3 this result is combined with the decomposition (3.3.1) to relate the arithmetic intersection of the cycles $\mathcal{T}(m)$ and $\mathcal{CM}_\Sigma$ on $\mathcal{M}$ to the m^{th} Fourier coefficient of the same Eisenstein series, after pulling back the derivative to a classical non-holomorphic modular form on the complex upper half-plane.

In Section 5 we take $\mathfrak{c} = \mathcal{O}_F$ and fix a CM type Σ of E. Let X_Σ denote the set of isomorphism classes of $\mathcal{O}_F$-polarized CM modules with CM type Σ, as defined in Section 2.3. Let $\mathcal{O}_\Sigma \subset \mathbb{C}$ (the maximal order in the reflex field of Σ) be the subring defined in Section 2.1. For a prime $\mathfrak{q}$ of $\mathcal{O}_\Sigma$ let $\mathbb{F}_\mathfrak{q}$ be the residue field of $\mathfrak{q}$ and let $\mathrm{Nm}(\mathfrak{q})$ be the cardinality of $\mathbb{F}_\mathfrak{q}$.

5.1. **Degrees of zero cycles.**

For a totally positive $\alpha \in (F^\sharp)^\times$, recall the algebraic stack $\mathcal{CM}_\Sigma(\alpha)$ of dimension zero defined in Section 3.3, and the finite set $\mathrm{Sppt}(\alpha)$ of places of $\mathbb{Q}$ defined by (3.3.4). The assumption that α is totally positive implies that $\infty \notin \mathrm{Sppt}(\alpha)$, and so $\mathrm{Sppt}(\alpha)$ consists of an odd number of finite primes of $\mathbb{Q}$. If $\#\mathrm{Sppt}(\alpha) > 1$ then Proposition 3.3.8 implies that $\mathcal{CM}_\Sigma(\alpha)$ is empty. If $\mathrm{Sppt}(\alpha) = \{p\}$ then $\mathcal{CM}_\Sigma(\alpha)$ is supported in characteristic p by Proposition 3.3.8, and has dimension 0 by Proposition 3.3.9. For a prime $\mathfrak{q}$ of $\mathcal{O}_\Sigma$ define

$$\deg_\mathfrak{q} \mathcal{CM}_\Sigma(\alpha) = \sum_{z \in \mathcal{CM}_\Sigma(\alpha)(\mathbb{F}_\mathfrak{q}^{\mathrm{alg}})} \frac{\mathrm{length}\big(\mathcal{O}^{\mathrm{sh}}_{\mathcal{CM}_\Sigma(\alpha),z}\big)}{\#\mathrm{Aut}(z)}$$

where $\mathcal{O}^{\mathrm{sh}}_{\mathcal{CM}_\Sigma(\alpha),z}$ is the strictly Henselian local ring at z (i.e. the local ring for the étale topology), and the automorphism group $\mathrm{Aut}(z)$ is computed in the category $\mathcal{CM}_\Sigma(\alpha)(\mathbb{F}_\mathfrak{q}^{\mathrm{alg}})$. Summing over all primes $\mathfrak{q}$ of $\mathcal{O}_\Sigma$, define the *arithmetic degree*

$$\widehat{\deg}\, \mathcal{CM}_\Sigma(\alpha) = \sum_{\mathfrak{q}} \log(\mathrm{Nm}(\mathfrak{q})) \cdot \deg_\mathfrak{q} \mathcal{CM}_\Sigma(\alpha).$$

From what we have said, if $\mathrm{Sppt}(\alpha) = \{p\}$ then only those $\mathfrak{q}$ above p contribute to the sum.

On the automorphic side, for each $\mathbf{T} \in X_\Sigma$, we have defined in Section 4.5 an incoherent Eisenstein series

$$E(\tau, s, \mathbf{T}) = \sum_{\alpha \in F^\sharp} E_\alpha(\tau, s, \mathbf{T})$$

($\tau = \mathbf{u} + i\mathbf{v} \in \mathcal{H}_{F^\sharp}$). Summing over all $\mathbf{T}$ we obtain a nonholomorphic weight one Hilbert modular Eisenstein series

$$E(\tau, s, \Sigma) = \sum_{\mathbf{T} \in X_\Sigma} E_\alpha(\tau, s, \mathbf{T})$$

whose derivative at $s = 0$ has a Fourier expansion

$$E'(\tau, 0, \Sigma) = \sum_{\alpha \in F^\sharp} c_\Sigma(\alpha, \mathbf{v}) \cdot q^\alpha,$$

where the coefficient $c_\Sigma(\alpha, \mathbf{v})$ is defined by the relation

$$c_\Sigma(\alpha, \mathbf{v}) \cdot q^\alpha = \sum_{\mathbf{T} \in X_\Sigma} E'_\alpha(\tau, 0, \mathbf{T}).$$

In the next subsection we will prove Theorem 5.1.1 below.

Theorem 5.1.1. *Suppose $\alpha \in (F^\sharp)^\times$ is totally positive, and that* $\mathrm{Sppt}(\alpha) = \{p\}$ *for some prime p satisfying both conditions of Hypothesis B. Then*

$$\widehat{\deg}\ \mathcal{CM}_\Sigma(\alpha) = -\frac{1}{W_E} \cdot c_\Sigma(\alpha, \mathbf{v}) \tag{5.1.1}$$

where (as always) W_E is the number of roots of unity in E. Note in particular that the right hand side is independent of $\mathbf{v}$.

Proposition 5.1.2. *Suppose $\alpha \in (F^\sharp)^\times$ is totally positive. If* $\#\mathrm{Sppt}(\alpha) > 1$ *then both sides of (5.1.1) are equal to* 0.

Proof. This is clear from Proposition 3.3.8 and Corollary 4.5.3. □

5.2. **Proof of Theorem 5.1.1.** Throughout Section 5.2 we fix a CM type Σ of E and a totally positive $\alpha \in (F^\sharp)^\times$ satisfying $\mathrm{Sppt}(\alpha) = \{p\}$ for a prime p satisfying both conditions of Hypothesis B. Let $\mathfrak{q}$ be a prime of $\mathcal{O}_\Sigma$ above p. The kernel of

$$\mathcal{O}_{E^\sharp} \xrightarrow{\phi_\Sigma} \mathcal{O}_\Sigma \to \mathcal{O}_\Sigma/\mathfrak{q}$$

is a prime of $E^\sharp$ which we denote by $\mathfrak{q}^\sharp$, the pullback of $\mathfrak{q}$ by ϕ_Σ. Let $\mathfrak{p}^\sharp$ be the prime of $F^\sharp$ below $\mathfrak{q}^\sharp$. Fix embeddings

$$\mathbb{Q}^{\mathrm{alg}} \to \mathbb{C} \qquad \mathbb{Q}^{\mathrm{alg}} \to \mathbb{Q}_p^{\mathrm{alg}}. \tag{5.2.1}$$

These choices allow us to view $\mathcal{O}_\Sigma$ as a subfield of $\mathbb{Q}_p^{\mathrm{alg}}$, and we assume that (5.2.1) are chosen so that the induced p-adic absolute value on $\mathcal{O}_\Sigma$ agrees with that determined by $\mathfrak{q}$. Let $\mathbb{C}_p$ be the metric completion of the algebraic closure of $\mathbb{Q}_p^{\mathrm{alg}}$ and let $\mathcal{O}_{\mathbb{C}_p} \subset \mathbb{C}_p$ be the valuation ring. We denote by $\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}$ the residue field of $\mathcal{O}_{\mathbb{C}_p}$, with its $\mathcal{O}_\Sigma$-algebra structure determined by the inclusion $\mathcal{O}_\Sigma \subset \mathcal{O}_{\mathbb{C}_p}$.

Remark 3.2.6 establishes a bijection $X_\Sigma \to \mathcal{CM}_\Sigma(\mathbb{C})$, denoted $\mathbf{T} \mapsto \mathbf{A}(\mathbf{T})$. By the theory of complex multiplication $\mathbf{A}(\mathbf{T})$ has a model over $\mathbb{Q}^{\mathrm{alg}}$ and has everywhere good reduction. Using the fixed embedding $\mathbb{Q}^{\mathrm{alg}} \to \mathbb{Q}_p^{\mathrm{alg}}$ we may reduce $\mathbf{A}(\mathbf{T})$ modulo p to obtain an $\mathcal{O}_F$-polarized CM abelian surface

$\tilde{\mathbf{A}}(\mathbf{T})$ over $\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}$. The construction $\mathbf{T} \mapsto \tilde{\mathbf{A}}(\mathbf{T})$ defines a $C_+(E)$-equivariant function

$$X_\Sigma \to \mathcal{CM}_\Sigma(\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}), \tag{5.2.2}$$

which is a bijection by [16, Theorem 2.2.1].

Proposition 5.2.1. *If $\mathfrak{p}^\sharp$ is nonsplit in $E^\sharp$ then every point of $\mathcal{CM}_\Sigma(\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}})$ is supersingular. If $\mathfrak{p}^\sharp$ is split in $E^\sharp$ then $\mathcal{CM}_\Sigma(\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}})$ contains no supersingular points.*

Proof. This will be proved in Section 6.1. Indeed, by the bijectivity of (5.2.2) any $\mathbf{A} \in \mathcal{CM}_\Sigma(\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}})$ has the form $\mathbf{A} = \tilde{\mathbf{A}}(\mathbf{T})$ for some $\mathbf{T} \in X_\Sigma$, and so we are in the situation considered in Section 6. By Proposition 6.1.1, $\mathbf{A}$ is supersingular if and only if $\mathfrak{p}^\sharp$ is nonsplit in $E^\sharp$. □

Proposition 3.3.7 implies that every $(\mathbf{A}, j) \in \mathcal{CM}_\Sigma(\alpha)(\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}})$ is supersingular. Therefore, by Proposition 5.2.1,

$$\mathfrak{p}^\sharp \text{ split in } E^\sharp \implies \mathcal{CM}_\Sigma(\alpha)(\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}) = \emptyset.$$

Proposition 5.2.2. *Assume $\mathfrak{p}^\sharp$ is nonsplit in $E^\sharp$. Fix a $\mathbf{T} \in X_\Sigma$, and set $\mathbf{A} = \tilde{\mathbf{A}}(\mathbf{T})$.*

(1) *For every prime $\ell \nmid p\infty$ there is an $E^\sharp_\ell$-linear isomorphism of $F^\sharp_\ell$-quadratic spaces*
$$\big(V_\ell(\mathbf{T}), Q^\sharp_{\mathbf{T}}\big) \cong \big(V_\ell(\mathbf{A}), Q^\sharp_{\mathbf{A}}\big)$$
taking $L_\ell(\mathbf{T})$ isomorphically to $L_\ell(\mathbf{A})$.

(2) *The archimedean place $w = \infty_\Sigma^-$ of $F^\sharp$ determined by the reflex map $\phi_\Sigma : E^\sharp \to \mathbb{C}$ is the unique archimedean place for which*
$$\big(V_w(\mathbf{T}), Q^\sharp_{\mathbf{T}}\big) \not\cong \big(V_w(\mathbf{A}), Q^\sharp_{\mathbf{A}}\big).$$

(3) *The prime $\mathfrak{p} = \mathfrak{p}^\sharp$ of $F^\sharp$ is the unique prime of $F^\sharp$ above p for which*
$$\big(V_{\mathfrak{p}}(\mathbf{T}), Q^\sharp_{\mathbf{T}}\big) \not\cong \big(V_{\mathfrak{p}}(\mathbf{A}), Q^\sharp_{\mathbf{A}}\big).$$

Proof. This will be proved in Section 6. See Proposition 6.2.4. □

Assume $\mathfrak{p}^\sharp$ is nonsplit in $E^\sharp$, fix a $\mathbf{T} \in X_\Sigma$, and set $\mathbf{A} = \tilde{\mathbf{A}}(\mathbf{T})$. Recalling that we assume $\mathrm{Sppt}(\alpha) = \{p\}$, the set $\mathrm{Diff}(\alpha, \mathbf{T})$ appearing in Lemma 4.5.2 contains a unique prime above p. If $\mathfrak{p}^\sharp \in \mathrm{Diff}(\alpha, \mathbf{T})$ then α is not represented by $(V_{\mathfrak{p}^\sharp}(\mathbf{T}), Q^\sharp_{\mathbf{T}})$, and the final claim of Proposition 5.2.2 implies that α is represented by $(V_{\mathfrak{p}^\sharp}(\mathbf{A}), Q^\sharp_{\mathbf{A}})$. If there is another prime $\mathfrak{p} \neq \mathfrak{p}^\sharp$ above p then $(V_{\mathfrak{p}}(\mathbf{T}), Q^\sharp_{\mathbf{T}})$ represents α (as $\mathfrak{p} \notin \mathrm{Diff}(\alpha, \mathbf{T})$), and the final claim Proposition 5.2.2 implies that α is also represented by $(V_{\mathfrak{p}}(\mathbf{A}), Q^\sharp_{\mathbf{A}})$. On the other hand, if $\mathfrak{p}^\sharp \notin \mathrm{Diff}(\alpha, \mathbf{T})$ then $(V_{\mathfrak{p}^\sharp}(\mathbf{T}), Q^\sharp_{\mathbf{T}})$ represents α, and so $(V_{\mathfrak{p}^\sharp}(\mathbf{A}), Q^\sharp_{\mathbf{A}})$ does not. In summary, we have proved

$$\big(V_p(\mathbf{A}), Q^\sharp_{\mathbf{A}}\big) \text{ represents } \alpha \iff \mathfrak{p}^\sharp \in \mathrm{Diff}(\alpha, \mathbf{T}). \tag{5.2.3}$$

Label the elements of our chosen CM type $\Sigma = \{\pi_3, \pi_4\}$, and set $\pi_1(x) = \pi_3(\overline{x})$ and $\pi_2(x) = \pi_4(\overline{x})$. Viewing each π_i as a map $E \to \mathbb{Q}_p^{\mathrm{alg}}$, all four take values in the subfield $\mathbb{E}_p$ defined in the introduction. The Galois group $\mathrm{Gal}(\mathbb{E}_p/\mathbb{Q}_p)$ acts on the set $\{\pi_1, \pi_2, \pi_3, \pi_4\}$ through rigid motions of the square

$$\begin{array}{ccc} \pi_1 & \text{———} & \pi_2 \\ \Big| & & \Big| \\ \pi_4 & \text{———} & \pi_3 \end{array}$$

and an exercise in Galois theory shows (see Proposition 6.1.1) that $\mathfrak{p}^\sharp$ is nonsplit in $E^\sharp$ if and only if the subgroup $\mathrm{Gal}(\mathbb{E}_p/\mathbb{Q}_p) \subset D_8$ contains an element that interchanges the top and bottom edges. As we assume that $[\mathbb{E}_p : \mathbb{Q}_p] \le 4$, there are five cases to consider:

- **Quadratic case I**: $\mathrm{Gal}(\mathbb{E}_p/\mathbb{Q}_p)$ is generated by rotation by $180°$,
- **Quadratic case II**: $\mathrm{Gal}(\mathbb{E}_p/\mathbb{Q}_p)$ is generated by reflection across the horizontal axis,
- **Cyclic quartic case**: $\mathrm{Gal}(\mathbb{E}_p/\mathbb{Q}_p)$ is cyclic of order four,
- **Klein four case I**: $\mathrm{Gal}(\mathbb{E}_p/\mathbb{Q}_p)$ is generated by the horizontal and vertical reflections,
- **Klein four case II**: $\mathrm{Gal}(\mathbb{E}_p/\mathbb{Q}_p)$ is generated by the diagonal reflections.

The following fundamental result, whose case-by-case proof will occupy the bulk of Section 6, is a technical exercise in Grothendieck-Messing theory (or Zink's theory of displays).

Proposition 5.2.3. *Assume $\mathfrak{p}^\sharp$ is nonsplit in $E^\sharp$, and define*

$$\nu_{\mathfrak{p}^\sharp}(\alpha) = \frac{\mathrm{ord}_{\mathfrak{p}^\sharp}(\alpha) + \mathrm{ord}_{\mathfrak{p}^\sharp}(\mathfrak{D}_{F^\sharp}) + 1}{2} \cdot \begin{cases} 1 & \text{if } \mathfrak{p}^\sharp \text{ is inert in } E^\sharp \\ 2 & \text{if } \mathfrak{p}^\sharp \text{ is ramified in } E^\sharp \end{cases}$$

where $\mathfrak{D}_{F^\sharp}$ is the different of $F^\sharp/\mathbb{Q}$. The strictly Henselian local ring of every geometric point $z \in \mathcal{CM}_\Sigma(\alpha)(\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}})$ has length

$$\mathrm{length}\big(\mathcal{O}^{\mathrm{sh}}_{\mathcal{CM}_\Sigma(\alpha), z}\big) = \nu_{\mathfrak{p}^\sharp}(\alpha).$$

In particular the length does not depend on z.

Proof. This will be proved in Section 6. If z corresponds to the pair $(\mathbf{A}, j)$, the completed strictly Henselian local ring of $\mathcal{CM}_\Sigma(\alpha)$ at z pro-represents the formal deformation functor $\mathrm{Def}_\Sigma(\mathbf{A}, j)$ defined in Section 6.2. Using Corollary 6.2.2, this formal deformation functor is computed, in the five cases listed above, in Propositions 6.4.1, 6.5.1, 6.6.1, 6.7.1, and 6.8.1, respectively. □

Remark 5.2.4. If the hypothesis $[\mathbb{E}_p : \mathbb{Q}_p] \le 4$ were dropped, we would be left to treat a sixth case: $\mathrm{Gal}(\mathbb{E}_p/\mathbb{Q}_p) \cong D_8$. There is no obvious reason why the methods used to prove Proposition 5.2.3 cannot be extended to treat this case as well, but the technical details involved seem quite formidable.

Fix a $\mathbf{T} \in X_\Sigma$ and abbreviate $\mathbf{A} = \tilde{\mathbf{A}}(\mathbf{T})$.

Proposition 5.2.5. *Assume $\mathfrak{p}^\sharp \in \mathrm{Diff}(\alpha, \mathbf{T})$, a hypothesis which implies both that $\mathfrak{p}^\sharp$ is nonsplit in $E^\sharp$ and that $W_{\alpha,p}(0, \mathbf{T}) = 0$.*

1. *If $L_p(\mathbf{A})$ represents α then $W_{\alpha,p}(0, \mathbf{A}) \neq 0$ and*
$$\frac{W'_{\alpha,p}(0, \mathbf{T})}{W_{\alpha,p}(0, \mathbf{A})} = -\frac{\nu_{\mathfrak{p}^\sharp}(\alpha)}{2} \cdot \log(\mathrm{Nm}(\mathfrak{q})).$$
2. *If $L_p(\mathbf{A})$ does not represent α then $W_{\alpha,p}(0, \mathbf{A})$ and $W'_{\alpha,p}(0, \mathbf{T})$ are both 0.*

Proof. This will be proved case-by-case in Section 6. See Propositions 6.4.4, 6.5.4, 6.6.4, 6.7.4, and 6.8.4. □

By combining the information above, we can express $\deg_{\mathfrak{q}} \mathcal{CM}_\Sigma(\alpha)$ in terms of central derivatives of incoherent Eisenstein series. Proposition 4.4.1 and the bijectivity of (5.2.2) imply

$$\begin{aligned}
&\sum_{z \in \mathcal{CM}_\Sigma(\alpha)(\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}})} \frac{1}{\#\mathrm{Aut}(z)} \cdot q^\alpha \\
&= \sum_{\mathbf{T} \in X_\Sigma} \sum_{\substack{j \in L(\mathbf{A}) \\ Q^\sharp_{\mathbf{A}}(j) = \alpha}} \frac{1}{\#\mathrm{Aut}(\mathbf{A})} \cdot q^\alpha \\
&= \frac{1}{\#C_0(E)} \sum_{\mathbf{T} \in X_\Sigma} \sum_{\mathbf{Z} \in C_0(E)} \sum_{\substack{j \in L(\mathbf{A} \otimes \mathbf{Z}) \\ Q^\sharp_{\mathbf{A} \otimes \mathbf{Z}}(j) = \alpha}} \frac{1}{\#\mathrm{Aut}(\mathbf{A} \otimes \mathbf{Z})} \cdot q^\alpha \\
&= \frac{1}{2W_E} \sum_{\mathbf{T} \in X_\Sigma} E_\alpha(\tau, 0, \mathbf{A}).
\end{aligned}$$

If $\mathfrak{p}^\sharp \not\in \mathrm{Diff}(\alpha, \mathbf{T})$, then (5.2.3) implies $E_\alpha(\tau, 0, \mathbf{A}) = 0$, and so the final sum may be restricted to those $\mathbf{T}$ for which $\mathfrak{p}^\sharp \in \mathrm{Diff}(\alpha, \mathbf{T})$. Applying Proposition 5.2.3 results in

$$\deg_{\mathfrak{q}} \mathcal{CM}_\Sigma(\alpha) \cdot q^\alpha = \frac{\nu_{\mathfrak{p}^\sharp}(\alpha)}{2W_E} \cdot \sum_{\substack{\mathbf{T} \in X_\Sigma \\ \mathfrak{p}^\sharp \in \mathrm{Diff}(\alpha, \mathbf{T})}} E_\alpha(\tau, 0, \mathbf{A}). \tag{5.2.4}$$

By the first claim of Proposition 5.2.2, for every rational prime $\ell \neq p$ there is an isomorphism $V_\ell(\mathbf{A}) \cong V_\ell(\mathbf{T})$ respecting the $F^\sharp_\ell$-quadratic forms and respecting the $\mathbb{Z}_\ell$-lattices. It follows that

$$W_{\alpha,\ell}(s, \mathbf{A}) = W_{\alpha,\ell}(s, \mathbf{T})$$

for all $\ell \neq p$ (including now $\ell = \infty$). Thus (4.1.2) implies

$$W_{\alpha,p}(s, \mathbf{A}) \cdot E_\alpha(\tau, s, \mathbf{T}) = W_{\alpha,p}(s, \mathbf{T}) \cdot E_\alpha(\tau, s, \mathbf{A}),$$

and Proposition 5.2.5 implies

$$-\frac{\nu_{\mathfrak{p}^\sharp}(\alpha)}{2}\cdot\log(\mathrm{Nm}(\mathfrak{q}))\cdot E_\alpha(\tau,0,\mathbf{A}) = E'_\alpha(\tau,0,\mathbf{T}). \tag{5.2.5}$$

Combining (5.2.4) and (5.2.5) shows that

$$\deg_{\mathfrak{q}}\mathcal{CM}_\Sigma(\alpha)\cdot\log(\mathrm{Nm}(\mathfrak{q}))\cdot q^\alpha = -\frac{1}{W_E}\sum_{\substack{\mathbf{T}\in X_\Sigma\\ \mathfrak{p}^\sharp\in\mathrm{Diff}(\alpha,\mathbf{T})}} E'_\alpha(\tau,0,\mathbf{T}). \tag{5.2.6}$$

To complete the proof of (5.1.1), it only remains to sum over all $\mathfrak{q}$.

Proof of Theorem 5.1.1. Assume, as we have been, that $\alpha\in(F^\sharp)^\times$ is totally positive, and that $\mathrm{Sppt}(\alpha)=\{p\}$ for a prime p satisfying both conditions of Hypothesis B. By Proposition 3.3.8, $\mathcal{CM}_\Sigma(\alpha)$ is supported in characteristic p, and so (5.2.6) implies

$$\widehat{\deg}\,\mathcal{CM}_\Sigma(\alpha)\cdot q^\alpha = -\frac{1}{W_E}\sum_{\mathfrak{q}\mid p}\sum_{\substack{\mathbf{T}\in X_\Sigma\\ \mathfrak{p}^\sharp\in\mathrm{Diff}(\alpha,\mathbf{T})}} E'_\alpha(\tau,0,\mathbf{T})$$

where the outer sum is over all primes $\mathfrak{q}\mid p$ of $\mathcal{O}_\Sigma$.

Suppose first that we are either in case (**cyclic**) or (**nongal**), so $E^\sharp$ is a quartic field and $\phi_\Sigma : E^\sharp\to E_\Sigma$ is an isomorphism. Thus

$$\widehat{\deg}\,\mathcal{CM}_\Sigma(\alpha)\cdot q^\alpha = -\frac{1}{W_E}\sum_{\mathfrak{q}\mid p}\sum_{\substack{\mathbf{T}\in X_\Sigma\\ \mathfrak{p}\in\mathrm{Diff}(\alpha,\mathbf{T})}} E'_\alpha(\tau,0,\mathbf{T})$$

where the outer sum is now over the primes $\mathfrak{q}\mid p$ of $E^\sharp$, and $\mathfrak{p}$ is the prime of $F^\sharp$ below $\mathfrak{q}$. By Lemma 4.5.2 the set $\mathrm{Diff}(\alpha,\mathbf{T})$ contains a unique prime $\mathfrak{p}$ of $F^\sharp$ above p, and that prime must be nonsplit in $E^\sharp$. The double sum on the right now simplifies to

$$\begin{aligned}\widehat{\deg}\,\mathcal{CM}_\Sigma(\alpha)\cdot q^\alpha &= -\frac{1}{W_E}\sum_{\mathfrak{p}\mid p}\#\{\mathfrak{q}\text{ above }\mathfrak{p}\}\sum_{\substack{\mathbf{T}\in X_\Sigma\\ \mathfrak{p}\in\mathrm{Diff}(\alpha,\mathbf{T})}} E'_\alpha(\tau,0,\mathbf{T})\\ &= \frac{-1}{W_E}\sum_{\mathbf{T}\in X_\Sigma} E'_\alpha(\tau,0,\mathbf{T})\end{aligned}$$

as desired.

Now suppose we are in case (**biquad**), so that $F^\sharp\cong\mathbb{Q}\times\mathbb{Q}$. Label the orthogonal idempotents $\epsilon_1,\epsilon_2\in F^\sharp$ in such a way that $\phi_\Sigma : E^\sharp\to E_\Sigma$ satisfies $\phi_\Sigma(\epsilon_2)=0$. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be the two primes of $F^\sharp$ above p, labeled so that $\mathfrak{p}_1$ lies on the summand $\epsilon_1F^\sharp$, and $\mathfrak{p}_2$ lies on the summand $\epsilon_2F^\sharp$. The map $\phi_\Sigma : E^\sharp\to E_\Sigma$ identifies $\epsilon_1E^\sharp\cong E_\Sigma$, and so for any prime $\mathfrak{q}$ of $\mathcal{O}_\Sigma$ we have $\mathfrak{p}^\sharp=\mathfrak{p}_1$. We now deduce

$$\widehat{\deg}\,\mathcal{CM}_\Sigma(\alpha)\cdot q^\alpha = -\frac{1}{W_E}\sum_{\mathfrak{q}_1\mid\mathfrak{p}_1}\sum_{\substack{\mathbf{T}\in X_\Sigma\\ \mathfrak{p}_1\in\mathrm{Diff}(\alpha,\mathbf{T})}} E'_\alpha(\tau,0,\mathbf{T})$$

where the outer sum is over primes $\mathfrak{q}_1 \mid \mathfrak{p}_1$ of $\epsilon_1 E^\sharp$. The inner sum is empty unless $\mathfrak{p}_1$ is nonsplit in $\epsilon_1 E^\sharp$, and so

$$\widehat{\deg}\, \mathcal{CM}_\Sigma(\alpha) \cdot q^\alpha = -\frac{1}{W_E} \sum_{\substack{\mathbf{T} \in X_\Sigma \\ \mathfrak{p}_1 \in \mathrm{Diff}(\alpha, \mathbf{T})}} E'_\alpha(\tau, 0, \mathbf{T}).$$

As $\mathrm{Diff}(\alpha, \mathbf{T})$ contains exactly one of $\mathfrak{p}_1$ or $\mathfrak{p}_2$ by Lemma 4.5.2, it now suffices to prove that $\mathfrak{p}_2 \in \mathrm{Diff}(\alpha, \mathbf{T})$ implies $E'_\alpha(\tau, 0, \mathbf{T}) = 0$. Consider the $\mathbb{Q}$-quadratic spaces $V_1(\mathbf{T}) = \epsilon_1 V(\mathbf{T})$ and $V_2(\mathbf{T}) = \epsilon_2 V(\mathbf{T})$ with their quadratic forms $Q^\sharp_{\mathbf{T}}$. Proposition 2.3.5 asserts that $V_1(\mathbf{T})$ has signature $(0, 2)$, while $V_2(\mathbf{T})$ has signature $(2, 0)$. As we assume $\mathfrak{p}_2 \in \mathrm{Diff}(\alpha, \mathbf{T})$, $V_2(\mathbf{T})$ does not represent $\epsilon_2 \alpha$ at p. As the set of rational places at which $\epsilon_2\alpha$ is not represented has even cardinality, and as $V_2(\mathbf{T})$ does represent $\epsilon_2\alpha$ at the archimedean place, there is a rational prime $\ell \neq p$ at which $V_2(\mathbf{T})$ does not represent $\epsilon_2\alpha$. This implies that both $W_{\alpha,p}(s, \mathbf{T})$ and $W_{\alpha,\ell}(s, \mathbf{T})$ vanish at $s = 0$, and so $E_\alpha(\tau, s, \mathbf{T})$ vanishes at $s = 0$ to at least order 2. ☐

5.3. **Arithmetic intersections.** In this subsection we assume that we are either in case (**cyclic**) or (**nongal**), and that Hypothesis B holds for *every* rational prime p. On the arithmetic threefold $\mathcal{M}$, we wish to compare the intersection multiplicity of the divisor $\mathcal{T}(m)$ and the codimension two cycle $\mathcal{CM}_\Sigma$ with the Fourier coefficients of a modular form.

For any nonzero $m \in \mathbb{Z}$ define an algebraic stack over $\mathcal{O}_\Sigma$ by

$$\mathcal{T}(m) \cap \mathcal{CM}_\Sigma = \mathcal{T}(m)_{/\mathcal{O}_\Sigma} \times_{\mathcal{M}_{/\mathcal{O}_\Sigma}} \mathcal{CM}_\Sigma,$$

and recall from (3.3.1) the decomposition

$$\mathcal{T}(m) \cap \mathcal{CM}_\Sigma = \bigsqcup_{\substack{\alpha \in F^\sharp \\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha) = m}} \mathcal{CM}_\Sigma(\alpha). \tag{5.3.1}$$

If $m < 0$ then both sides of (5.3.1) are empty. As $F^\sharp$ is a field, only those α with $\alpha \in (F^\sharp)^\times$ contribute to the right hand side, and so both sides are zero dimensional by Proposition 3.3.9. We may further restrict to α totally positive, as $Q^\sharp_{\mathbf{A}}$ is totally positive definite for any $\mathcal{O}_F$-polarized CM abelian surface $\mathbf{A}$. If $\mathfrak{q}$ is a prime of $\mathcal{O}_\Sigma$ and

$$z \in (\mathcal{T}(m) \cap \mathcal{CM}_\Sigma)(\mathbb{F}_\mathfrak{q}^{\mathrm{alg}})$$

is a geometric point, define the *Serre intersection multiplicity* at z by

$$\langle \mathcal{T}(m) : \mathcal{CM}_\Sigma \rangle_z = \sum_{\ell \geq 0} \mathrm{length}_{\mathcal{O}^{\mathrm{sh}}_{\mathcal{M},z}} \mathrm{Tor}_\ell^{\mathcal{O}^{\mathrm{sh}}_{\mathcal{M},z}} \big(\mathcal{O}^{\mathrm{sh}}_{\mathcal{T}(m),z}, \mathcal{O}^{\mathrm{sh}}_{\mathcal{CM}_\Sigma, z} \big).$$

On the right hand side we have shortened $\mathcal{M}_{/\mathcal{O}_\Sigma}$ and $\mathcal{T}(m)_{/\mathcal{O}_\Sigma}$ simply to $\mathcal{M}$ and $\mathcal{T}(m)$ to ease notation. The *finite intersection multiplicity* is defined by

$$\langle \mathcal{T}(m) : \mathcal{CM}_\Sigma \rangle_{\mathrm{fin}} = \sum_{\mathfrak{q}} \log(\mathrm{Nm}(\mathfrak{q})) \sum_{z \in (\mathcal{T}(m) \cap \mathcal{CM}_\Sigma)(\mathbb{F}_\mathfrak{q}^{\mathrm{alg}})} \frac{\langle \mathcal{T}(m) : \mathcal{CM}_\Sigma \rangle_z}{\# \mathrm{Aut}(z)}$$

Theorem 5.3.1. *For any nonzero $m \in \mathbb{Z}$,*

$$\langle \mathcal{T}(m) : \mathcal{CM}_\Sigma \rangle_{\mathrm{fin}} = -\frac{1}{W_E} \sum_{\substack{\alpha \in F^\sharp,\ \alpha \gg 0 \\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha)=m}} c_\Sigma(\alpha, \mathbf{v}).$$

Proof. The strictly Henselian local rings of $\mathcal{T}(m)$ and $\mathcal{CM}_\Sigma$ are Cohen-Macaulay by Propositions 3.3.6 and 3.3.5, respectively, and so [41, p.111] implies that only the $\ell = 0$ term contributes to the Serre intersection multiplicity at z. In other words, $\langle \mathcal{T}(m) : \mathcal{CM}_\Sigma \rangle_z$ is simply the length of the strictly Henselian local ring of $\mathcal{T}(m) \cap \mathcal{CM}_\Sigma$ at z. It follows from (5.3.1) that

$$\langle \mathcal{T}(m) : \mathcal{CM}_\Sigma \rangle_{\mathrm{fin}} = \sum_{\substack{\alpha \in F^\sharp,\ \alpha \gg 0 \\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha)=m}} \widehat{\deg}\, \mathcal{CM}_\Sigma(\alpha),$$

and applying Theorem 5.1.1 and Proposition 5.1.2 completes the proof. □

To appreciate the meaning of Theorem 5.3.1, let $\mathcal{H}$ be the usual complex upper half-plane, and let

$$i_\Delta : \mathcal{H} \to \mathcal{H} \times \mathcal{H} \cong \mathcal{H}_{F^\sharp}$$

be the diagonal embedding. The pullback of $E(\tau, s, \Sigma)$ to $\mathcal{H}$ is a nonholomorphic modular form of weight 2, whose central derivative has a Fourier expansion (for $\tau = \mathbf{u} + i\mathbf{v} \in \mathcal{H}$)

$$E'(i_\Delta(\tau), 0, \Sigma) = \sum_{m \in \mathbb{Z}} b_\Sigma(m, \mathbf{v}) \cdot q^m$$

in which

$$b_\Sigma(m, \mathbf{v}) = \sum_{\substack{\alpha \in F^\sharp \\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha)=m}} c_\Sigma(\alpha, \mathbf{v}).$$

Theorem 5.3.1 exhibits the finite intersection $\langle \mathcal{T}(m) : \mathcal{CM}_\Sigma \rangle_{\mathrm{fin}}$ as a part of the Fourier coefficient $b_\Sigma(m, \mathbf{v})$. Our next goal is to account for the rest of $b_\Sigma(m, \mathbf{v})$. That is to say, we seek an arithmetic interpretation of $c_\Sigma(\alpha, \mathbf{v})$ for α not totally positive.

Abbreviate $\infty^\pm$ for the archimedean place $\infty_\Sigma^\pm$ of Proposition 2.3.5. If $x \in F^\sharp$ let $x_\pm$ be the image of x in the completion of $F^\sharp$ at $\infty^\pm$, and identify this completion with $\mathbb{R}$.

Lemma 5.3.2. *Fix $\tau \in \mathcal{H}_{F^\sharp}$, $\mathbf{T} \in X_\Sigma$, and $\alpha \in (F^\sharp)^\times$. If $\alpha_+ > 0$ and $\alpha_- < 0$ then*

$$\beta_1(4\pi |\mathbf{v}\alpha|_{\infty_\Sigma^-}) \sum_{\mathbf{T} \in X_\Sigma} \sum_{\substack{j \in L(\mathbf{T}) \\ Q_{\mathbf{T}}^\sharp(j)=\alpha}} \frac{1}{W_E} \cdot q^\alpha = -\frac{1}{W_E} \sum_{\mathbf{T} \in X_\Sigma} E'_\alpha(\tau, 0, \mathbf{T}).$$

Otherwise the left hand side is equal to 0.

Proof. If either $\alpha_+ < 0$ or $\alpha_- > 0$, then Proposition 2.3.5 shows that $(V(\mathbf{T}), Q^\sharp_{\mathbf{T}})$ does not represent α. This proves the final claim, and we now assume that $\alpha_+ > 0$ and $\alpha_- < 0$, so that $\infty^- \in \mathrm{Diff}(\alpha, \mathbf{T})$. Recall from Section 4.5 that we have attached two Eisenstein series to $\mathbf{T}$. The first is the incoherent Eisenstein series $E(\tau, s, \mathbf{T})$ constructed from $\mathcal{C}(\mathbf{T})$, and the second is the coherent Eisenstein series $E(g, s, V(\mathbf{T})$ constructed from the adelization of $V(\mathbf{T})$. These two $\mathbb{A}_{F^\sharp}$-quadratic spaces are isomorphic at every place of $F^\sharp$ except ∞^-, and it follows from the constructions that

$$W_{\alpha,v}(g, s, \mathbf{T}) = W_{\alpha,v}(g, s, V(\mathbf{T}))$$

for every place $v \neq \infty^-$. Therefore

$$\frac{E_\alpha(\tau, s, \mathbf{T})}{E_\alpha(\tau, s, V(\mathbf{T}))} = \frac{W_{\alpha_-,\mathbb{R}}(\tau_-, s, \Phi^1_{\mathbb{R}})}{W_{\alpha_-,\mathbb{R}}(\tau_-, s, \Phi^{-1}_{\mathbb{R}})},$$

i.e.,

$$E_\alpha(\tau, s, \mathbf{T}) = \frac{W_{\alpha_-,\mathbb{R}}(\tau_-, s, \Phi^1_{\mathbb{R}})}{W_{\alpha_-,\mathbb{R}}(\tau_-, s, \Phi^{-1}_{\mathbb{R}})} E_\alpha(\tau, s, V(\mathbf{T})).$$

The assumption $\alpha_- < 0$ implies $W_{\alpha_-,\mathbb{R}}(\tau_-, 0, \Phi^1_{\mathbb{R}}) = 0$. Differentiating both sides at $s = 0$ and applying [25, Proposition 2.6] show that

$$E'_\alpha(\tau, 0, \mathbf{T}) = -\frac{1}{2\mathbf{v}_-}\beta_1(4\pi|\mathbf{v}\alpha|_{\infty^-})e^{-4\pi\mathbf{v}_-\alpha_-}E_\alpha(\tau, 0, V(\mathbf{T})). \qquad (5.3.2)$$

The stated equality now follows by combining (5.3.2) with Proposition 4.5.4. □

Theorem 5.3.3. *For any nonzero $m \in \mathbb{Z}$ and any positive $\mathbf{v} \in \mathbb{R}$,*

$$\frac{1}{2} \cdot \mathrm{Gr}(m, \mathbf{v}, \mathcal{CM}_\Sigma) = -\frac{1}{W_E} \sum_{\substack{\alpha \in F^\sharp,\ \alpha \not\gg 0 \\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha) = m}} c_\Sigma(\alpha, \mathbf{v}).$$

Recall that the left hand side was defined by (3.4.2).

Proof. Proposition 3.4.1 implies

$$\mathrm{Gr}(m, \mathbf{v}, \mathcal{CM}_\Sigma) = \sum_{1 \le k \le 4} \sum_{\substack{\alpha \in F^\sharp \\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha) = m}} \beta_1(4\pi|\mathbf{v}\alpha|_{\infty^-_{\Sigma_k}}) \sum_{\mathbf{T} \in X_{\Sigma_k}} \sum_{\substack{j \in L(\mathbf{T}) \\ Q^\sharp_{\mathbf{T}}(j) = \alpha}} \frac{1}{W_E},$$

in which $\Sigma_1, \ldots, \Sigma_4$ are the four CM types of E, labeled so that $\Sigma_1 = \Sigma$ and $\Sigma_2 = \overline{\Sigma}$. If we let $\alpha_\pm \in \mathbb{R}$ be the image of α in the completion of $F^\sharp$ at $\infty^\pm_\Sigma$, then Lemma 5.3.2 implies

$$\begin{aligned}\mathrm{Gr}(m,\mathbf{v},\mathcal{CM}_\Sigma)\cdot q^m &= -\frac{1}{W_E}\sum_{k\in\{1,2\}}\sum_{\substack{\alpha\in F^\sharp\\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha)=m\\ \alpha_+>0,\ \alpha_-<0}}\sum_{\mathbf{T}\in X_{\Sigma_k}} E'_\alpha(i_\Delta(\tau),0,\mathbf{T})\\ &\quad -\frac{1}{W_E}\sum_{k\in\{3,4\}}\sum_{\substack{\alpha\in F^\sharp\\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha)=m\\ \alpha_+<0,\ \alpha_->0}}\sum_{\mathbf{T}\in X_{\Sigma_k}} E'_\alpha(i_\Delta(\tau),0,\mathbf{T})\end{aligned}$$

for all $\tau=\mathbf{u}+i\mathbf{v}\in\mathcal{H}$.

Returning to Section 2.4, define an element $\mathbf{Z}=(\mathcal{O}_E,-1)\in C(E)$. If $\mathbf{T}=(T,\kappa_T,\lambda_T)$ is a polarized CM-module, then $\mathbf{T}\otimes\mathbf{Z}=(T,\kappa_T,-\lambda_T)$, and $\mathbf{T}\mapsto\mathbf{T}\otimes\mathbf{Z}$ defines bijections $X_{\Sigma_1}\to X_{\Sigma_2}$ and $X_{\Sigma_3}\to X_{\Sigma_4}$. There is an isomorphism of $F^\sharp$-quadratic spaces

$$(V(\mathbf{T}),Q^\sharp_{\mathbf{T}})\cong(V(\mathbf{T}\otimes\mathbf{Z}),Q^\sharp_{\mathbf{T}\otimes\mathbf{Z}})$$

identifying $L(\mathbf{T})\cong L(\mathbf{T}\otimes\mathbf{Z})$, and it follows from the construction of $E(\tau,s,\mathbf{T})$ that $E(\tau,s,\mathbf{T})=E(\tau,s,\mathbf{T}\otimes\mathbf{Z})$. Suppose instead we choose $\mathbf{Z}\in C(E)$ as in Proposition 2.4.3. After possibly interchanging Σ_3 and Σ_4, $\mathbf{T}\mapsto\mathbf{T}\otimes\mathbf{Z}$ defines bijections $X_{\Sigma_1}\to X_{\Sigma_3}$ and $X_{\Sigma_2}\to X_{\Sigma_4}$, and again we have $E(\tau,s,\mathbf{T})=E(\tau,s,\mathbf{T}\otimes\mathbf{Z})$. From this discussion it is clear that the sum

$$\sum_{\mathbf{T}\in X_{\Sigma_k}} E(\tau,s,\mathbf{T})$$

is in fact independent of k.

If $\alpha\in F^\sharp$ is totally negative then $\mathrm{Diff}(\alpha,\mathbf{T})$ contains both archimedean places of $F^\sharp$, and it follows that $E(\tau,s,\mathbf{T})$ vanishes to at least order 2 at $s=0$, as (4.3.3) implies that the local Whittaker function vanishes at $s=0$ for both archimedean places. Combining all of this information we deduce that

$$\mathrm{Gr}(m,\mathbf{v},\mathcal{CM}_\Sigma)\cdot q^m=-\frac{2}{W_E}\sum_{\substack{\alpha\in F^\sharp\\ \mathrm{Tr}_{F^\sharp/\mathbb{Q}}(\alpha)=m\\ \alpha\not\gg 0}}\sum_{\mathbf{T}\in X_\Sigma} E'_\alpha(i_\Delta(\tau),0,\mathbf{T})$$

and the claim follows. □

Remark 5.3.4. Nowhere in the proof of Theorem 5.3.3 have we used our standing assumption that Hypothesis B holds for all primes.

As in the introduction, define an arithmetic divisor

$$\widehat{\mathcal{T}}(m,\mathbf{v})=\big(\mathcal{T}(m),\mathrm{Gr}(m,\mathbf{v},\cdot)\big)$$

on $\mathcal{M}$, and set

$$\langle\widehat{\mathcal{T}}(m,\mathbf{v}):\mathcal{CM}_\Sigma\rangle=\langle\mathcal{T}(m):\mathcal{CM}_\Sigma\rangle_{\mathrm{fin}}+\frac{1}{2}\cdot\mathrm{Gr}(m,\mathbf{v},\mathcal{CM}_\Sigma).$$

Corollary 5.3.5. *For any nonzero $m \in \mathbb{Z}$ and any positive $\mathbf{v} \in \mathbb{R}$,*

$$\langle \widehat{\mathcal{T}}(m, \mathbf{v}) : \mathcal{CM}_\Sigma \rangle = -\frac{1}{W_E} \cdot b_\Sigma(\alpha, \mathbf{v})$$

Proof. This is immediate from Theorems 5.3.1 and 5.3.3. □

6. Local calculations

This section contains the technical core of this work, and makes heavy use of the theory of Dieudonné modules, the Grothendieck-Messing deformation theory of p-divisible groups, and Zink's theory of displays.

Fix a fractional $\mathcal{O}_F$-ideal $\mathfrak{c} \supset \mathcal{O}_F$ and a CM type Σ of E. Exactly as in Section 5.2, fix a prime $\mathfrak{q} \subset \mathcal{O}_\Sigma$ and embeddings (5.2.1) such that the resulting embedding $\mathcal{O}_\Sigma \to \mathbb{Q}_p^{\mathrm{alg}}$ lies above $\mathfrak{q}$. Write $\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}$ for the residue field of $\mathbb{Q}_p^{\mathrm{alg}}$, to emphasize its $\mathcal{O}_{\Sigma,\mathfrak{q}}$-algebra structure. Throughout Section 6 we fix a $\mathfrak{c}$-polarized CM module $\mathbf{T}$ with CM type Σ. The $\mathfrak{c}$-polarized CM abelian surface over $\mathbb{C}$ (or over $\mathbb{Q}^{\mathrm{alg}}$ or $\mathbb{C}_p$) associated to $\mathbf{T}$ by Remark 3.2.6 will be denoted $\mathbf{A}(\mathbf{T})$. Denote by $\mathbf{A}$ the reduction of $\mathbf{A}(\mathbf{T})$ to the residue field of $\mathbb{C}_p$. Thus $\mathbf{A}$ is a $\mathfrak{c}$-polarized CM abelian surface over $\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}$ satisfying the Σ-Kottwitz condition. The above choices and notation remain fixed throughout Section 6.

Section 6 has three goals. The first is to compare the two $F_p^\sharp$-quadratic spaces $\big(V_p(\mathbf{T}), Q^\sharp_{\mathbf{T}}\big)$ and $\big(V_p(\mathbf{A}), Q^\sharp_{\mathbf{A}}\big)$, in order to complete the proof of Proposition 5.2.2. The second goal is to compute the formal deformation space of the pair $(\mathbf{A}, j)$ for any $j \in L(\mathbf{A})$, completing the proof of Proposition 5.2.3. The third goal is the calculation of the integral structures $L_p(\mathbf{T}) \subset V_p(\mathbf{T})$ and $L_p(\mathbf{A}) \subset V_p(\mathbf{A})$, which lead to the proof of Proposition 5.2.5.

Let $\mathfrak{q}^\sharp$ be the pullback of $\mathfrak{q}$ by $\phi_\Sigma : \mathcal{O}_{E^\sharp} \to \mathcal{O}_\Sigma$, and let $\mathfrak{p}^\sharp$ be the prime of $F^\sharp$ below $\mathfrak{q}^\sharp$. The elements of the fixed CM type are denoted $\Sigma = \{\pi_3, \pi_4\}$, and we define

$$\pi_1(x) = \pi_3(\overline{x}) \qquad \pi_2(x) = \pi_4(\overline{x}). \tag{6.0.3}$$

Let $\mathfrak{D}_{F^\sharp}$ be the different of $F^\sharp/\mathbb{Q}$, and let $\mathfrak{D}_{E^\sharp/F^\sharp}$ be the different of $E^\sharp/F^\sharp$. If $L_0 \subset L$ are finite extensions of $\mathbb{Q}_p$ then $e(L/L_0)$ denotes the ramification degree of L/L_0. As p is fixed, we abbreviate $\mathbb{E}$ for the subfield $\mathbb{E}_p \subset \mathbb{Q}_p^{\mathrm{alg}}$ of the introduction.

6.1. **Extended Dieudonné modules.** The CM types of E are $\{\pi_1, \pi_2\}$, $\{\pi_3, \pi_4\}$, $\{\pi_1, \pi_4\}$, and $\{\pi_2, \pi_3\}$, and the field $\mathbb{E}$ contains the images of the four reflex homomorphisms $E^\sharp \to \mathbb{Q}_p^{\mathrm{alg}}$ defined in Section 2.1:

$$\begin{aligned} \phi_{12}(x) &= \phi_{\{\pi_1,\pi_2\}}(x) & \qquad \phi_{23}(x) &= \phi_{\{\pi_2,\pi_3\}}(x) \\ \phi_{34}(x) &= \phi_{\{\pi_3,\pi_4\}}(x) & \qquad \phi_{14}(x) &= \phi_{\{\pi_1,\pi_4\}}(x). \end{aligned}$$

The group $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ acts faithfully on each of the sets $\mathrm{Hom}(E, \mathbb{Q}_p^{\mathrm{alg}})$ and $\mathrm{Hom}(E^\sharp, \mathbb{Q}_p^{\mathrm{alg}})$, and to understand these two actions we draw the square

$$\begin{array}{ccc} \pi_1 & \overset{\phi_{12}}{\text{———}} & \pi_2 \\ \phi_{14} \Big| & & \Big| \phi_{23} \\ \pi_4 & \underset{\phi_{34}}{\text{———}} & \pi_3 \end{array} \tag{6.1.1}$$

with vertices $\mathrm{Hom}(E, \mathbb{Q}_p^{\mathrm{alg}})$ and edges $\mathrm{Hom}(E^\sharp, \mathbb{Q}_p^{\mathrm{alg}})$. The relations (6.0.3) imply that $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ acts through rigid motions of (6.1.1), and we view $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ as a subgroup of the dihedral group.

Proposition 6.1.1. *The following are equivalent:*

(1) *the* $\mathfrak{c}$*-polarized CM abelian surface* $\mathbf{A}$ *is supersingular,*
(2) *the prime* $\mathfrak{p}^\sharp$ *is nonsplit in* $E^\sharp$*,*
(3) *the edges* ϕ_{12} *and* ϕ_{34} *of (6.1.1) lie in the same* $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$*-orbit.*

Proof. The maps $\phi_{12}, \phi_{34} : E^\sharp \to \mathbb{E}$ are complex conjugates, and both lie above the prime $\mathfrak{p}^\sharp$ of $F^\sharp$. The equivalence of the second and third conditions is clear from this.

For any prime $\mathfrak{Q}$ of E above p, let $H_{\mathfrak{Q}} \subset \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C}_p)$ be the subset of maps inducing the prime $\mathfrak{Q}$. The proof of the Shimura-Taniyama formula (see for example [9, Corollary 4.3]) shows that

$$\frac{\dim(A[\mathfrak{Q}^\infty])}{\mathrm{height}(A[\mathfrak{Q}^\infty])} = \frac{\#(\Sigma \cap H_{\mathfrak{Q}})}{\# H_{\mathfrak{Q}}}. \tag{6.1.2}$$

By [16, Proposition 2.1.1] the image of $\mathcal{O}_{E,\mathfrak{Q}}$ in the endomorphism ring of $A[\mathfrak{Q}^\infty]$ is its own centralizer, and it follows from the proof of [16, Proposition 2.1.1] that $A[\mathfrak{Q}^\infty]$ is isoclinic: the slope sequence of its Dieudonné module is constant. The simple (up to isogeny) Dieudonné module of slope s/t has dimension s and height t, and so the unique slope of $A[\mathfrak{Q}^\infty]$ is (6.1.2). Therefore $\mathbf{A}$ is supersingular if and only if (6.1.2) is equal to $1/2$ for every $\mathfrak{Q}$. If $[\pi]$ denotes the $\mathrm{Aut}(\mathbb{E}/\mathbb{Q}_p)$-orbit of a map $\pi \in \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{E})$, then this result may be restated as

$$\frac{1}{2} = \frac{\#(\{\pi_3, \pi_4\} \cap [\pi])}{\#[\pi]} \text{ for every } \pi \in \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{E}) \tag{6.1.3}$$

if and only if $\mathbf{A}$ is supersingular.

It only remains to check that the condition (6.1.3) is equivalent to the third condition in the statement of the proposition: simply verify by brute force the desired equivalence for every possible subgroup $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ of the dihedral group of the square (6.1.1). $\square$

Let $W = W(\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}})$ be the Witt ring of $\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}$, and let $x \mapsto x^{\mathrm{Fr}}$ be the unique ring automorphism of W inducing $x \mapsto x^p$ on the residue field. There is a unique ring homomorphism $W \to \mathcal{O}_{\mathbb{C}_p}$ inducing the identity on the common residue field $\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}$, and this homomorphism realizes $W_{\mathbb{Q}} = W \otimes_{\mathbb{Z}} \mathbb{Q}$ as the completion of the maximal unramified extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$. The (covariant) Dieudonné module of A is denoted (D, F, V), so that D is a free W-module of rank four and F and V are continuous group homomorphisms $F, V : D \to D$ satisfying $FV = p = VF$ and

$$w^{\mathrm{Fr}} F(x) = F(wx)$$

for every $w \in W$. The *Lie algebra* of D is the $\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}$-vector space

$$\mathrm{Lie}(D) = D/VD, \tag{6.1.4}$$

which is canonically isomorphic to the Lie algebra of A (using the isomorphism [52, (157)]). The action $\kappa_A : \mathcal{O}_E \to \mathrm{End}(A)$ induces an action $\kappa_D : \mathcal{O}_E \to \mathrm{End}(D)$, and the polarization λ_A induces an alternating W-bilinear form $\lambda_D : D \times D \to W$ satisfying

$$\lambda_D(Fx, y) = \lambda_D(x, Vy)^{\mathrm{Fr}}.$$

Let $\mathcal{W}_{\mathbb{Q}}$ denote the completion of the maximal unramified extension of $\mathbb{E}$ in $\mathbb{C}_p$, a finite extension of $W_{\mathbb{Q}}$, and let $\mathcal{W} \subset \mathcal{W}_{\mathbb{Q}}$ be its valuation ring. In order to diagonalize the action of $\mathcal{O}_E$ on D we define

$$\mathcal{D} = D \otimes_W \mathcal{W} \qquad \mathcal{D}_{\mathbb{Q}} = D \otimes_W \mathcal{W}_{\mathbb{Q}}.$$

The action κ_D extends uniquely to $\kappa_{\mathcal{D}} : \mathcal{O}_E \to \mathrm{End}_{\mathcal{W}}(\mathcal{D})$, and the pairing λ_D extends uniquely to a $\mathcal{W}$-linear pairing $\lambda_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} \to \mathcal{W}$. The operators F and V induce a family of operators on $\mathcal{D}$, which we now describe.

Let $\mathrm{Fr}^{\mathbb{Z}} \cong \mathbb{Z}$ be the group of field automorphisms of $W_{\mathbb{Q}}$ generated by Fr. We will say that $\gamma \in \mathrm{Aut}(\mathcal{W}_{\mathbb{Q}}/\mathbb{Q}_p)$ is *algebraic* if the restriction of γ to $\mathrm{Aut}(W_{\mathbb{Q}}/\mathbb{Q}_p)$ lies in $\mathrm{Fr}^{\mathbb{Z}}$, and when this is the case we define $k(\gamma) \in \mathbb{Z}$ by the relation

$$\gamma|_{W_{\mathbb{Q}}} = \mathrm{Fr}^{k(\gamma)}.$$

Let $\Gamma \subset \mathrm{Aut}(\mathcal{W}_{\mathbb{Q}}/\mathbb{Q}_p)$ be the subgroup of algebraic elements. There are obvious surjective homomorphisms $\Gamma \to \mathbb{Z}$ and $\Gamma \to \mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$, the first defined by $\gamma \mapsto k(\gamma)$ and the second by restriction. The fiber over $k \in \mathbb{Z}$ of the first map is denoted $\Gamma(k)$. The second map identifies $\Gamma(0) = \mathrm{Gal}(\mathcal{W}_{\mathbb{Q}}/W_{\mathbb{Q}})$ with the inertia subgroup of $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$. For each $\gamma \in \Gamma$ there is a pair of operators $\mathcal{F}_\gamma$ and $\mathcal{V}_\gamma$ on $\mathcal{D}$ defined by

$$\mathcal{F}_\gamma(x \otimes w) = (F^{k(\gamma)}x) \otimes (w^{\gamma})$$
$$\mathcal{V}_\gamma(x \otimes w) = (V^{k(\gamma)}x) \otimes (w^{\gamma^{-1}}).$$

These operators commute with the action $\kappa_{\mathcal{D}}$ and satisfy

$$\mathcal{F}_\gamma \circ \mathcal{V}_\gamma = p^{k(\gamma)} = \mathcal{V}_\gamma \circ \mathcal{F}_\gamma.$$

One may recover $D = \mathcal{D}^{\Gamma(0)}$ from $\mathcal{D}$ as the W-submodule of elements fixed by $\mathcal{F}_\gamma$ and $\mathcal{V}_\gamma$ for every $\gamma \in \Gamma(0)$, and then F and V agree with the restrictions to D of $\mathcal{F}_\gamma$ and $\mathcal{V}_\gamma$ for any $\gamma \in \Gamma(1)$. The $\mathcal{W}$-module $\mathcal{D}$ with its family of operators $\{\mathcal{F}_\gamma : \gamma \in \Gamma\} \cup \{\mathcal{V}_\gamma : \gamma \in \Gamma\}$ is the *extended Dieudonné module of* A. The pairing $\lambda_{\mathcal{D}}$ satisfies

$$\lambda_{\mathcal{D}}(\mathcal{F}_\gamma x, y) = \lambda_{\mathcal{D}}(x, \mathcal{V}_\gamma y)^{\gamma} \text{ for all } \gamma \in \Gamma \tag{6.1.5}$$

and

$$\mathfrak{c}\mathcal{D} = \{w \in \mathcal{D}_{\mathbb{Q}} : \lambda_{\mathcal{D}}(w, z) \in \mathcal{W} \text{ for all } z \in \mathcal{D}\}. \tag{6.1.6}$$

In order to do explicit calculations, we now put coordinates on $\mathcal{D}$. Fix, once and for all, an isomorphism of $\mathcal{O}_E \otimes_{\mathbb{Z}} W$-modules $D \cong \mathcal{O}_E \otimes_{\mathbb{Z}} W$ (see

[16, Proposition 2.1.1] for the existence of such an isomorphism). Of course this identifies $\mathcal{D} \cong \mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{W}$. Let $e_1, e_2, e_3, e_4 \in \mathcal{W}^4$ be the standard basis elements (written as column vectors). The $\mathcal{W}$-linear map $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{W} \to \mathcal{W}^4$ determined by

$$x \otimes 1 \mapsto \begin{bmatrix} \pi_1(x) \\ \pi_2(x) \\ \pi_3(x) \\ \pi_4(x) \end{bmatrix} \tag{6.1.7}$$

extends $\mathcal{W}_{\mathbb{Q}}$-linearly to an isomorphism

$$\mathcal{D}_{\mathbb{Q}} \cong \mathcal{W}_{\mathbb{Q}}^4, \tag{6.1.8}$$

which we use to identify $\mathcal{D}$ with a submodule of $\mathcal{W}^4$. If p is unramified in E then $\mathcal{D} = \mathcal{W}^4$, but in general the inclusion is proper. Under the identification (6.1.8) the action $\kappa_{\mathcal{D}} : \mathcal{O}_E \to \mathrm{End}_{\mathcal{W}}(\mathcal{D})$ takes the form

$$\kappa_{\mathcal{D}}(x) = \begin{pmatrix} \pi_1(x) & & & \\ & \pi_2(x) & & \\ & & \pi_3(x) & \\ & & & \pi_4(x) \end{pmatrix}. \tag{6.1.9}$$

The operators $\mathcal{F}_\gamma$ and $\mathcal{V}_\gamma$ with $\gamma \in \Gamma(0)$ each act on $\mathcal{W}^4$ as the product of a permutation matrix with a Galois automorphism:

$$\gamma \circ \pi_i = \pi_j \implies \mathcal{F}_\gamma(we_i) = w^\gamma \cdot e_j. \tag{6.1.10}$$

for every $w \in \mathcal{W}$. For a general $\gamma \in \Gamma$ we cannot know the precise form of $\mathcal{F}_\gamma$ without explicit knowledge of the operator F from which it is derived. However, if $\gamma \circ \pi_i = \pi_j$ then for any $x \in \mathcal{O}_E$ we have

$$\begin{aligned} \kappa_{\mathcal{D}}(x)\mathcal{F}_\gamma(e_i) &= \mathcal{F}_\gamma(\kappa_{\mathcal{D}}(x)e_i) \\ &= \mathcal{F}_\gamma(\pi_i(x)e_i) \\ &= \pi_i(x)^\gamma \mathcal{F}_\gamma(e_i) \\ &= \pi_j(x)\mathcal{F}_\gamma(e_i), \end{aligned}$$

and it follows that $\mathcal{F}_\gamma(e_i)$ is a scalar multiple of e_j. This shows that

$$\gamma \circ \pi_i = \pi_j \implies \mathcal{F}_\gamma(we_i) = w^\gamma x_{i,j} \cdot e_j \tag{6.1.11}$$

for some $x_{i,j} \in \mathcal{W}_{\mathbb{Q}}^\times$. Thus $\mathcal{F}_\gamma$ has the form $\mathcal{F}_\gamma = X \circ \gamma$ for some matrix $X \in M_4(\mathcal{W}_{\mathbb{Q}})$ having a unique nonzero entry in each row and each column, and the location of the nonzero entries may be read off from the action of γ on the diagram (6.1.1).

It follows from Lemma 3.2.2 that

$$\lambda_{\mathcal{D}}(\kappa_{\mathcal{D}}(t)x, y) = \lambda_{\mathcal{D}}(x, \kappa_{\mathcal{D}}(\bar{t})y)$$

for all $t \in \mathcal{O}_E$, and so the pairing $\lambda_{\mathcal{D}}$ has the form

$$\lambda_{\mathcal{D}}(x,y) = {}^t x \cdot \begin{pmatrix} & & \rho_1 & \\ & & & \rho_2 \\ -\rho_1 & & & \\ & -\rho_2 & & \end{pmatrix} \cdot y \tag{6.1.12}$$

for some $\rho_1, \rho_2 \in \mathcal{W}_{\mathbb{Q}}^{\times}$. Every special endomorphism $j \in V_p(\mathbf{A})$ induces an endomorphism of $\mathcal{D}_{\mathbb{Q}}$, which is given by the action of some element of $M_4(\mathcal{W}_{\mathbb{Q}})$. The conditions $j = j^*$ and $\kappa_A(x) \circ j = j \circ \kappa_A(x^\sigma)$ for $x \in \mathcal{O}_F$ imply that this matrix has the form

$$j = \begin{pmatrix} & \rho_2 b & & \rho_2 c \\ \rho_1 a & & -\rho_1 c & \\ & -\rho_2 d & & \rho_2 a \\ \rho_1 d & & \rho_1 b & \end{pmatrix} \in M_4(\mathcal{W}_{\mathbb{Q}}). \tag{6.1.13}$$

The condition that j commutes with all operators $\mathcal{F}_\gamma$ and $\mathcal{V}_\gamma$ imposes additional relations among the matrix entries, and the condition $j \in L_p(\mathbf{A})$ (which is equivalent to $j \cdot \mathcal{D} \subset \mathcal{D}$) imposes still more relations. All of these additional relations depend on the action of Γ on (6.1.1), and will be determined on a case-by-case basis. Using the description (6.1.9) of the action of $\mathcal{O}_E$ on $\mathcal{D}_{\mathbb{Q}}$, the action of $E_p^\sharp$ on $V_p(\mathbf{A})$ defined in Section 3.2 takes the explicit form
(6.1.14)

$$x \bullet j = \begin{pmatrix} & \rho_2 b \cdot \phi_{14}(x) & & \rho_2 c \cdot \phi_{12}(x) \\ \rho_1 a \cdot \phi_{23}(x) & & -\rho_1 c \cdot \phi_{12}(x) & \\ & -\rho_2 d \cdot \phi_{34}(x) & & \rho_2 a \cdot \phi_{23}(x) \\ \rho_1 d \cdot \phi_{34}(x) & & \rho_1 b \cdot \phi_{14}(x) & \end{pmatrix}.$$

The construction of the extended Dieudonné module of $\mathbf{A}$ has an analogue for $\mathbf{T}$. Define

$$\mathcal{T} = T \otimes_{\mathbb{Z}} \mathcal{W} \qquad \mathcal{T}_{\mathbb{Q}} = \mathcal{T} \otimes_{\mathcal{W}} \mathcal{W}_{\mathbb{Q}}.$$

The action κ_T induces a $\mathcal{W}$-linear action $\kappa_{\mathcal{T}} : \mathcal{O}_E \to \mathrm{End}_{\mathcal{W}}(\mathcal{T})$, and the symplectic form λ_T extends to a $\mathcal{W}$-symplectic form $\lambda_{\mathcal{T}} : \mathcal{T} \times \mathcal{T} \to \mathcal{W}$. For each $\gamma \in \Gamma$ there are operators $\mathcal{F}_\gamma$ and $\mathcal{V}_\gamma$ on $\mathcal{T}$ defined by

$$\mathcal{F}_\gamma(x \otimes w) = x \otimes (w^\gamma) \qquad \mathcal{V}_\gamma(x \otimes w) = x \otimes (w^{\gamma^{-1}})$$

and satisfying $\mathcal{F}_\gamma \circ \mathcal{V}_\gamma = \mathrm{id} = \mathcal{V}_\gamma \circ \mathcal{F}_\gamma$. Furthermore

$$\lambda_{\mathcal{T}}(\mathcal{F}_\gamma x, y) = \lambda_{\mathcal{T}}(x, \mathcal{V}_\gamma y)^\gamma \quad \forall \gamma \in \Gamma. \tag{6.1.15}$$

We may recover $\mathbf{T}$ from $\mathcal{T}$ as the $\mathbb{Z}_p$-submodule of elements fixed by $\mathcal{F}_\gamma$ for every $\gamma \in \Gamma$. Fix, once and for all, an isomorphism of $\mathcal{O}_E$-modules $T \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathcal{O}_{E,p}$. This induces an isomorphism $\mathcal{T} \cong \mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{W}$, and the map (6.1.7) then determines a $\mathcal{W}_{\mathbb{Q}}$-linear isomorphism $\mathcal{T}_{\mathbb{Q}} \to \mathcal{W}_{\mathbb{Q}}^4$. Using this isomorphism we view $\mathcal{T}$ as a submodule of $\mathcal{W}^4$. As submodules of $\mathcal{W}^4$ we have $\mathcal{D} = \mathcal{T}$, and the action $\kappa_{\mathcal{T}}$ is given by the same formula (6.1.9) as $\kappa_{\mathcal{D}}$. The pairing $\lambda_{\mathcal{T}}$ has the same form (6.1.12) as $\lambda_{\mathcal{D}}$, although the values of ρ_1 and ρ_2 for $\lambda_{\mathcal{T}}$ need not be the same as those for $\lambda_{\mathcal{D}}$; it will always be clear from context whether ρ_1 and ρ_2 refer to the pairing $\lambda_{\mathcal{D}}$ or $\lambda_{\mathcal{T}}$. Every $j \in V_p(\mathbf{T})$ induces a $\mathcal{W}$-linear endomorphism of $\mathcal{T}$ of the form (6.1.13) which commutes with all operators $\mathcal{F}_\gamma$ and $\mathcal{V}_\gamma$, and satisfies

$$j \in L_p(\mathbf{T}) \iff j \cdot \mathcal{T} \subset \mathcal{T}.$$

The action of $E_p^\sharp$ on $V_p(\mathbf{T})$ again has the explicit form (6.1.14). For every $\gamma \in \Gamma$ each of $\mathcal{F}_\gamma$ and $\mathcal{V}_\gamma$ (now acting on $\mathcal{W}_{\mathbb{Q}}^4$) may be written as the product of a permutation matrix and a Galois automorphism. More precisely, (6.1.10) holds for every $\gamma \in \Gamma$ (not just $\gamma \in \Gamma(0)$).

Proposition 6.1.2. *For any $j \in V_p(\mathbf{A})$, written in the form (6.1.13),*

$$e(\mathbb{E}/F^\sharp_{\mathfrak{p}^\sharp}) \cdot \mathrm{ord}_{\mathfrak{p}^\sharp}\big(Q^\sharp_{\mathbf{A}}(j)\big) = \mathrm{ord}_{\mathcal{W}}(\rho_1\rho_2) + \mathrm{ord}_{\mathcal{W}}(cd).$$

If there is another prime $\mathfrak{p}^\sharp_* \neq \mathfrak{p}^\sharp$ of $F^\sharp$ above p then

$$e(\mathbb{E}/F^\sharp_{\mathfrak{p}^\sharp_*}) \cdot \mathrm{ord}_{\mathfrak{p}^\sharp_*}\big(Q^\sharp_{\mathbf{A}}(j)\big) = \mathrm{ord}_{\mathcal{W}}(\rho_1\rho_2) + \mathrm{ord}_{\mathcal{W}}(ab).$$

The same statement holds with $\mathbf{A}$ replaced everywhere by $\mathbf{T}$.

Proof. Label the orthogonal idempotents $\epsilon_{23}, \epsilon_{14}, \epsilon_{12}, \epsilon_{34}$ in

$$E_p^\sharp \otimes_{\mathbb{Q}_p} \mathcal{W}_{\mathbb{Q}} \cong \mathcal{W}_{\mathbb{Q}}^4$$

in such a way that $(x \otimes 1) \cdot \epsilon_{ij} = 1 \otimes \phi_{ij}(x)$ for all $x \in E_p^\sharp$. If we identify $V_p(\mathbf{A}) \otimes_{\mathbb{Q}_p} \mathcal{W}_{\mathbb{Q}}$ with the space of all matrices (6.1.13), then (6.1.14) shows that

$$\epsilon_{23} \bullet j = \begin{pmatrix} & 0 & & 0 \\ \rho_1 a & & 0 & \\ & 0 & & \rho_2 a \\ 0 & & 0 & \end{pmatrix} \qquad \epsilon_{14} \bullet j = \begin{pmatrix} & \rho_2 b & & 0 \\ 0 & & 0 & \\ & 0 & & 0 \\ 0 & & \rho_1 b & \end{pmatrix}$$

$$\epsilon_{12} \bullet j = \begin{pmatrix} & 0 & & \rho_2 c \\ 0 & & -\rho_1 c & \\ & 0 & & 0 \\ 0 & & 0 & \end{pmatrix} \qquad \epsilon_{34} \bullet j = \begin{pmatrix} & 0 & & 0 \\ 0 & & 0 & \\ & -\rho_2 d & & 0 \\ \rho_1 d & & 0 & \end{pmatrix}.$$

Let $f : F_p^\sharp \to \mathcal{W}_{\mathbb{Q}}$ be the common restriction to $F_p^\sharp$ of the two maps $\phi_{34}, \phi_{12} : E_p^\sharp \to \mathcal{W}_{\mathbb{Q}}$, so that f induces the prime $\mathfrak{p}^\sharp$ of $F_p^\sharp$. If we set $\epsilon = \epsilon_{12} + \epsilon_{34}$ and extend f to a map $F_p^\sharp \otimes_{\mathbb{Q}_p} \mathcal{W}_{\mathbb{Q}} \to \mathcal{W}_{\mathbb{Q}}$, then

$$f(Q^\sharp_{\mathbf{A}}(j)) = f(\epsilon Q^\sharp_{\mathbf{A}}(j)) = f(Q_{\mathbf{A}}(\epsilon j)) = \rho_1\rho_2 cd.$$

Therefore

$$\mathrm{ord}_{\mathcal{W}}(f(Q^\sharp_{\mathbf{A}}(j))) = \mathrm{ord}_{\mathcal{W}}(\rho_1\rho_2) + \mathrm{ord}_{\mathcal{W}}(cd).$$

This proves the first claim, and the proof of the remaining claims is similar. □

Lemma 6.1.3. *If $\mathfrak{c} = \mathcal{O}_F$ then*

$$\frac{\mathrm{ord}_{\mathcal{W}}(\mathrm{disc}(E/\mathbb{Q}))}{2} = \mathrm{length}_{\mathcal{W}}(\mathcal{W}^4/\mathcal{D}) = -\mathrm{ord}_{\mathcal{W}}(\rho_1\rho_2).$$

The same statement holds with $\mathcal{D}$ replaced by $\mathcal{T}$.

Proof. The isomorphism $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{W}_{\mathbb{Q}} \to \mathcal{W}_{\mathbb{Q}}^4$ defined by (6.1.7) identifies the $\mathcal{W}_{\mathbb{Q}}$-bilinear form $q(x_1 \otimes 1, x_2 \otimes 1) = \mathrm{Tr}_{E/\mathbb{Q}}(x_1 x_2)$ on the left with the usual dot product on the right. If $\mathcal{D}^\vee$ denotes the dual lattice of $\mathcal{D}$ with respect to the dot product on $\mathcal{W}_{\mathbb{Q}}^4$, it follows that

$$\mathrm{ord}_{\mathcal{W}}(\mathrm{disc}(E/\mathbb{Q})) = \mathrm{length}_{\mathcal{W}}(\mathcal{D}^\vee/\mathcal{D}) = 2 \cdot \mathrm{length}_{\mathcal{W}}(\mathcal{W}^4/\mathcal{D}).$$

This proves the first equality. For the second, (6.1.6) implies that $\mathcal{D}$ is self-dual with respect to the pairing (6.1.12), while the dual lattice $\mathcal{L}$ of $\mathcal{W}^4$ with respect to the same pairing satisfies $\mathrm{length}_{\mathcal{W}}(\mathcal{D}/\mathcal{L}) = \mathrm{length}_{\mathcal{W}}(\mathcal{W}^4/\mathcal{D})$ and $\mathrm{length}_{\mathcal{W}}(\mathcal{W}^4/\mathcal{L}) = -2 \cdot \mathrm{ord}_{\mathcal{W}}(\rho_1\rho_2)$. □

6.2. **Deformation theory.** The image of $\phi_\Sigma = \phi_{34} : E_p^\sharp \to \mathbb{Q}_p^{\mathrm{alg}}$ is contained in $\mathbb{E}$, and is denoted $\mathbb{E}_\Sigma$. Recalling that $\mathfrak{q}^\sharp$ is the prime of $E^\sharp$ corresponding to ϕ_Σ, there is an isomorphism of $\mathbb{Q}_p$-algebras $E^\sharp_{\mathfrak{q}^\sharp} \to \mathbb{E}_\Sigma$. Let $\mathcal{W}_{\Sigma,\mathbb{Q}}$ be the completion of the maximal unramified extension of $\mathbb{E}_\Sigma$ in $\mathbb{C}_p$, and let $\mathcal{W}_\Sigma \subset \mathcal{W}_{\Sigma,\mathbb{Q}}$ be the valuation ring. Thus we have inclusions $W \subset \mathcal{W}_\Sigma \subset \mathcal{W}$. Let $\mathbf{Art}_\Sigma$ be the category of local Artinian $\mathcal{W}_\Sigma$-algebras with residue field $\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}$.

Proposition 6.2.1. *Suppose R is any object of $\mathbf{Art}_\Sigma$, and $\mathbf{A}$ is a $\mathfrak{c}$-polarized CM abelian surface over $\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}$ satisfying the Σ-Kottwitz condition. There is a unique way to lift $\mathbf{A}$ to R in such a way that the lift again satisfies the Σ-Kottwitz condition.*

Proof. This is a special case of [16, Theorem 2.2.1]. □

For any $j \in L(\mathbf{A})$ let $\mathrm{Def}_\Sigma(\mathbf{A}, j)$ be the functor that assigns to every object R of $\mathbf{Art}_\Sigma$ the set of isomorphism classes of deformations (satisfying the Σ-Kottwitz condition) of $(\mathbf{A}, j)$ to R. Let $\mathfrak{Q}_\Sigma$ be the maximal ideal of $\mathcal{W}_\Sigma$. In the situation of Proposition 6.2.1, the unique lift of $\mathbf{A}$ is the *canonical lift* of $\mathbf{A}$ to R. For any positive integer m let $\mathbf{A}_m^{\mathrm{can}}$ be the canonical lift of $\mathbf{A}$ to $\mathcal{W}_\Sigma/\mathfrak{Q}_\Sigma^m$.

Corollary 6.2.2. *For any $j \in L(\mathbf{A})$ the deformation functor $\mathrm{Def}_\Sigma(\mathbf{A}, j)$ is pro-represented by $\mathcal{W}_\Sigma/\mathfrak{Q}_\Sigma^m$ where m is the largest positive integer (including possibly $m = \infty$) for which j lifts to $L(\mathbf{A}_m^{\mathrm{can}})$.*

Proof. Let $\mathrm{Def}_\Sigma(\mathbf{A})$ be the functor that assigns to every object R of $\mathbf{Art}_\Sigma$ the set of isomorphism classes of deformations (satisfying the Σ-Kottwitz condition) of $\mathbf{A}$ to R. By Proposition 6.2.1 $\mathrm{Def}_\Sigma(\mathbf{A})$ is pro-represented by $\mathcal{W}_\Sigma$. It now follows from [39, Proposition 2.9] the functor $\mathrm{Def}_\Sigma(\mathbf{A}, j)$ is pro-represented by a quotient of $\mathcal{W}_\Sigma$, and the claim follows easily. □

Define a rank two $\mathcal{W}$-direct summand $\mathcal{D}^1 \subset \mathcal{D}$ by

$$\mathcal{D}^1 = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \in \mathcal{D} : \begin{array}{c} w_3 = 0 \\ w_4 = 0 \end{array} \right\}$$

and set $\mathcal{D}^1_{\mathbb{Q}} = \mathcal{D}^1 \otimes_{\mathcal{W}} \mathcal{W}_{\mathbb{Q}}$. For any $j \in L_p(\mathbf{A})$ define, following Zink [52, Section 2.5],

$$\mathrm{obst}(j) : \mathcal{D}^1 \to \mathcal{D}/\mathcal{D}^1$$

as the composition $\mathcal{D}^1 \to \mathcal{D} \xrightarrow{j} \mathcal{D} \to \mathcal{D}/\mathcal{D}^1$ where the first arrow is the inclusion and the third arrow is the quotient map. Extending scalars to $\mathcal{W}_{\mathbb{Q}}$, the submodule $\mathcal{D}^1_{\mathbb{Q}} \subset \mathcal{D}_{\mathbb{Q}}$ is spanned by e_1 and e_2, and the $\mathcal{W}_{\mathbb{Q}}$-module map $q : \mathcal{D}_{\mathbb{Q}} \to \mathcal{W}^2_{\mathbb{Q}}$ defined by

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \mapsto \begin{bmatrix} w_3 \\ w_4 \end{bmatrix} \tag{6.2.1}$$

identifies $\mathcal{D}_{\mathbb{Q}}/\mathcal{D}^1_{\mathbb{Q}} \cong \mathcal{W}^2_{\mathbb{Q}}$. The map $\mathrm{obst}(j) : \mathcal{D}^1_{\mathbb{Q}} \to \mathcal{D}_{\mathbb{Q}}/\mathcal{D}^1_{\mathbb{Q}}$ can be read off from the lower left 2×2 block of (6.1.13), and has the explicit form

$$\mathrm{obst}(j)(e_1) = \begin{bmatrix} -\rho_2 d \\ 0 \end{bmatrix} \qquad \mathrm{obst}(j)(e_2) = \begin{bmatrix} 0 \\ \rho_1 d \end{bmatrix}. \tag{6.2.2}$$

Let $\mathfrak{Q}$ denote the maximal ideal of $\mathcal{W}$, and define

$$\mathrm{obst}_m(j) : \mathcal{D}^1 \otimes_{\mathcal{W}} \mathcal{W}/\mathfrak{Q}^m \to (\mathcal{D}/\mathcal{D}^1) \otimes_{\mathcal{W}} \mathcal{W}/\mathfrak{Q}^m$$

to be the reduction of $\mathrm{obst}(j)$ modulo $\mathfrak{Q}^m$. The next proposition shows that $\mathrm{obst}(j)$ measures is the obstruction to lifting $j \in L_p(\mathbf{A})$ to a special endomorphism of the canonical lift of $\mathbf{A}$.

Proposition 6.2.3. *Assume $e(\mathbb{E}_\Sigma/\mathbb{Q}_p) < p$, and that $\mathbf{A}$ is supersingular. For every $j \in L(\mathbf{A})$ and $m \in \mathbb{Z}^+$ we have*

$$j \textit{ lifts to } L(\mathbf{A}^{\mathrm{can}}_m) \iff \mathrm{obst}_{em}(j) = 0,$$

where $e = e(\mathbb{E}/\mathbb{E}_\Sigma)$.

Proof. Let $\mathcal{D}_{\mathrm{crys}}$ be the covariant crystal associated to A by Grothendieck-Messing theory [14, 31], or by Zink's theory of displays [32, 52]. Thus for any object R of $\mathbf{Art}_\Sigma$ for which the kernel of $R \to \mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}$ is equipped with divided powers (compatible with the canonical divided powers on pR), $\mathcal{D}_{\mathrm{crys}}(R)$ is a free R-module equipped with an action of $\mathcal{O}_E$ and a canonical isomorphism [52, Proposition 51] $\mathcal{D}_{\mathrm{crys}}(R) \cong D \otimes_W R$. Let $\mathfrak{Q}_\Sigma$ be the maximal ideal of $\mathcal{W}_\Sigma$. The hypothesis $e(\mathbb{E}_\Sigma/\mathbb{Q}_p) < p$ implies that the maximal ideal of $R = \mathcal{W}_\Sigma/\mathfrak{Q}_\Sigma^m$ is equipped with canonical divided powers [14, Chapter IV.1.3]. Thus if we set $\mathcal{D}_\Sigma = D \otimes_W \mathcal{W}_\Sigma$, there is a canonical isomorphism

$$\mathcal{D}_{\mathrm{crys}}(R) \cong \mathcal{D}_\Sigma \otimes_{\mathcal{W}_\Sigma} R.$$

The unique lift $\mathbf{A}^{\mathrm{can}}_m$ of $\mathbf{A}$ to R corresponds to the unique $\mathcal{O}_E$-stable R-direct summand

$$\mathrm{Fil}^1 \mathcal{D}_{\mathrm{crys}}(R) \subset \mathcal{D}_{\mathrm{crys}}(R)$$

lifting the Hodge filtration

$$\mathrm{Fil}^1 \mathcal{D}_{\mathrm{crys}}(\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}) = \ker\big(\mathcal{D}_{\mathrm{crys}}(\mathbb{F}_{\mathfrak{q}}^{\mathrm{alg}}) \to \mathrm{Lie}(A)\big).$$

This unique lift of the Hodge filtration is determined in [16, §2.1], and has the following form. Let J_Σ be the kernel of the $\mathcal{W}_\Sigma$-algebra map

$$\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{W}_\Sigma \to \mathbb{C}_p \times \mathbb{C}_p$$

defined by $x \otimes 1 \mapsto (\pi_3(x), \pi_4(x))$. Then J_Σ is an ideal in $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{W}_\Sigma$, and

$$\mathrm{Fil}^1 \mathcal{D}_{\mathrm{crys}}(R) = J_\Sigma \mathcal{D}_{\mathrm{crys}}(R).$$

The special endomorphism j of $\mathbf{A}$ induces an endomorphism j_{crys} of $\mathcal{D}_{\mathrm{crys}}(R)$, and j lifts to $L(\mathbf{A}_m^{\mathrm{can}})$ if and only if j_{crys} preserves the direct summand $J_\Sigma \mathcal{D}_{\mathrm{crys}}(R)$ (see [52, Theorem 48]). This latter condition is equivalent to the vanishing of the map

$$J_\Sigma \mathcal{D}_{\mathrm{crys}}(R) \xrightarrow{j_{\mathrm{crys}}} \mathcal{D}_{\mathrm{crys}}(R)/J_\Sigma \mathcal{D}_{\mathrm{crys}}(R). \tag{6.2.3}$$

The extension $\mathcal{W}_\Sigma \to \mathcal{W}$ is faithfully flat, so the vanishing of (6.2.3) is equivalent to the vanishing of

$$(J_\Sigma \mathcal{D}) \otimes_{\mathcal{W}} \mathcal{W}/\mathfrak{Q}^{em} \xrightarrow{j} (\mathcal{D}/J_\Sigma \mathcal{D}) \otimes_{\mathcal{W}} \mathcal{W}/\mathfrak{Q}^{em}.$$

As $J_\Sigma \mathcal{D} = \mathcal{D}^1$, the vanishing of this latter map is equivalent to the vanishing of $\mathrm{obst}_{em}(j)$. □

Proposition 6.2.4. *Assume that* $\mathbf{A}$ *is supersingular.*

(1) *For every prime* $\ell \neq p$ *there is an* $E_\ell^\sharp$*-linear isomorphism of* $F_\ell^\sharp$*-quadratic spaces*

$$\big(V_\ell(\mathbf{T}), Q_{\mathbf{T}}^\sharp\big) \cong \big(V_\ell(\mathbf{A}), Q_{\mathbf{A}}^\sharp\big)$$

taking $L_\ell(\mathbf{T})$ *isomorphically to* $L_\ell(\mathbf{A})$.

(2) *The archimedean place* $w = \infty_\Sigma^-$ *of* $F^\sharp$ *determined by the reflex map* $\phi_\Sigma : E^\sharp \to \mathbb{C}$ *is the unique archimedean place for which*

$$\big(V_w(\mathbf{T}), Q_{\mathbf{T}}^\sharp\big) \not\cong \big(V_w(\mathbf{A}), Q_{\mathbf{A}}^\sharp\big).$$

(3) *The prime* $\mathfrak{p} = \mathfrak{p}^\sharp$ *of* $F^\sharp$ *is the unique prime of* $F^\sharp$ *above* p *for which*

$$\big(V_{\mathfrak{p}}(\mathbf{T}), Q_{\mathbf{T}}^\sharp\big) \not\cong \big(V_{\mathfrak{p}}(\mathbf{A}), Q_{\mathbf{A}}^\sharp\big).$$

Proof. Suppose $\ell \neq p$ and let $\mathrm{Ta}_\ell(A)$ be the ℓ-adic Tate module of A. As $\mathbf{T}$ is identified with the first homology of $\mathbf{A}(\mathbf{T})$, and reduction modulo p induces an isomorphism on ℓ-adic Tate modules, there is canonical isomorphism

$$\mathrm{Ta}_\ell(A) \cong T \otimes_{\mathbb{Z}} \mathbb{Z}_\ell,$$

and hence a canonical injection $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathrm{End}(T) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. It is easy to see that this injection has torsion-free cokernel, and restricts to an injection $L_\ell(\mathbf{A}) \to L_\ell(\mathbf{T})$ with torsion-free cokernel. Both sides are free $\mathbb{Z}_\ell$-modules of rank four (by Propositions 2.2.2 and 3.1.3), and so this map is an isomorphism. This proves the first claim.

The second claim is clear from Proposition 2.3.5 and the fact that $Q_{\mathbf{A}}^\sharp$ is totally positive definite.

For the third claim, first suppose that $\mathfrak{p} = \mathfrak{p}^\sharp$ is the unique prime of $F^\sharp$ above p. The global quadratic spaces $(V(\mathbf{T}), Q^\sharp_{\mathbf{T}})$ and $(V(\mathbf{A}), Q^\sharp_{\mathbf{A}})$ are nonisomorphic at an even number of places of $F^\sharp$, and so the previous two claims imply that they are nonisomorphic at $\mathfrak{p}$.

Now suppose there are two primes of $F^\sharp$ above p, $\mathfrak{p}^\sharp$ and $\mathfrak{p}$, and let $\epsilon_\mathfrak{p} \in F^\sharp_p \cong \mathbb{Q}_p \times \mathbb{Q}_p$ be the idempotent satisfying $\epsilon_\mathfrak{p}\mathfrak{p} = \mathfrak{p}$. For any $j \in V_p(\mathbf{A})$ let $j_p \in \mathrm{End}(A[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ be the corresponding quasi-endomorphism of the p-divisible group of A. By the proof of Proposition 6.1.2, the action of $\epsilon_\mathfrak{p} j_p$ on the extended Dieudonné module $\mathcal{D}$ is through a matrix of the form

$$\epsilon_\mathfrak{p} j_p = \begin{pmatrix} & \rho_2 b & & 0 \\ \rho_1 a & & 0 & \\ & 0 & & \rho_2 a \\ 0 & & \rho_1 b & \end{pmatrix}.$$

In particular $\epsilon_\mathfrak{p} j_p$ preserves the submodule $\mathcal{D}^1$. If we repeat the proof of Proposition 6.2.3 replacing A by $A[p^\infty]$ everywhere, we see that $\epsilon_\mathfrak{p} j_p$ lifts uniquely to a quasi-endomorphism of the p-divisible group of $\mathbf{A}^{\mathrm{can}}_m$ for every $m \in \mathbb{Z}^+$. By applying Grothendieck's formal existence theorem [8, Section 3] to truncated p-divisible groups, we see that $\epsilon_\mathfrak{p} j_p$ lifts to a quasi-endomorphism of the p-divisible group of the unique deformation of $\mathbf{A}$ to $\mathcal{W}_\Sigma$. Applying base change from $\mathcal{W}_\Sigma$ to $\mathbb{C}_p$ shows that $\epsilon_\mathfrak{p} j_p$ lifts to a quasi-endomorphism $(\epsilon_\mathfrak{p} j_p)^\sim$ of the p-divisible group of $\mathbf{A}(\mathbf{T})$, which is none other than $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. We have now constructed an injection $\epsilon_\mathfrak{p} j \mapsto (\epsilon_\mathfrak{p} j_p)^\sim$ from $V_\mathfrak{p}(\mathbf{A})$ to $V_\mathfrak{p}(\mathbf{T})$. As both are free of rank two over $F^\sharp_\mathfrak{p}$ (by (2.3.4) and Proposition 3.2.4), this map is an isomorphism. The proof that $(V(\mathbf{T}), Q^\sharp_{\mathbf{T}})$ and $(V(\mathbf{A}), Q^\sharp_{\mathbf{A}})$ are not isomorphic locally at $\mathfrak{p}^\sharp$ is now exactly as in the previous paragraph. □

6.3. **Strategy.** In the remaining subsections of Section 6 we examine the quadratic spaces $V_p(\mathbf{A})$ and $V_p(\mathbf{T})$, together with the deformation functor $\mathrm{Def}_\Sigma(\mathbf{A}, j)$, on a case-by-case basis depending on the action of $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ on (6.1.1). The results of the preceding subsections reduce everything to direct (if very tedious) linear algebra. The calculation of $\mathrm{Def}_\Sigma(\mathbf{A}, j)$ follows the same argument for each possible $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p) \subset D_8$, and in this subsection we explain the general structure of this argument.

Hypothesis 6.3.1. For the remainder of Section 6 we assume

1. $\mathbf{A}$ is supersingular (equivalently, $\mathfrak{p}^\sharp$ is nonsplit in $E^\sharp$),
2. $\mathfrak{c} = \mathcal{O}_F$,
3. $[\mathbb{E} : \mathbb{Q}_p] \leq 4$,
4. $e(\mathbb{E}/\mathbb{Q}_p) < p$.

Because we assume that $\mathbf{A}$ supersingular, Proposition 6.1.1 tells us that the two edges ϕ_{12} and ϕ_{34} of (6.1.1) lie in the same orbit under the action of the subgroup $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p) \subset D_8$. For each subgroup with this property we carry out the following steps.

(1) Determine the submodules $\mathcal{D}$ and $\mathcal{T}$ of $\mathcal{W}^4$. As both are equal to the image of $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{W}$ under the map (6.1.7), the theory of higher ramification groups (especially Proposition 4 of [40, Chapter IV.1] and its corollary) implies that the coordinates of points in $\mathcal{D}$ (and $\mathcal{T}$) satisfy certain congruences. The most important example of such a congruence is the following. Suppose there is a $\gamma \in \mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ of order 2, and let $\mathbb{E}_0$ be the fixed field of γ. If Δ generates the relative different of $\mathbb{E}/\mathbb{E}_0$ then $\pi(x) \equiv \pi(x)^\gamma \pmod{\Delta\mathcal{W}}$ for any $x \in \mathcal{O}_E$ and any $\pi : \mathcal{O}_E \to \mathcal{W}$. Thus if, say, $\pi_1 = \gamma \circ \pi_3$ then any point in $\mathcal{D}$ must have its first and third coordinates congruent modulo $\Delta\mathcal{W}$. In every case the strategy is first to find congruences satisfied by the coordinates of $\mathcal{D}$. These congruences determine a submodule $\mathcal{D}' \subset \mathcal{W}^4$ containing $\mathcal{D}$, and after finding sufficiently many congruences one may verify $\mathcal{D}' = \mathcal{D}$ by computing the index $[\mathcal{W}^4 : \mathcal{D}']$ and comparing with Lemma 6.1.3.

(2) Compute $\mathrm{ord}_{\mathcal{W}}(\rho_1)$ and $\mathrm{ord}_{\mathcal{W}}(\rho_2)$ both for $\lambda_{\mathcal{D}}$ and $\lambda_{\mathcal{T}}$. Once the $\mathcal{W}$-lattice $\mathcal{D} \subset \mathcal{W}^4$ is known, this is a straightforward calculation using (6.1.6) and the same relation with $\mathcal{D}$ replaced by $\mathcal{T}$. One may simplify this calculation by using Lemma 6.1.3.

(3) Find generators of Γ, and determine the operators $\mathcal{F}_\gamma$ and $\mathcal{V}_\gamma$ on $\mathcal{D}$ and $\mathcal{T}$ for each of the chosen generators $\gamma \in \Gamma$. Finding generators for Γ is trivial: find generators for $\Gamma(0)$ (the inertia subgroup of $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$) and then add any element of $\Gamma(1)$. As explained earlier, the operators $\mathcal{F}_\gamma$ and $\mathcal{V}_\gamma$ on $\mathcal{D}$ for $\gamma \in \Gamma(0)$ have a simple form: each is a permutation matrix composed with a Galois automorphism, and the permutation matrix can be read off directly from the action of $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ on (6.1.1), as in (6.1.10). For $\gamma \in \Gamma(1)$ recall from (6.1.11) that $\mathcal{V}_\gamma$ has a single nonzero entry in each row and column, and that the positions of the entries can be read off from the action of $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ on (6.1.1). The exact values of the nonzero entries are impossible to determine, but recalling (6.1.4) and using the fact that the action of $\mathcal{O}_E$ on

$$\mathcal{D}^{\Gamma(0)}/\mathcal{V}_\gamma\mathcal{D}^{\Gamma(0)} \cong D/VD \cong \mathrm{Lie}(A)$$

must satisfy the Σ-Kottwitz condition, one obtains some information about the image of $\mathcal{V}_\gamma$. In particular, this information is enough to determine the p-adic valuations of the coefficients of the matrix $\mathcal{V}_\gamma \circ \gamma$. The operators $\mathcal{F}_\gamma$ and $\mathcal{V}_\gamma$ on $\mathcal{T}$ may be computed in a similar manner. As noted before, each such operator is a permutation matrix composed with a Galois automorphism Both for $\mathcal{D}$ and $\mathcal{T}$ we only keep track of the $\mathcal{V}_\gamma$ operators, as $\mathcal{F}_\gamma$ can be easily recovered from $\mathcal{V}_\gamma$.

(4) Among the elements of Γ there is at least one, say γ, that interchanges the edges ϕ_{12} and ϕ_{34}. Such a γ is either reflection across the horizontal axis in (6.1.1), or is rotation by 180°, and accordingly the corresponding operator

$\mathcal{F}_\gamma$ has, by (6.1.11), one of the two forms

$$\begin{pmatrix} & & & * \\ & & * & \\ & * & & \\ * & & & \end{pmatrix} \circ \gamma \qquad \begin{pmatrix} & & * & \\ & & & * \\ * & & & \\ & * & & \end{pmatrix} \circ \gamma$$

where the unknown entries will have known p-adic valuations. Given a $j \in L_p(\mathbf{A})$ written in the form (6.1.13), the condition that j commutes with $\mathcal{F}_\gamma$ then implies some relation between $\mathrm{ord}_{\mathcal{W}}(c)$ and $\mathrm{ord}_{\mathcal{W}}(d)$.

Once these calculations are done, the deformations space $\mathrm{Def}_\Sigma(\mathbf{A}, j)$, for any nonzero $j \in L(\mathbf{A})$, is easy to compute. Knowledge of the $\mathcal{W}$-modules $\mathcal{D}^1 \subset \mathcal{D} \subset \mathcal{W}^4$ together with (6.2.2) allows one to compute the maximal m for which $\mathrm{obst}_m(j) = 0$ in terms of $\mathrm{ord}_{\mathcal{W}}(d)$. Proposition 6.1.2 expresses $\mathrm{ord}_{\mathfrak{p}^\sharp}(Q^\sharp_{\mathbf{A}}(j))$ in terms of $\mathrm{ord}_{\mathcal{W}}(\rho_1)$, $\mathrm{ord}_{\mathcal{W}}(\rho_2)$, $\mathrm{ord}_{\mathcal{W}}(c)$, and $\mathrm{ord}_{\mathcal{W}}(d)$. Combining all of these relations gives a formula for the largest m for which $\mathrm{obst}_m(j) = 0$ in terms of $\mathrm{ord}_{\mathfrak{p}^\sharp}(Q^\sharp_{\mathbf{A}}(j))$. Combining this formula with Proposition 6.2.3 one determines the largest m for which j lifts to a special endomorphism of $\mathbf{A}^{\mathrm{can}}_m$, and then the functor $\mathrm{Def}_\Sigma(\mathbf{A}, j)$ is known by Corollary 6.2.2.

To be more concrete, consider the case of $\mathbb{E}/\mathbb{Q}_p$ unramified and fix a nonzero $j \in L(\mathbf{A})$. Lemma 6.1.3 implies $\mathcal{D} = \mathcal{W}^4$, and (6.1.6) implies $\mathrm{ord}_{\mathcal{W}}(\rho_1) = 0$ and $\mathrm{ord}_{\mathcal{W}}(\rho_2) = 0$. The submodule $\mathcal{D}^1 \subset \mathcal{D}$ is spanned by e_1 and e_2, and the $\mathcal{W}$-module map $q : \mathcal{D} \to \mathcal{W}^2$ defined by (6.2.1) identifies $\mathcal{D}/\mathcal{D}^1 \cong \mathcal{W}^2$. By (6.2.2) the map $\mathrm{obst}(j)$ has the explicit form

$$\mathrm{obst}(j)(e_1) = \begin{bmatrix} -\rho_2 d \\ 0 \end{bmatrix} \qquad \mathrm{obst}(j)(e_2) = \begin{bmatrix} 0 \\ \rho_1 d \end{bmatrix}.$$

It follows immediately that

$$\mathrm{obst}_m(j) = 0 \iff m \le \mathrm{ord}_{\mathcal{W}}(d).$$

On the other hand, Proposition 6.1.2 implies $\mathrm{ord}_{\mathfrak{p}^\sharp}(Q^\sharp_{\mathbf{A}}(j)) = \mathrm{ord}_{\mathcal{W}}(cd)$. The only missing ingredient, which we will prove on a case-by-case basis, is the relation $\mathrm{ord}_{\mathcal{W}}(c) + 1 = \mathrm{ord}_{\mathcal{W}}(d)$. Once this is known we immediately deduce from Proposition 6.2.3

$$j \text{ lifts to } L(\mathbf{A}^{\mathrm{can}}_m) \iff \mathrm{obst}_m(j) = 0 \iff m \le \frac{\mathrm{ord}_{\mathfrak{p}^\sharp}\big(Q^\sharp_{\mathbf{A}}(j)\big) + 1}{2}$$

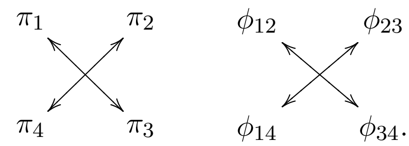
and conclude that j lifts to $L(\mathbf{A}^{\mathrm{can}}_m)$ but not to $L(\mathbf{A}^{\mathrm{can}}_{m+1})$ where

$$m = \frac{\mathrm{ord}_{\mathfrak{p}^\sharp}(Q^\sharp_{\mathbf{A}}(j)) + 1}{2}.$$

If $\mathrm{ord}_{\mathfrak{p}^\sharp}(Q^\sharp_{\mathbf{A}}(j)) = \infty$ then j lifts to $L(\mathbf{A}^{\mathrm{can}}_m)$ for every m.

The cases in which $\mathbb{E}/\mathbb{Q}_p$ is ramified are more technically involved, but the arguments follow the same general structure.

6.4. **The quadratic case, part I.** Suppose that $\mathbb{E}/\mathbb{Q}_p$ is a degree two extension, and that the nontrivial element $\gamma \in \mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ acts on the sets $\mathrm{Hom}(E, \mathbb{Q}_p^{\mathrm{alg}})$ and $\mathrm{Hom}(E^\sharp, \mathbb{Q}_p^{\mathrm{alg}})$ by



This implies $\mathbb{E} = \mathbb{E}_\Sigma$, and that there are isomorphisms (which we do not fix)

$$F_p \cong \mathbb{Q}_p \times \mathbb{Q}_p \qquad E_p \cong \mathbb{E} \times \mathbb{E}$$
$$F_p^\sharp \cong \mathbb{Q}_p \times \mathbb{Q}_p \qquad E_p^\sharp \cong \mathbb{E} \times \mathbb{E}.$$

Fix a generator $\Delta_\mathbb{E}$ of the different of $\mathbb{E}/\mathbb{Q}_p$. The fixed isomorphism (6.1.8) identifies

$$\mathcal{D} \cong \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \in \mathcal{W}^4 : \begin{array}{c} w_1 - w_3 \in \Delta_\mathbb{E}\mathcal{W} \\ w_2 - w_4 \in \Delta_\mathbb{E}\mathcal{W} \end{array} \right\}.$$

The condition (6.1.6) implies

$$\mathrm{ord}_\mathcal{W}(\rho_1\Delta_\mathbb{E}) = 0 \qquad \mathrm{ord}_\mathcal{W}(\rho_2\Delta_\mathbb{E}) = 0. \tag{6.4.1}$$

There are two primes of $F^\sharp$ above p. One of them is $\mathfrak{p}^\sharp$, and we denote the other prime by $\mathfrak{p}^\sharp_*$.

First suppose that $\mathbb{E}/\mathbb{Q}_p$ is unramified, so that $\mathcal{W} = W$ and $\Delta_\mathbb{E} \in \mathcal{W}^\times$. There is a unique lift of γ to $\gamma \in \Gamma(1)$, and this γ generates Γ. By (6.1.11) the operator $\mathcal{V}_\gamma$ on $\mathcal{D}$ has the form

$$\mathcal{V}_\gamma = \begin{pmatrix} & & u_1 & \\ & & & u_2 \\ \varpi_1 & & & \\ & \varpi_2 & & \end{pmatrix} \circ \gamma^{-1}$$

for some $u_1, u_2, \varpi_1, \varpi_2 \in \mathcal{W}$. The characteristic polynomial of $x \in \mathcal{O}_E$ acting on $\mathcal{D}/\mathcal{V}_\gamma\mathcal{D}$ is

$$(T - \pi_1(x))^{\mathrm{ord}_\mathcal{W}(u_1)} \cdot (T - \pi_2(x))^{\mathrm{ord}_\mathcal{W}(u_2)}$$
$$\cdot (T - \pi_3(x))^{\mathrm{ord}_\mathcal{W}(\varpi_1)} \cdot (T - \pi_4(x))^{\mathrm{ord}_\mathcal{W}(\varpi_1)} \in \mathbb{F}_\mathfrak{q}^{\mathrm{alg}}[T],$$

and so the Σ-Kottwitz condition implies, as the notation suggests, that ϖ_1 and ϖ_2 are uniformizing parameters of $\mathcal{W}$, while $u_1, u_2 \in \mathcal{W}^\times$. The relation (6.1.5) is equivalent to

$$p\rho_1 = -(u_1\varpi_1\rho_1)^\gamma \qquad p\rho_2 = -(u_2\varpi_2\rho_2)^\gamma.$$

The condition that (6.1.13) commutes with $\mathcal{V}_\gamma$ is equivalent to

$$a = \left(\frac{-pb}{u_1\varpi_2}\right)^\gamma \qquad b = \left(\frac{-pa}{u_2\varpi_1}\right)^\gamma \qquad c = \left(\frac{pd}{\varpi_1\varpi_2}\right)^\gamma \qquad d = \left(\frac{pc}{u_1u_2}\right)^\gamma.$$

Now consider the triple $(\mathcal{T}, \kappa_{\mathcal{T}}, \lambda_{\mathcal{T}})$. The operator $\mathcal{V}_\gamma$ acts on $\mathcal{T}$ as

$$\mathcal{V}_\gamma = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \gamma^{-1}$$

and the relation (6.1.15) is equivalent to $\rho_1 = -\rho_1^\gamma$ and $\rho_2 = -\rho_2^\gamma$. The condition that (6.1.13) commutes with $\mathcal{V}_\gamma$ is equivalent to $a, b, c, d \in \mathbb{E}$ together with $a = -b^\gamma$ and $c = d^\gamma$.

Now suppose $\mathbb{E}/\mathbb{Q}_p$ is ramified, so that $[\mathcal{W}_\mathbb{Q} : W_\mathbb{Q}] = 2$ and $\mathbb{E}$ and $W_\mathbb{Q}$ are linearly disjoint over $\mathbb{Q}_p$. There is a unique lift of γ to $\gamma \in \Gamma(0)$, and a unique lift of the identity in $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ to $\epsilon \in \Gamma(1)$. The elements $\gamma, \epsilon \in \Gamma$ commute and generate all of Γ. As $\mathbb{E} = \mathbb{E}_\Sigma$, Hypothesis 6.3.1 implies $p > 2$, and it follows that $\Delta_\mathbb{E}$ is a uniformizer of $\mathcal{W}$. From (6.1.10), (6.1.11), and the fact that $\mathcal{V}_\gamma$ and $\mathcal{V}_\epsilon$ commute, we deduce that the operators $\mathcal{V}_\gamma$ and $\mathcal{V}_\epsilon$ act on $\mathcal{D}$ as

$$\mathcal{V}_\gamma = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \gamma^{-1} \qquad \mathcal{V}_\epsilon = \begin{pmatrix} \varpi_1 & & & \\ & \varpi_2 & & \\ & & \varpi_1^\gamma & \\ & & & \varpi_2^\gamma \end{pmatrix} \circ \epsilon^{-1}$$

for some $\varpi_1, \varpi_2 \in \mathcal{W}$. As in the paragraph above, by computing the characteristic polynomial of $x \in \mathcal{O}_E$ acting on $\mathcal{D}^{\Gamma(0)}/\mathcal{V}_\epsilon \mathcal{D}^{\Gamma(0)}$ and comparing with $c_{\Sigma,x}$ one can show that ϖ_1 and ϖ_2 are uniformizing parameters of $\mathcal{W}$. The relation (6.1.5) is equivalent to $\rho_1^\gamma = -\rho_1$ and $\rho_2^\gamma = -\rho_2$, together with

$$p\rho_1 = (\rho_1 \varpi_1 \varpi_1^\gamma)^\epsilon \qquad p\rho_2 = (\rho_2 \varpi_2 \varpi_2^\gamma)^\epsilon.$$

The condition that (6.1.13) commutes with $\mathcal{V}_\gamma$ and $\mathcal{V}_\epsilon$ is equivalent to the relations $c = d^\gamma$ and $a = -b^\gamma$ together with

$$a = \left(\frac{pa}{\varpi_1^\gamma \varpi_2}\right)^\epsilon \qquad b = \left(\frac{pb}{\varpi_1 \varpi_2^\gamma}\right)^\epsilon \qquad c = \left(\frac{pc}{\varpi_1 \varpi_2}\right)^\epsilon \qquad d = \left(\frac{pd}{\varpi_1^\gamma \varpi_2^\gamma}\right)^\epsilon.$$

Now consider the triple $(\mathcal{T}, \kappa_{\mathcal{T}}, \lambda_{\mathcal{T}})$. The commuting operators $\mathcal{V}_\gamma$ and $\mathcal{V}_\epsilon$ act on $\mathcal{T}$ as

$$\mathcal{V}_\gamma = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \gamma^{-1} \qquad \mathcal{V}_\epsilon = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \circ \epsilon^{-1}.$$

The relation (6.1.15) implies that $\rho_1, \rho_2 \in \mathbb{E}$ and satisfy $\rho_1^\gamma = -\rho_1$ and $\rho_2^\gamma = -\rho_2$. The condition that (6.1.13) commutes with $\mathcal{V}_\epsilon$ and $\mathcal{V}_\gamma$ is equivalent to $a, b, c, d \in \mathbb{E}$ together with $c = d^\gamma$ and $a = -b^\gamma$

Proposition 6.4.1. *For any $j \in L_p(\mathbf{A})$, j lifts to $L_p(\mathbf{A}^{\mathrm{can}}_m)$ but not to $L_p(\mathbf{A}^{\mathrm{can}}_{m+1})$ where*

$$m = \frac{\mathrm{ord}_{\mathfrak{p}^\sharp}(Q^\sharp_{\mathbf{A}}(j)) + 1}{2} \cdot \begin{cases} 1 & \text{if } \mathfrak{p}^\sharp \text{ is inert in } E^\sharp \\ 2 & \text{if } \mathfrak{p}^\sharp \text{ is ramified in } E^\sharp. \end{cases}$$

Proof. If $\mathbb{E}/\mathbb{Q}_p$ is unramified then the calculations above prove $\mathrm{ord}_{\mathcal{W}}(d/c) = 1$, and so the claim follows from the discussion of Section 6.3. Assume now that $\mathbb{E}/\mathbb{Q}_p$ is ramified. The submodule $\mathcal{D}^1 \subset \mathcal{D}$ is free on the generators $\Delta_{\mathbb{E}} e_1$ and $\Delta_{\mathbb{E}} e_2$, and the map (6.2.1) identifies the quotient $\mathcal{D}/\mathcal{D}^1$ with $\mathcal{W}^2$. By (6.2.2) the obstruction $\mathrm{obst}(j) : \mathcal{D}^1 \to \mathcal{W}^2$ is given

$$\Delta_{\mathbb{E}} e_1 \mapsto \begin{bmatrix} 0 \\ \Delta_{\mathbb{E}} \rho_1 d \end{bmatrix} \qquad \Delta_{\mathbb{E}} e_2 \mapsto \begin{bmatrix} -\Delta_{\mathbb{E}} \rho_2 d \\ 0 \end{bmatrix}$$

and it follows from (6.4.1) that $\mathrm{obst}_m(j) = 0$ if and only if $m \le \mathrm{ord}_{\mathcal{W}}(d)$. We saw above that $c/d \in \mathcal{W}^\times$, and so Proposition 6.1.2 implies

$$\mathrm{ord}_{\mathcal{W}}(d) = \mathrm{ord}_{\mathfrak{p}^\sharp}\big(Q^\sharp_{\mathbf{A}}(j)\big) + 1.$$

As $\mathfrak{p}^\sharp$ is ramified in $E^\sharp$, the claim follows from Proposition 6.2.3. □

By direct linear algebra we can determine the conditions on the matrix (6.1.13) which ensure that $j \in L_p(\mathbf{A})$. This leads to the following result.

Proposition 6.4.2. *For some $\beta_p(\mathbf{A}) \in F^\sharp_p$ satisfying $\mathrm{ord}_{\mathfrak{p}^\sharp_*}(\beta_p(\mathbf{A})) = 0$ and*

$$\mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{A})) = \begin{cases} 1 & \text{if } \mathfrak{p}^\sharp \text{ is unramified in } E^\sharp \\ 0 & \text{otherwise} \end{cases}$$

there is an $E^\sharp_p$-linear isomorphism of $F^\sharp_p$-quadratic spaces

$$\big(V_p(\mathbf{A}), Q^\sharp_{\mathbf{A}}\big) \cong \big(E^\sharp_p, \beta_p(\mathbf{A}) x x^\dagger\big)$$

identifying $L_p(\mathbf{A}) \cong \mathcal{O}_{E^\sharp,p}$.

Proof. Suppose $j \in V_p(\mathbf{A})$ is written in the form (6.1.13). As a $\mathcal{W}$-module, $\mathcal{D}$ is generated by the elements $e_1 + e_3$, $e_2 + e_4$, $\Delta_{\mathbb{E}} e_3$, and $\Delta_{\mathbb{E}} e_4$, and so $j \cdot \mathcal{D} \subset \mathcal{D}$ if and only if

$$\begin{bmatrix} 0 \\ \rho_1 a - \rho_1 c \\ 0 \\ \rho_1 d + \rho_1 b \end{bmatrix} \qquad \begin{bmatrix} \rho_2 b + \rho_2 c \\ 0 \\ -\rho_2 d + \rho_2 a \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ -\Delta_{\mathbb{E}} \rho_1 c \\ 0 \\ \Delta_{\mathbb{E}} \rho_1 b \end{bmatrix} \qquad \begin{bmatrix} \Delta_{\mathbb{E}} \rho_2 c \\ 0 \\ \Delta_{\mathbb{E}} \rho_2 a \\ 0 \end{bmatrix}$$

all lie in $\mathcal{D}$. This is easily seen to be equivalent to $a, b, c, d \in \mathcal{W}$ and satisfying the congruences

$$\begin{aligned} a - b - c - d &\equiv 0 \pmod{\Delta_{\mathbb{E}}^2 \mathcal{W}} \\ a \equiv -b &\equiv c \pmod{\Delta_{\mathbb{E}} \mathcal{W}}. \end{aligned}$$

If $\mathbb{E}/\mathbb{Q}_p$ is unramified then these congruences are automatically satisfied. If $\mathbb{E}/\mathbb{Q}_p$ is ramified then the relations $a = -b^\gamma$ and $c = d^\gamma$ imply that

$$a \equiv -b \pmod{\Delta_\mathbb{E}\mathcal{W}}$$
$$c \equiv d \pmod{\Delta_\mathbb{E}\mathcal{W}}$$

and $a - b - c - d = (a - c) + (a - c)^\gamma \in W$. Therefore if $a \equiv c \pmod{\Delta_\mathbb{E}\mathcal{W}}$ then $a - b - c - d \in pW \subset \Delta_\mathbb{E}^2\mathcal{W}$. It follows that the above system of congruences can be replaced by the single congruence $a \equiv c \pmod{\Delta_\mathbb{E}\mathcal{W}}$, and so

$$j \in L_p(\mathbf{A}) \iff a, b, c, d \in \mathcal{W} \text{ and } a \equiv c \pmod{\Delta_\mathbb{E}\mathcal{W}}.$$

The next claim is that

$$j \in L_p(\mathbf{A}) \iff a, b, c, d \in \Delta_\mathbb{E}\mathcal{W}. \tag{6.4.2}$$

This is clear if $\mathbb{E}/\mathbb{Q}_p$ is unramified, so assume $\mathbb{E}/\mathbb{Q}_p$ is ramified. The implication $\Longleftarrow$ is obvious from the previous paragraph. The implication $\Longrightarrow$ is more subtle, and we give two proofs. First, if $j \in L_p(\mathbf{A})$ then (trivially) j lifts to $L_p(\mathbf{A}_1^{\mathrm{can}})$. By Proposition 6.2.3 the map $\mathrm{obst}_1(j)$ must vanish, and the proof of Proposition 6.4.1 shows that $1 \le \mathrm{ord}_\mathcal{W}(d)$. As $\Delta_\mathbb{E}$ is a uniformizer of $\mathcal{W}$ and $c = d^\gamma$, we deduce that $c, d \in \Delta_\mathbb{E}\mathcal{W}$. We already saw that $a \equiv c \pmod{\Delta_\mathbb{E}\mathcal{W}}$, and using $a = -b^\gamma$ we have proved $a, b, c, d \in \Delta_\mathbb{E}\mathcal{W}$ as desired. For the second proof we give a different argument that $1 \le \mathrm{ord}_\mathcal{W}(d)$. If not then $d \in \mathcal{W}^\times$, and so also $a, b, c, d \in \mathcal{W}^\times$. The congruence $a \equiv c \pmod{\Delta_\mathbb{E}\mathcal{W}}$ and the relations

$$a = \left(\frac{pa}{\varpi_1^\gamma\varpi_2}\right)^\gamma \qquad c = \left(\frac{pc}{\varpi_1\varpi_2}\right)^\gamma$$

imply that

$$\frac{\varpi_1}{\varpi_1^\gamma} \equiv 1 \pmod{\Delta_\mathbb{E}\mathcal{W}}.$$

This implies $\mathrm{ord}_\mathcal{W}(\varpi_1 - \varpi_1^\gamma) > \mathrm{ord}_\mathcal{W}(\Delta_\mathbb{E})$. As $\mathcal{W} = W[\varpi_1]$, it follows that $\mathrm{ord}_\mathcal{W}(x - x^\gamma) > \mathrm{ord}_\mathcal{W}(\Delta_\mathbb{E})$ for every $x \in \mathcal{W}$, contradicting [40, Proposition IV.1.4].

Having proved (6.4.2), fix any $E_p^\sharp$-module generator $j \in V_p(\mathbf{A})$, so that

$$\mathrm{ord}_\mathcal{W}(d) = \mathrm{ord}_\mathcal{W}(c) + \begin{cases} 1 & \text{if } \mathbb{E}/\mathbb{Q}_p \text{ is unramified} \\ 0 & \text{otherwise} \end{cases}$$

and $\mathrm{ord}_\mathcal{W}(a) = \mathrm{ord}_\mathcal{W}(b)$. The map

$$E_p^\sharp \xrightarrow{\phi_{23}\times\phi_{12}} \mathbb{E} \times \mathbb{E}$$

is an isomorphism, and any uniformizer of $\mathcal{O}_\mathbb{E}$ is a uniformizer of $\mathcal{W}$. Using this and (6.1.14), we may multiply j by an element of $E_p^\sharp$ to assume that a

and c each generate the ideal $\Delta_{\mathbb{E}}\mathcal{W}$. Then also b generates $\Delta_{\mathbb{E}}\mathcal{W}$. The map $x \mapsto x \bullet j$ defines an $E_p^\sharp$-linear isomorphism

$$\big(E_p^\sharp, \beta_p(\mathbf{A})xx^\dagger\big) \to \big(V_p(\mathbf{A}), Q_{\mathbf{A}}^\sharp\big)$$

where $\beta_p(\mathbf{A}) = Q_{\mathbf{A}}^\sharp(j)$, and Proposition 6.1.2 shows that this $\beta_p(\mathbf{A})$ has the desired valuations at $\mathfrak{p}^\sharp$ and $\mathfrak{p}_*^\sharp$. Finally, it follows from (6.4.2) and (6.1.14) that $x \bullet j \in L_p(\mathbf{A})$ if and only if $\phi(x) \in \mathcal{W}$ for every $\phi \in \mathrm{Hom}(E_p^\sharp, \mathbb{Q}_p^{\mathrm{alg}})$. This is equivalent to $x \in \mathcal{O}_{E^\sharp,p}$, and so $x \mapsto x \bullet j$ identifies $\mathcal{O}_{E^\sharp,p} \cong L_p(\mathbf{A})$. □

As we have identified $\mathcal{T} = \mathcal{D}$ as submodules of $\mathcal{W}^4$, it is straightforward to modify the proof of Proposition 6.4.2 to obtain the following result.

Proposition 6.4.3. *For some $\beta_p(\mathbf{T}) \in F_p^\sharp$ satisfying*

$$\mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{T})) = -\mathrm{ord}_{\mathfrak{p}^\sharp}(\mathfrak{D}_{E^\sharp/F^\sharp})$$
$$\mathrm{ord}_{\mathfrak{p}_*^\sharp}(\beta_p(\mathbf{T})) = -\mathrm{ord}_{\mathfrak{p}_*^\sharp}(\mathfrak{D}_{E^\sharp/F^\sharp})$$

there is an $E_p^\sharp$-linear isomorphism of $F_p^\sharp$-quadratic spaces

$$\big(V_p(\mathbf{T}), Q_{\mathbf{T}}^\sharp\big) \cong \big(E_p^\sharp, \beta_p(\mathbf{T})xx^\dagger\big)$$

identifying $L_p(\mathbf{T}) \cong \mathbb{Z}_p + \mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p}$.

Proof. Suppose $j \in V_p(\mathbf{T})$. The proof of Proposition 6.4.2 (plus the earlier calculation $a, b, c, d \in \mathbb{E}$) shows that $j \cdot \mathcal{T} \subset \mathcal{T}$ if and only if $a, b, c, d \in \mathcal{O}_{\mathbb{E}}$ and satisfy the congruence $a \equiv c \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}$. The equivalence (6.4.2), however, is false in this setting.

Fix a $j \in L_p(\mathbf{T})$ such that $a, b, c, d \in \mathcal{O}_{\mathbb{E}}^\times$. For example, first choose any $a, c \in \mathcal{O}_{\mathbb{E}}^\times$ satisfying $a \equiv c \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}$; by the calculation of the operators $\{\mathcal{V}_\gamma : \gamma \in \Gamma\}$ above, if we set $b = -a^\gamma$ and $d = c^\gamma$ then (6.1.13) defines the desired $j \in L_p(\mathbf{A})$. Consider the function $E_p^\sharp \to V_p(\mathbf{T})$ defined by $x \mapsto x \bullet j$. For any $x \in E_p^\sharp$, the results of the paragraph above and the explicit action (6.1.14) show that $x \bullet j \in L_p(\mathbf{T})$ if and only if $x \in \mathcal{O}_{E^\sharp,p}$ and

$$a\phi_{23}(x) \equiv c\phi_{12}(x) \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}.$$

Of course this is equivalent to $\phi_{23}(x) \equiv \phi_{12}(x) \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}$, which is equivalent to

$$x \in \mathbb{Z}_p + \mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p}.$$

Taking $\beta_p(\mathbf{T}) = Q_{\mathbf{T}}^\sharp(j)$ and using Proposition 6.1.2 to compute the valuation of $\beta_p(\mathbf{T})$ at $\mathfrak{p}^\sharp$ and $\mathfrak{p}_*^\sharp$, we see that $x \mapsto x \bullet j$ defines the desired isomorphism $E_p^\sharp \to V_p(\mathbf{A})$. □

Suppose $\alpha \in (F^\sharp)^\times$ and $p \in \mathrm{Sppt}(\alpha)$, and recall the quantity $\nu_{\mathfrak{p}^\sharp}(\alpha)$ of Proposition 5.2.3. Assume $\mathfrak{p}^\sharp \in \mathrm{Diff}(\alpha, \mathbf{T})$, so that $W_{\alpha,p}(0, \mathbf{T}) = 0$ by (4.3.3).

Proposition 6.4.4.

(1) *If $L_p(\mathbf{A})$ represents α then $W_{\alpha,p}(0,\mathbf{A}) \neq 0$ and*

$$\frac{W'_{\alpha,p}(0,\mathbf{T})}{W_{\alpha,p}(0,\mathbf{A})} = -\frac{\nu_{\mathfrak{p}^\sharp}(\alpha)}{2} \cdot \log(\mathrm{Nm}(\mathfrak{q})).$$

(2) *If $L_p(\mathbf{A})$ does not represent α then $W_{\alpha,p}(0,\mathbf{A})$ and $W'_{\alpha,p}(0,\mathbf{T})$ are both* 0.

Proof. Abbreviate $\mathfrak{p} = \mathfrak{p}^\sharp$ and $\mathfrak{p}_* = \mathfrak{p}^\sharp_*$. Identify $E^\sharp_p = E^\sharp_\mathfrak{p} \times E^\sharp_{\mathfrak{p}_*}$ and $F^\sharp_p = F^\sharp_\mathfrak{p} \times F^\sharp_{\mathfrak{p}_*}$, and write

$$\beta_p(\mathbf{T}) = (\beta_\mathfrak{p}(\mathbf{T}), \beta_{\mathfrak{p}_*}(\mathbf{T}))$$
$$\beta_p(\mathbf{A}) = (\beta_\mathfrak{p}(\mathbf{A}), \beta_{\mathfrak{p}_*}(\mathbf{A})).$$

We divide the proof of (1) into two cases. First assume that $\mathfrak{p}$ is inert in $E^\sharp$. This implies that $\mathfrak{p}_*$ is also inert in $E^\sharp$. In this case, both $L_p(\mathbf{T})$ and $L_p(\mathbf{A})$ can be identified with $\mathcal{O}_{E^\sharp,p}$, by Propositions 6.4.3 and 6.4.2 respectively, and so there are factorizations

$$W_{\alpha,p}(s,\mathbf{T}) = W_{\alpha,\mathfrak{p}}(s,\Phi^0_{\beta_\mathfrak{p}(\mathbf{T})}) \cdot W_{\alpha,\mathfrak{p}_*}(s,\Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{T})})$$
$$W_{\alpha,p}(s,\mathbf{A}) = W_{\alpha,\mathfrak{p}}(s,\Phi^0_{\beta_\mathfrak{p}(\mathbf{A})}) \cdot W_{\alpha,\mathfrak{p}_*}(s,\Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{A})}),$$

where Φ^0_β is in the notation of Section 4.6. Those same propositions imply

$$(\mathcal{O}_{E^\sharp,\mathfrak{p}_*}, \beta_{\mathfrak{p}_*}(\mathbf{T})xx^\dagger) \cong (\mathcal{O}_{E^\sharp,\mathfrak{p}_*}, \beta_{\mathfrak{p}_*}(\mathbf{A})xx^\dagger),$$

and so

$$W_{\alpha,\mathfrak{p}_*}(s,\Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{T})}) = W_{\alpha,\mathfrak{p}_*}(s,\Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{A})}).$$

The hypothesis that $L_{\mathfrak{p}_*}(\mathbf{A})$ represents α implies that $\mathrm{ord}_{\mathfrak{p}_*}(\alpha/\beta_{\mathfrak{p}_*}(\mathbf{A}))$ is both even and positive, and so Proposition 4.6.2 implies

$$W^*_{\alpha,\mathfrak{p}_*}(0,\Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{A})}) = \gamma(V_{\beta_{\mathfrak{p}_*}(\mathbf{A})}).$$

The same reasoning shows that

$$W^*_{\alpha,\mathfrak{p}}(0,\Phi^0_{\beta_\mathfrak{p}(\mathbf{A})}) = \gamma(V_{\beta_\mathfrak{p}(\mathbf{A})}).$$

On the other hand, Propositions 6.4.3 and 4.6.2 imply

$$W^{*,\prime}_{\alpha,\mathfrak{p}}(0,\Phi^0_{\beta_\mathfrak{p}(\mathbf{T})}) = \frac{\mathrm{ord}_\mathfrak{p}(\alpha)+1}{2} \cdot \gamma(V_{\beta_\mathfrak{p}(\mathbf{T})}) \cdot \log \mathrm{Nm}(\mathfrak{p}).$$

Lemma 4.2.1 implies $\gamma(V_{\beta_\mathfrak{p}(\mathbf{T})}) = -\gamma(V_{\beta_\mathfrak{p}(\mathbf{A})})$, and combining this with $\mathrm{Nm}(\mathfrak{q}) = \mathrm{Nm}(\mathfrak{p})^2$ shows

$$\frac{W'_{\alpha,p}(0,\mathbf{T})}{W_{\alpha,p}(0,\mathbf{A})} = \frac{W^{*,\prime}_{\alpha,\mathfrak{p}}(0,\Phi^0_{\beta_\mathfrak{p}(\mathbf{T})})}{W^*_{\alpha,\mathfrak{p}}(0,\Phi^0_{\beta_\mathfrak{p}(\mathbf{A})})} = -\frac{\nu_\mathfrak{p}(\alpha)}{2} \log \mathrm{Nm}(\mathfrak{q}).$$

This proves (1) under the assumption that $\mathfrak{p}$ is inert in $E^\sharp$.

Next we assume that $\mathfrak{p}$ is ramified in $E^\sharp$. In this case $\mathfrak{p}_*$ is also ramified in $E^\sharp$, and $p \neq 2$. Let ϖ and ϖ_* be uniformizing parameters of $F^\sharp_\mathfrak{p}$ and $F^\sharp_{\mathfrak{p}_*}$, respectively. Under the identification $E^\sharp_p = E^\sharp_\mathfrak{p} \times E^\sharp_{\mathfrak{p}_*}$ we have

$$\mathbb{Z}_p + \mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p} = \bigsqcup_{a \in \mathbb{Z}/p\mathbb{Z}} (a + \varpi\mathcal{O}_{E^\sharp,\mathfrak{p}}) \times (a + \varpi_*\mathcal{O}_{E^\sharp,\mathfrak{p}_*}).$$

Define

$$\tilde{\beta}_p(\mathbf{T}) = (\tilde{\beta}_\mathfrak{p}(\mathbf{T}), \tilde{\beta}_{\mathfrak{p}_*}(\mathbf{T})) = (\varpi\varpi^\dagger\beta_\mathfrak{p}(\mathbf{T}), \varpi_*\varpi_*^\dagger\beta_{\mathfrak{p}_*}(\mathbf{T})).$$

The map $(x_1, x_2) \mapsto (x_1\varpi^{-1}, x_2\varpi_*^{-1})$ gives an isomorphism of quadratic spaces

$$(E^\sharp_p, \beta_p(\mathbf{T})xx^\dagger) \cong (E^\sharp_p, \tilde{\beta}_p(\mathbf{T})xx^\dagger)$$

which identifies $\mathbb{Z}_p + \mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p}$ with

$$\bigsqcup_{a \in \mathbb{Z}/p\mathbb{Z}} \left(\frac{a}{\varpi} + \mathcal{O}_{E^\sharp,\mathfrak{p}}\right) \times \left(\frac{a}{\varpi_*} + \mathcal{O}_{E^\sharp,\mathfrak{p}_*}\right),$$

and therefore

$$W^*_{\alpha,p}(s, \mathbf{T}) = \sum_{a \in \mathbb{Z}/p\mathbb{Z}} W^*_{\alpha,\mathfrak{p}}\big(s, \Phi^{\frac{a}{\varpi}}_{\tilde{\beta}_\mathfrak{p}(\mathbf{T})}\big) \cdot W^*_{\alpha,\mathfrak{p}_*}\big(s, \Phi^{\frac{a}{\varpi_*}}_{\tilde{\beta}_{\mathfrak{p}_*}(\mathbf{T})}\big). \tag{6.4.3}$$

If $a \neq 0$ and

$$\alpha \in \tilde{\beta}_\mathfrak{p}(\mathbf{T})\frac{a^2}{\varpi\varpi^\dagger} + \mathcal{O}_{F^\sharp,\mathfrak{p}}$$

then

$$\frac{\alpha}{a^2\beta_\mathfrak{p}(\mathbf{T})} \in 1 + \frac{1}{a^2\beta_\mathfrak{p}(\mathbf{T})}\mathcal{O}_F,$$

and it follows (using Proposition 6.4.3) that $\chi_\mathfrak{p}(\alpha) = \chi_\mathfrak{p}(\beta_\mathfrak{p}(\mathbf{T}))$. But this implies that $V_\mathfrak{p}(\mathbf{T})$ represents α, contradicting $\mathfrak{p} \in \mathrm{Diff}(\alpha, \mathbf{T})$. Invoking Proposition 4.6.4 now shows that if $a \neq 0$ then

$$W^*_{\alpha,\mathfrak{p}}(s, \Phi^{\frac{a}{\varpi}}_{\tilde{\beta}_\mathfrak{p}(\mathbf{T})}) = 0.$$

We have now shown that only the $a = 0$ term contributes to (6.4.3), and so

$$W^*_{\alpha,p}(s, \mathbf{T}) = W^*_{\alpha,\mathfrak{p}}(s, \Phi^0_{\tilde{\beta}_\mathfrak{p}(\mathbf{T})}) \cdot W^*_{\alpha,\mathfrak{p}_*}(s, \Phi^0_{\tilde{\beta}_{\mathfrak{p}_*}(\mathbf{T})}).$$

As we assume that $L_p(\mathbf{A})$ represents α, Proposition 6.4.2 implies that $\alpha \in \mathcal{O}_{F^\sharp,p}$. The same proposition allows us to identify $L_p(\mathbf{A}) \cong \mathcal{O}_{E^\sharp,p}$, and so there is a factorization

$$W^*_{\alpha,p}(s, \mathbf{A}) = W^*_{\alpha,\mathfrak{p}}(s, \Phi^0_{\beta_\mathfrak{p}(\mathbf{A})}) \cdot W^*_{\alpha,\mathfrak{p}_*}(s, \Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{A})}).$$

Proposition 4.6.3 shows that both factors in the right hand side are non-vanishing at $s = 0$. Lemma 4.5.2 and the hypothesis $\mathfrak{p} \in \mathrm{Diff}(\alpha, \mathbf{T})$ imply that $\mathfrak{p}_* \notin \mathrm{Diff}(\alpha, \mathbf{T})$, and so α is represented by $V_{\mathfrak{p}^*}(\mathbf{T})$ as well as $V_{\mathfrak{p}^*}(\mathbf{A})$. Combining this with

$$\mathrm{ord}_{\mathfrak{p}_*}(\beta_{\mathfrak{p}_*}(\mathbf{A})) = \mathrm{ord}_{\mathfrak{p}_*}(\beta_{\mathfrak{p}_*}(\mathbf{T})) = 0$$

shows that $\beta_{\mathfrak{p}^*}(\mathbf{A})/\beta_{\mathfrak{p}^*}(\mathbf{T})$ is a norm from $\mathcal{O}^\times_{E^\sharp,\mathfrak{p}^*}$, which implies

$$W^*_{\alpha,\mathfrak{p}_*}(s,\Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{A})}) = W^*_{\alpha,\mathfrak{p}_*}(s,\Phi^0_{\tilde{\beta}_{\mathfrak{p}_*}(\mathbf{T})}).$$

From this we deduce

$$\frac{W'_{\alpha,p}(0,\mathbf{T})}{W_{\alpha,p}(0,\mathbf{A})} = \frac{W^{*,\prime}_{\alpha,p}(0,\mathbf{T})}{W^*_{\alpha,p}(0,\mathbf{A})} = \frac{W^{*,\prime}_{\alpha,\mathfrak{p}}(0,\Phi^0_{\tilde{\beta}_{\mathfrak{p}}(\mathbf{T})})}{W^*_{\alpha,\mathfrak{p}}(0,\Phi^0_{\beta_{\mathfrak{p}}(\mathbf{A})})}.$$

But Propositions 6.4.2, 6.4.3, and 4.6.3 imply

$$\frac{W^{*,\prime}_{\alpha,\mathfrak{p}}(0,\Phi^0_{\tilde{\beta}_{\mathfrak{p}}(\mathbf{T})})}{W^*_{\alpha,\mathfrak{p}}(0,\Phi^0_{\beta_{\mathfrak{p}}(\mathbf{A})})} = -\frac{\nu_{\mathfrak{p}}(\alpha)}{2}\log \mathrm{Nm}(\mathfrak{p}).$$

Since $\mathfrak{p}$ is ramified in $E^\sharp$, we see $\mathrm{Nm}(\mathfrak{p}) = \mathrm{Nm}(\mathfrak{q})$. This proves (1) in the ramified case.

Now we prove (2). Assume that α is not represented by $L_p(\mathbf{A})$. If $\alpha \notin \mathcal{O}_{F^\sharp,p}$, then

$$W_{\alpha,p}(s,\mathbf{T}) = W_{\alpha,p}(s,A) = 0$$

by Propositions 6.4.2, 6.4.3, 4.6.3 and 4.6.4, so we may assume that $\alpha \in \mathcal{O}_{F^\sharp,p}$. The assumption $\mathfrak{p} \in \mathrm{Diff}(\alpha,\mathbf{T})$ implies that $V_{\mathfrak{p}}(\mathbf{T})$ does not represent α. Proposition 6.2.4 now implies that $V_{\mathfrak{p}}(\mathbf{A})$ does represent α, and Proposition 6.4.2 (together with the hypothesis that $\alpha \in \mathcal{O}_{F^\sharp,p}$) implies that α is represented by $L_{\mathfrak{p}}(\mathbf{A})$. It follows that $L_{\mathfrak{p}_*}(\mathbf{A})$ does not represent α, and so

$$W_{\alpha,\mathfrak{p}_*}(0,\Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{A})}) = 0$$

by Proposition 4.6.3. Therefore

$$W_{\alpha,p}(0,\mathbf{A}) = W_{\alpha,\mathfrak{p}}(0,\Phi^0_{\beta_{\mathfrak{p}}(\mathbf{A})}) \cdot W_{\alpha,\mathfrak{p}_*}(0,\Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{A})}) = 0.$$

The proof of (1) gives the first equality in

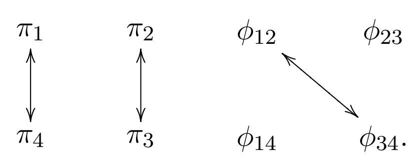
$$W_{\alpha,\mathfrak{p}_*}(0,\Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{T})}) = W_{\alpha,\mathfrak{p}_*}(0,\Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{A})}) = 0$$

and also shows that

$$W'_{\alpha,p}(0,\mathbf{T}) = W'_{\alpha,\mathfrak{p}}(0,\Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{T})}) \cdot W_{\alpha,\mathfrak{p}_*}(0,\Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{T})}) = 0.$$

□

6.5. **The quadratic case, part II.** Suppose that $\mathbb{E}/\mathbb{Q}_p$ is a degree two extension, and that the nontrivial element $\gamma \in \mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ acts on the sets $\mathrm{Hom}(E,\mathbb{Q}_p^{\mathrm{alg}})$ and $\mathrm{Hom}(E^\sharp,\mathbb{Q}_p^{\mathrm{alg}})$ by



This implies $\mathbb{E} = \mathbb{E}_\Sigma$, and that

$$F_p \cong \mathbb{E} \qquad E_p \cong \mathbb{E} \times \mathbb{E}$$
$$F_p^\sharp \cong \mathbb{Q}_p \times \mathbb{Q}_p \qquad E_p^\sharp \cong \mathbb{E} \times (\mathbb{Q}_p \times \mathbb{Q}_p).$$

Fix a generator $\Delta_\mathbb{E}$ of the different of $\mathbb{E}/\mathbb{Q}_p$. The fixed isomorphism (6.1.8) identifies

$$\mathcal{D} \cong \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \in \mathcal{W}^4 : \begin{array}{c} w_1 - w_4 \in \Delta_\mathbb{E}\mathcal{W} \\ w_2 - w_3 \in \Delta_\mathbb{E}\mathcal{W} \end{array} \right\}.$$

The condition (6.1.6) implies

$$\mathrm{ord}_\mathcal{W}(\rho_1\Delta_\mathbb{E}) = 0 \qquad \mathrm{ord}_\mathcal{W}(\rho_2\Delta_\mathbb{E}) = 0.$$

There are two primes of $F^\sharp$ above p. One of them, $\mathfrak{p}^\sharp$, is nonsplit in $E^\sharp$. The other, which we will denote by $\mathfrak{p}^\sharp_*$, is split in $E^\sharp$.

First suppose that $\mathbb{E}/\mathbb{Q}_p$ is unramified, so that Γ is generated by the unique lift of γ to $\gamma \in \Gamma(1)$. The operator $\mathcal{V}_\gamma$ has the form

$$\mathcal{V}_\gamma = \begin{pmatrix} & & & u_2 \\ & & u_1 & \\ & \varpi_1 & & \\ \varpi_2 & & & \end{pmatrix} \circ \gamma^{-1}$$

for some uniformizing parameters $\varpi_1, \varpi_2 \in \mathcal{W}$ and units $u_1, u_2 \in \mathcal{W}^\times$, and the relation (6.1.5) is equivalent to

$$p\rho_2 = -(u_2\varpi_1\rho_1)^\gamma \qquad p\rho_1 = -(u_1\varpi_2\rho_2)^\gamma.$$

The condition that (6.1.13) commutes with $\mathcal{V}_\gamma$ is equivalent to

$$a = \left(\frac{-pa}{u_1\varpi_1}\right)^\gamma \qquad b = \left(\frac{-pb}{u_2\varpi_2}\right)^\gamma \qquad c = \left(\frac{-pd}{\varpi_1\varpi_2}\right)^\gamma \qquad d = \left(\frac{-pc}{u_1u_2}\right)^\gamma.$$

Now consider the triple $(\mathcal{T}, \kappa_\mathcal{T}, \lambda_\mathcal{T})$. The operator $\mathcal{V}_\gamma$ is equal to

$$\mathcal{V}_\gamma = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \circ \gamma^{-1}$$

and the relation (6.1.15) is equivalent to $\rho_2 = -\rho_1^\gamma$ and $\rho_1 = -\rho_2^\gamma$. The condition that (6.1.13) commutes with $\mathcal{V}_\gamma$ is equivalent to

$$a = -a^\gamma \qquad b = -b^\gamma \qquad c = -d^\gamma \qquad d = -c^\gamma.$$

Now suppose $\mathbb{E}/\mathbb{Q}_p$ is ramified. The group Γ is generated by two commuting elements: the unique lift of γ to $\gamma \in \Gamma(0)$, and the unique lift of the identity element of $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ to $\epsilon \in \Gamma(1)$. commuting elements $\epsilon = (\mathrm{Fr}, \mathrm{id})$

and $\gamma = (\mathrm{id}, \gamma)$. Hypothesis 6.3.1 implies $p > 2$, and hence $\Delta_{\mathbb{E}}$ is a uniformizer of $\mathbb{E}$. The commuting operators $\mathcal{V}_\gamma$ and $\mathcal{F}_\gamma$ on $\mathcal{D}$ have the form

$$\mathcal{V}_\gamma = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \circ \gamma^{-1} \qquad \mathcal{V}_\epsilon = \begin{pmatrix} \varpi_1 & & & \\ & \varpi_2 & & \\ & & \varpi_2^\gamma & \\ & & & \varpi_1^\gamma \end{pmatrix} \circ \epsilon^{-1}$$

for some some uniformizers $\varpi_1, \varpi_2 \in \mathcal{W}$. The relation (6.1.5) implies that ρ_1 and ρ_2 satisfy $\rho_1^\gamma = -\rho_2$ and $\rho_2^\gamma = -\rho_1$, together with

$$p\rho_1 = (\rho_1 \varpi_1 \varpi_2^\gamma)^\epsilon \qquad p\rho_2 = (\rho_2 \varpi_2 \varpi_1^\gamma)^\epsilon.$$

The condition that (6.1.13) commutes with $\mathcal{V}_\gamma$ is equivalent to

$$a = -a^\gamma \qquad b = -b^\gamma \qquad c = -d^\gamma \qquad d = -c^\gamma$$

and the condition that (6.1.13) commutes with $\mathcal{V}_\epsilon$ is equivalent to the further relations

$$a = \left(\frac{pa}{\varpi_2 \varpi_2^\gamma} \right)^\epsilon \qquad b = \left(\frac{pb}{\varpi_1 \varpi_1^\gamma} \right)^\epsilon \qquad c = \left(\frac{pc}{\varpi_1 \varpi_2} \right)^\epsilon \qquad d = \left(\frac{pd}{\varpi_1^\gamma \varpi_2^\gamma} \right)^\epsilon.$$

Now consider the triple $(\mathcal{T}, \kappa_{\mathcal{T}}, \lambda_{\mathcal{T}})$. The operators $\mathcal{V}_\gamma$ and $\mathcal{V}_\epsilon$ on $\mathcal{T}$ are equal to

$$\mathcal{V}_\gamma = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \circ \gamma^{-1} \qquad \mathcal{V}_\epsilon = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \circ \epsilon^{-1}.$$

The relation (6.1.15) implies that $\rho_1, \rho_2 \in \mathbb{E}$ and satisfy $\rho_1^\gamma = -\rho_2$ and $\rho_2^\gamma = -\rho_1$. The condition that (6.1.13) commutes with $\mathcal{V}_\epsilon$ and $\mathcal{V}_\gamma$ is equivalent to $a, b, c, d \in \mathbb{E}$ satisfying

$$a = -a^\gamma \qquad b = -b^\gamma \qquad c = -d^\gamma \qquad d = -c^\gamma.$$

Proposition 6.5.1. *For any $j \in L_p(\mathbf{A})$, j lifts to $L_p(\mathbf{A}_m^{\mathrm{can}})$ but not to $L_p(\mathbf{A}_{m+1}^{\mathrm{can}})$ where*

$$m = \frac{\mathrm{ord}_{\mathfrak{p}^\sharp}(Q_{\mathbf{A}}^\sharp(j)) + 1}{2} \cdot \begin{cases} 1 & \text{if } \mathfrak{p}^\sharp \text{ is inert in } E^\sharp \\ 2 & \text{if } \mathfrak{p}^\sharp \text{ is ramified in } E^\sharp \end{cases}$$

Proof. After the calculations done above, the proof is the same, word-for-word, as the proof of Proposition 6.4.1. □

Proposition 6.5.2. *For some $\beta_p(\mathbf{A}) \in F_p^\sharp$ satisfying* $\mathrm{ord}_{\mathfrak{p}_*^\sharp}(\beta_p(\mathbf{A})) = 0$ *and*

$$\mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{A})) = \begin{cases} 1 & \text{if } \mathfrak{p}^\sharp \text{ is inert in } E^\sharp \\ 0 & \text{if } \mathfrak{p}^\sharp \text{ is ramified in } E^\sharp \end{cases}$$

there is an $E_p^\sharp$-linear isomorphism of $F_p^\sharp$-quadratic spaces

$$\big(V_p(\mathbf{A}), Q_{\mathbf{A}}^\sharp\big) \to \big(E_p^\sharp, \beta_p(\mathbf{A}) x x^\dagger\big)$$

which identifies $L_p(\mathbf{A})$ *with the maximal order* $\mathcal{O}_{E^\sharp,p}$.

Proof. Suppose $j \in V_p(\mathbf{A})$ and write j in the form (6.1.13). As a $\mathcal{W}$-module, $\mathcal{D}$ is generated by $e_1 + e_4$, $e_2 + e_3$, $\Delta_{\mathbb{E}} e_3$, and $\Delta_{\mathbb{E}} e_4$. Applying j to each of these vectors shows that $j \cdot \mathcal{D} \subset \mathcal{D}$ if and only if $a, b, c, d \in \Delta_{\mathbb{E}}\mathcal{W}$, and satisfy the congruences

$$\begin{aligned} \rho_2 c - \rho_1 d &\equiv 0 \pmod{\Delta_{\mathbb{E}}\mathcal{W}} \\ \rho_2 b - \rho_1 b &\equiv 0 \pmod{\Delta_{\mathbb{E}}\mathcal{W}} \\ \rho_1 a - \rho_2 a &\equiv 0 \pmod{\Delta_{\mathbb{E}}\mathcal{W}} \\ \rho_1 c - \rho_2 d &\equiv 0 \pmod{\Delta_{\mathbb{E}}\mathcal{W}}. \end{aligned}$$

If $\mathbb{E}/\mathbb{Q}_p$ is unramified then these congruences are automatically satisfied. If $\mathbb{E}/\mathbb{Q}_p$ is ramified then we may choose $\Delta_{\mathbb{E}}$ so that $\Delta_{\mathbb{E}}^\gamma = -\Delta_{\mathbb{E}}$, and write $\rho_1 = u\Delta_{\mathbb{E}}^{-1}$ for some $u \in \mathcal{W}^\times$. The relations $\rho_1^\gamma = -\rho_2$ and $u^\gamma \equiv u \pmod{\Delta_{\mathbb{E}}\mathcal{W}}$ imply $\rho_1 - \rho_2 \in \mathcal{W}$. This shows that the second and third congruences are automatically satisfied. The relations $\rho_1 d = (\rho_2 c)^\gamma$ and $\rho_2 d = (\rho_1 c)^\gamma$ show that the first and fourth congruences are automatically satisfied. Therefore $j \in L_p(\mathbf{A})$ if and only if $a, b, c, d \in \Delta_{\mathbb{E}}\mathcal{W}$. The rest of the proof is exactly the same as Proposition 6.4.2. □

Proposition 6.5.3. *For some* $\beta_p(\mathbf{T}) \in F_p^\sharp$ *satisfying*

$$\mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{T})) = 0 \qquad \mathrm{ord}_{\mathfrak{p}^\sharp_*}(\beta_p(\mathbf{T})) = 0$$

there is an $E_p^\sharp$*-linear isomorphism of* $F_p^\sharp$*-quadratic spaces*

$$\big(V_p(\mathbf{T}), Q^\sharp_{\mathbf{T}}\big) \to \big(E_p^\sharp, \beta_p(\mathbf{T}) x x^\dagger\big)$$

which identifies $L_p(\mathbf{T})$ *with the maximal order* $\mathcal{O}_{E^\sharp,p}$.

Proof. Using the equality $\mathcal{T} = \mathcal{D}$ of submodules of $\mathcal{W}^4$, the proof of Proposition 6.5.2 shows that every $j \in V_p(\mathbf{T})$ satisfies $j \in L_p(\mathbf{T})$ if and only if $a, b, c, d \in \Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}$. One may construct a $j \in L_p(\mathbf{T})$ in such a way that a, b, c, and d each generate $\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}$ (start by choosing a, b, c so that $a^\gamma = -a$, $b^\gamma = -b$, and each of a, b, c generates $\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}$; then define $d = -c^\gamma$), and then (6.1.14) shows that $x \bullet j \in L_p(\mathbf{T})$ if and only if $\phi(x) \in \mathcal{O}_{\mathbb{E}}$ for every $\phi : E_p^\sharp \to \mathbb{E}$. This implies

$$x \bullet j \in L_p(\mathbf{T}) \iff x \in \mathcal{O}_{E^\sharp,p}.$$

Taking $\beta_p(\mathbf{T}) = Q^\sharp_{\mathbf{T}}(j)$, the completion of the proof is exactly is in Proposition 6.4.3. □

Suppose $\alpha \in (F^\sharp)^\times$ and $p \in \mathrm{Sppt}(\alpha)$, and recall the quantity $\nu_{\mathfrak{p}^\sharp}(\alpha)$ of Proposition 5.2.3. Assume $\mathfrak{p}^\sharp \in \mathrm{Diff}(\alpha, \mathbf{T})$, so that $W_{\alpha,p}(0, \mathbf{T}) = 0$ by (4.3.3).

Proposition 6.5.4.

(1) *If* $L_p(\mathbf{A})$ *represents* α *then* $W_{\alpha,p}(0,\mathbf{A}) \neq 0$ *and*

$$\frac{W'_{\alpha,p}(0,\mathbf{T})}{W_{\alpha,p}(0,\mathbf{A})} = -\frac{\nu_{\mathfrak{p}^\sharp}(\alpha)}{2} \cdot \log(\mathrm{Nm}(\mathfrak{q})).$$

(2) *If* $L_p(\mathbf{A})$ *does not represent* α *then* $W_{\alpha,p}(0,\mathbf{A})$ *and* $W'_{\alpha,p}(0,\mathbf{T})$ *are both* 0.

Proof. (sketch) The proof is similar to that of Proposition 6.4.4 and a little simpler. Keep the notation in Proposition 6.4.4. Then propositions 6.5.2 and 6.5.3 imply

$$(L_p(\mathbf{A}), Q^\sharp_{\mathbf{A}}) \cong (\mathcal{O}_{E^\sharp,\mathfrak{p}}, \beta_{\mathfrak{p}}(\mathbf{A})xx^\dagger) \oplus (\mathcal{O}_{E^\sharp,\mathfrak{p}_*}, \beta_{\mathfrak{p}_*}(\mathbf{A})xx^\dagger)$$

and the same with $\mathbf{A}$ replaced by $\mathbf{T}$. Therefore

$$\begin{aligned} W_{\alpha,p}(s,\mathbf{A}) &= W_{\alpha,\mathfrak{p}}(s, \Phi^0_{\beta_{\mathfrak{p}}(\mathbf{A})}) W_{\alpha,\mathfrak{p}_*}(s, \Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{A})}) \\ W_{\alpha,p}(s,\mathbf{T}) &= W_{\alpha,\mathfrak{p}}(s, \Phi^0_{\beta_{\mathfrak{p}}(\mathbf{T})}) W_{\alpha,\mathfrak{p}_*}(s, \Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{T})}). \end{aligned}$$

Moreover, the same propositions give

$$(\mathcal{O}_{E^\sharp,\mathfrak{p}_*}, \beta_{\mathfrak{p}_*}(\mathbf{A})xx^\dagger) \cong (\mathcal{O}_{E^\sharp,\mathfrak{p}_*}, \beta_{\mathfrak{p}_*}(\mathbf{T})xx^\dagger)$$

and so

$$W_{\alpha,\mathfrak{p}_*}(s, \Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{A})}) = W_{\alpha,\mathfrak{p}_*}(s, \Phi^0_{\beta_{\mathfrak{p}_*}(\mathbf{T})}).$$

The rest of the proof goes exactly as that of Proposition 6.4.4, and is left to the reader. □

6.6. **The cyclic quartic case.** Suppose that $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ is cyclic of order 4. Let $\gamma \in \mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ be a generator inducing the absolute Frobenius $x \mapsto x^p$ on the residue field of $\mathbb{E}$. As the CM type $\Sigma = \{\pi_3, \pi_4\}$ is an unordered pair, we are free to interchange the indices of π_3 and π_4 (which requires also interchanging π_1 and π_2) in order to assume that γ acts on $\mathrm{Hom}(E, \mathbb{Q}_p^{\mathrm{alg}})$ and $\mathrm{Hom}(E^\sharp, \mathbb{Q}_p^{\mathrm{alg}})$ by

$$\begin{array}{ccc} \pi_1 & \longrightarrow & \pi_2 \\ \uparrow & & \downarrow \\ \pi_4 & \longleftarrow & \pi_3 \end{array} \qquad \begin{array}{ccc} \phi_{12} & \longrightarrow & \phi_{23} \\ \uparrow & & \downarrow \\ \phi_{14} & \longleftarrow & \phi_{34}. \end{array}$$

If we let $\mathbb{E}_0 \subset \mathbb{E}$ be the fixed field of γ^2 then $\mathbb{E} = \mathbb{E}_\Sigma$ and

$$\begin{aligned} F_p &\cong \mathbb{E}_0 & \qquad E_p &\cong \mathbb{E} \\ F^\sharp_p &\cong \mathbb{E}_0 & \qquad E^\sharp_p &\cong \mathbb{E}. \end{aligned}$$

Let $\Delta_{\mathbb{E}} \in \mathbb{E}$ be a generator of the different of $\mathbb{E}/\mathbb{Q}_p$. The fixed field of the inertia subgroup of $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ is either $\mathbb{E}$, $\mathbb{E}_0$, or $\mathbb{Q}_p$. In the first case $\mathbb{E}/\mathbb{Q}_p$ is unramified. In the second case $\mathbb{E}_0/\mathbb{Q}_p$ is unramified, $\mathbb{E}/\mathbb{E}_0$ is ramified, $p > 2$ (by Hypothesis 6.3.1), and $\Delta_{\mathbb{E}}$ is a uniformizing parameter of $\mathbb{E}$. In the third case $\mathbb{E}/\mathbb{Q}_p$ is totally ramified, $p > 3$, and (as $\mathbb{E}/\mathbb{Q}_p$ is tamely ramified) $\mathrm{ord}_{\mathcal{W}}(\Delta_{\mathbb{E}}) = 3$.

If $\mathbb{E}/\mathbb{Q}_p$ is not totally ramified then the fixed isomorphism (6.1.8) identifies

$$\mathcal{D} \cong \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \in \mathcal{W}^4 : \begin{array}{c} w_1 - w_3 \in \Delta_{\mathbb{E}}\mathcal{W} \\ w_2 - w_4 \in \Delta_{\mathbb{E}}\mathcal{W} \end{array} \right\}.$$

If $\mathbb{E}/\mathbb{Q}_p$ is totally ramified then the situation is more complicated. As $p > 2$, W contains a primitive 4^{th} root of unity. By Kummer theory there is a uniformizer $\Pi \in \mathcal{W}$ such that $\Pi^4 \in W$. As already noted $\Delta_{\mathbb{E}}\mathcal{W} = \Pi^3\mathcal{W}$. There is a unique extension of $\gamma \in \mathrm{Aut}(\mathcal{O}_{\mathbb{E}}/\mathbb{Z}_p)$ to $\gamma \in \mathrm{Aut}(\mathcal{W}/W)$, and we let $\zeta \in W^\times$ be the 4^{th} root of unity determined by $\Pi^\gamma = \zeta\Pi$. Then $\mathcal{D} \subset \mathcal{W}^4$ is generated as a W-module by the span of the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} \Pi \\ \zeta\Pi \\ -\Pi \\ -\zeta\Pi \end{bmatrix} \quad \begin{bmatrix} \Pi^2 \\ -\Pi^2 \\ \Pi^2 \\ -\Pi^2 \end{bmatrix} \quad \begin{bmatrix} \Pi^3 \\ -\zeta\Pi^3 \\ -\Pi^3 \\ \zeta\Pi^3 \end{bmatrix}.$$

From this it is not hard to show that

$$\mathcal{D} = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \in \mathcal{W}^4 : \begin{array}{c} \forall i, j \ w_i - w_j \in \Pi \cdot \mathcal{W} \\ w_1 - w_2 + w_3 - w_4 \in \Pi^2 \cdot \mathcal{W} \\ w_1 + \zeta w_2 - w_3 - \zeta w_4 \in \Pi^3 \cdot \mathcal{W} \end{array} \right\}.$$

Regardless of the ramification of $\mathbb{E}/\mathbb{Q}_p$, the relation (6.1.6) implies

$$\mathrm{ord}_{\mathcal{W}}(\rho_1\Delta_{\mathbb{E}}) = 0 \qquad \mathrm{ord}_{\mathcal{W}}(\rho_2\Delta_{\mathbb{E}}) = 0.$$

Suppose first that $\mathbb{E}/\mathbb{Q}_p$ is unramified. The group Γ is generated by the unique lift of γ to $\gamma \in \Gamma(1)$, and the operator $\mathcal{V}_\gamma$ on $\mathcal{D}$ has the form

$$\mathcal{V}_\gamma = \begin{pmatrix} & u_1 & & \\ & & u_2 & \\ & & & \varpi_1 \\ \varpi_2 & & & \end{pmatrix} \circ \gamma^{-1}$$

for some units $u_1, u_2 \in \mathcal{W}^\times$ and some uniformizers $\varpi_1, \varpi_2 \in \mathcal{W}$. The relation (6.1.5) is equivalent to

$$-p\rho_1 = (\varpi_2 u_2 \rho_2)^\gamma \qquad p\rho_2 = (\varpi_1 u_1 \rho_1)^\gamma.$$

The condition that (6.1.13) commutes with $\mathcal{V}_\gamma$ is equivalent to the conditions

$$a = \left(\frac{-pc}{u_1u_2}\right)^\gamma \qquad b = \left(\frac{pd}{\varpi_1\varpi_2}\right)^\gamma \qquad c = \left(\frac{pb}{u_1\varpi_2}\right)^\gamma \qquad d = \left(\frac{-pa}{\varpi_1 u_2}\right)^\gamma.$$

Now consider the triple $(\mathcal{T}, \kappa_{\mathcal{T}}, \lambda_{\mathcal{T}})$. The operator $\mathcal{V}_\gamma$ on $\mathcal{T}$ is equal to

$$\mathcal{V}_\gamma = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{pmatrix} \circ \gamma^{-1}$$

and the relation (6.1.15) is equivalent to $\rho_1 = -\rho_2^\gamma$ and $\rho_2 = -\rho_1^\gamma$. The condition that (6.1.13) commutes with $\mathcal{V}_\gamma$ is equivalent to $a, b, c, d \in \mathbb{E}$ satisfying

$$a = -c^\gamma \qquad b = d^\gamma \qquad c = b^\gamma \qquad d = -a^\gamma.$$

Next suppose that $\mathbb{E}_0/\mathbb{Q}_p$ is unramified, while $\mathbb{E}/\mathbb{E}_0$ is ramified. The group Γ is generated by two commuting elements: the unique lift of γ to $\gamma \in \Gamma(1)$, and the unique lift of γ^2 to $\tau \in \Gamma(0)$. The commuting operators $\mathcal{V}_\tau$ and $\mathcal{V}_\gamma$ on $\mathcal{D}$ have the form

$$\mathcal{V}_\tau = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \tau^{-1} \qquad \mathcal{V}_\gamma = \begin{pmatrix} & \varpi_1 & & \\ & & \varpi_2^\tau & \\ & & & \varpi_1^\tau \\ \varpi_2 & & & \end{pmatrix} \circ \gamma^{-1}$$

for some uniformizers $\varpi_1, \varpi_2 \in \mathcal{W}$. The relation (6.1.5) is equivalent to $\rho_1^\tau = -\rho_1$ and $\rho_2^\tau = -\rho_2$, together with

$$-p\rho_1 = (\rho_2 \varpi_2 \varpi_2^\tau)^\gamma \qquad p\rho_2 = (\rho_1 \varpi_1 \varpi_1^\tau)^\gamma.$$

The condition that (6.1.13) commutes with $\mathcal{V}_\gamma$ and $\mathcal{V}_\tau$ is equivalent to $a = -b^\tau$, $d = c^\tau$, and

$$a = \left(\frac{-pc}{\varpi_1 \varpi_2^\tau}\right)^\gamma \quad b = \left(\frac{pd}{\varpi_1^\tau \varpi_2}\right)^\gamma \quad c = \left(\frac{pb}{\varpi_1 \varpi_2}\right)^\gamma \quad d = \left(\frac{-pa}{\varpi_1^\tau \varpi_2^\tau}\right)^\gamma.$$

Now consider the triple $(\mathcal{T}, \kappa_\mathcal{T}, \lambda_\mathcal{T})$. The operators $\mathcal{V}_\tau$ and $\mathcal{V}_\gamma$ on $\mathcal{T}$ are equal to

$$\mathcal{V}_\tau = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \tau^{-1} \qquad \mathcal{V}_\gamma = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{pmatrix} \circ \gamma^{-1}.$$

The relation (6.1.15) is equivalent to

$$\rho_1^\tau = -\rho_1 \qquad \rho_2^\tau = -\rho_2 \qquad \rho_1^\gamma = \rho_2 \qquad \rho_2^\gamma = -\rho_1.$$

The condition that (6.1.13) commutes with $\mathcal{V}_\gamma$ and with $\mathcal{V}_\tau$ is equivalent to $a, b, c, d \in \mathbb{E}$ satisfying $a = -b^\tau$, $d = c^\tau$, and

$$a = -c^\gamma \qquad b = d^\gamma \qquad c = b^\gamma \qquad d = -a^\gamma.$$

Finally suppose that $\mathbb{E}/\mathbb{Q}_p$ is totally ramified. Then Γ is generated by two commuting elements: the unique lift of γ to $\gamma \in \Gamma(0)$, and the unique lift of the identity in $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ to $\epsilon \in \Gamma(1)$. The operators $\mathcal{V}_\gamma$ and $\mathcal{V}_\epsilon$ have the form

$$\mathcal{V}_\gamma = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{pmatrix} \circ \gamma^{-1} \qquad \mathcal{V}_\epsilon = \begin{pmatrix} z & & & \\ & z^\gamma & & \\ & & z^{\gamma^2} & \\ & & & z^{\gamma^3} \end{pmatrix} \circ \epsilon^{-1}$$

for some $z \in \mathcal{W}$ satisfying $\mathrm{ord}_{\mathcal{W}}(z) = 2$. The condition (6.1.5) is equivalent to $\rho_1^\gamma = \rho_2$ and $\rho_2^\gamma = -\rho_1$, together with

$$p\rho_1 = (\rho_1 z z^{\gamma^2})^\epsilon \qquad p\rho_2 = (\rho_2 z^\gamma z^{\gamma^3})^\epsilon.$$

The condition that (6.1.13) commutes with $\mathcal{V}_\epsilon$ and $\mathcal{V}_\gamma$ is equivalent to

$$a^\gamma = -d \qquad b^\gamma = c \qquad c^\gamma = -a \qquad d^\gamma = b$$

together with the extra condition

$$a = \Big(\frac{ap}{z^\gamma z^{\gamma^2}}\Big)^\epsilon.$$

Now consider the triple $(\mathcal{T}, \kappa_{\mathcal{T}}, \lambda_{\mathcal{T}})$. The operators $\mathcal{V}_\gamma$ and $\mathcal{V}_\epsilon$ on $\mathcal{T}$ are equal to

$$\mathcal{V}_\gamma = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{pmatrix} \circ \gamma^{-1} \quad \mathcal{V}_\epsilon = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \circ \epsilon^{-1}.$$

The condition (6.1.15) is equivalent to $\rho_1, \rho_2 \in \mathbb{E}$ together with $\rho_1^\gamma = \rho_2$ and $\rho_2^\gamma = -\rho_1$. The condition that (6.1.13) commutes with $\mathcal{V}_\epsilon$ and $\mathcal{V}_\gamma$ is equivalent to $a, b, c, d \in \mathbb{E}$ satisfying

$$a^\gamma = -d \qquad b^\gamma = c \qquad c^\gamma = -a \qquad d^\gamma = b.$$

Proposition 6.6.1. *For any $j \in L_p(\mathbf{A})$, j lifts to $L_p(\mathbf{A}_m^{\mathrm{can}})$ but not to $L_p(\mathbf{A}_{m+1}^{\mathrm{can}})$ where*

$$m = \frac{\mathrm{ord}_{\mathfrak{p}^\sharp}(Q_{\mathbf{A}}^\sharp(j)) + \mathrm{ord}_{\mathfrak{p}^\sharp}(\mathfrak{D}_{F^\sharp}) + 1}{2} \cdot \begin{cases} 1 & \text{if } \mathfrak{p}^\sharp \text{ is inert in } E^\sharp \\ 2 & \text{if } \mathfrak{p}^\sharp \text{ is ramified in } E^\sharp. \end{cases}$$

Proof. If $\mathbb{E}/\mathbb{Q}_p$ is not totally ramified then the proof is nearly identical to that of Proposition 6.4.1, so we assume that $\mathbb{E}/\mathbb{Q}_p$ is totally ramified. The submodule $\mathcal{D}^1 \subset \mathcal{D}$ is

$$\mathcal{D}^1 = \{w_1 e_1 + w_2 e_2 \in \mathcal{W}^4 : w_1, w_2 \in \Pi^2 \mathcal{W} \text{ and } w_1 + \zeta w_2 \in \Pi^3 \mathcal{W}\}$$

and the map (6.2.1) identifies

$$\mathcal{D}/\mathcal{D}^1 = \left\{ \begin{bmatrix} w_3 \\ w_4 \end{bmatrix} \in \mathcal{W}^2 : w_3 - w_4 \in \Pi \mathcal{W} \right\}.$$

After (6.2.2) the obstruction $\mathrm{obst}(j) : \mathcal{D}^1 \to \mathcal{D}/\mathcal{D}^1$ is given by

$$\mathrm{obst}(j)(w_1 e_1 + w_2 e_2) = \begin{bmatrix} -w_2 \rho_2 d \\ w_1 \rho_1 d \end{bmatrix}.$$

The submodule $\mathcal{D}^1$ is generated by $\Pi^3 e_1$ and $\zeta \Pi^2 e_1 - \Pi^2 e_2$ and so the image of $\mathrm{obst}(j)$ is generated by

$$s = \begin{bmatrix} 0 \\ \Pi^3 \rho_1 d \end{bmatrix} \qquad t = \begin{bmatrix} \Pi^2 \rho_2 d \\ \zeta \Pi^2 \rho_1 d \end{bmatrix}.$$

Of course $\mathrm{obst}_m(j)$ vanishes if and only if both s and t are divisible by Π^m in $\mathcal{D}/\mathcal{D}^1$, which is equivalent to

$$\mathrm{ord}_{\mathcal{W}}(\Pi^3\rho_1 d) \geq m+1$$
$$\mathrm{ord}_{\mathcal{W}}(\Pi^2\rho_2 d - \zeta\Pi^2\rho_1 d) \geq m+1.$$

Rewrite these inequalities as

$$\mathrm{ord}_{\mathcal{W}}(d) \geq m+1$$
$$\mathrm{ord}_{\mathcal{W}}(d) + \mathrm{ord}_{\mathcal{W}}(1-\zeta\rho_1\rho_2^{-1}) \geq m+2.$$

There are $x_1, x_2 \in \mathcal{W}^\times$ such that $\rho_1\Pi^3 = x_1$ and $\rho_2\Pi^3 = x_2$. Using $\rho_1^\gamma = \rho_2$ and $x_1 - x_1^\gamma \in \Pi\mathcal{W}$, we see that $x_1 \equiv x_2\zeta^3 \pmod{\Pi\mathcal{W}}$, and so

$$\rho_1\rho_2^{-1} \equiv \zeta^3 \pmod{\Pi\mathcal{W}}.$$

We deduce that $\mathrm{ord}_{\mathcal{W}}(1-\zeta\rho_1\rho_2^{-1}) \geq 1$, and so the second inequality is a consequence of the first. Thus $\mathrm{obst}_m(j) = 0$ if and only if $\mathrm{ord}_{\mathcal{W}}(d) \geq m+1$. But $\mathrm{ord}_{\mathcal{W}}(c/d) = 0$, and so Proposition 6.1.2 implies

$$\mathrm{ord}_{\mathcal{W}}(d) - 3 = \mathrm{ord}_{\mathfrak{p}^\sharp}(Q^\sharp_{\mathbf{A}}(j)).$$

By Proposition 6.2.3, j lifts to $L_p(\mathbf{A}_m^{\mathrm{can}})$ if and only if $m \leq \mathrm{ord}_{\mathfrak{p}^\sharp}(Q^\sharp_{\mathbf{A}}(j))+2$, as desired. □

Proposition 6.6.2. *For some $\beta_p(\mathbf{A}) \in F_p^\sharp$ satisfying*

$$\mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{A})) = \begin{cases} 1 & \text{if } \mathfrak{p}^\sharp \text{ is inert in } E^\sharp \\ -\mathrm{ord}_{\mathfrak{p}^\sharp}(\mathfrak{D}_{F^\sharp}) & \text{if } \mathfrak{p}^\sharp \text{ is ramified in } E^\sharp \end{cases}$$

there is an $E_p^\sharp$-linear isomorphism of $F_p^\sharp$-quadratic spaces

$$\big(V_p(\mathbf{A}), Q^\sharp_{\mathbf{A}}\big) \to \big(E_p^\sharp, \beta_p(\mathbf{A})xx^\dagger\big)$$

identifying $L_p(\mathbf{A}) \cong \mathcal{O}_{E^\sharp,p}$.

Proof. Assume first that $\mathbb{E}/\mathbb{Q}_p$ is not totally ramified. For any $j \in V_p(\mathbf{A})$, an argument similar to that used in the proof of Proposition 6.4.2 shows that

$$j \in L_p(\mathbf{A}) \iff a,b,c,d \in \Delta_{\mathbb{E}}\mathcal{W},$$

and the rest of the proof follows as in the proof Proposition 6.4.2.

Now assume that $\mathbb{E}/\mathbb{Q}_p$ is totally ramified. Decompose (6.1.13) as

$$(6.6.1)\quad j = \begin{pmatrix} \rho_2 b &&& \\ & -\rho_1 c && \\ && \rho_2 a & \\ &&& \rho_1 d \end{pmatrix}\begin{pmatrix} & 1 && \\ && 1 & \\ &&& 1 \\ 1 &&& \end{pmatrix} + \begin{pmatrix} \rho_2 c &&& \\ & \rho_1 a && \\ && -\rho_2 d & \\ &&& \rho_1 b \end{pmatrix}\begin{pmatrix} &&& 1 \\ 1 &&& \\ & 1 && \\ && 1 & \end{pmatrix}.$$

Recall that we have fixed an $\mathcal{O}_E$-linear isomorphism of W-modules $D \cong \mathcal{O}_E \otimes_{\mathbb{Z}} W$. If we now identify $\mathcal{O}_E \otimes_{\mathbb{Z}} W \cong \mathcal{W}$ using $x \otimes w \mapsto \pi_1(x)w$ then j may be viewed as a W-linear endomorphism of $\mathcal{W}_{\mathbb{Q}}$. Using the relations computed earlier between ρ_1 and ρ_2, and among a, b, c, d, we find that (6.6.1), restricted to $D_{\mathbb{Q}} \subset \mathcal{D}_{\mathbb{Q}}$ and viewed as an endomorphism of $\mathcal{W}_{\mathbb{Q}}$, is

$$j(x) = \rho_2 b \cdot x^{\gamma} + \rho_2 c \cdot x^{\gamma^3}. \tag{6.6.2}$$

It follows that for any $j \in V_p(\mathbf{A})$, $j \in L_p(\mathbf{A})$ if and only if the W-linear endomorphism (6.6.2) of $\mathcal{W}_{\mathbb{Q}}$ stabilizes $\mathcal{W}$.

The next claim is that every $j \in V_p(\mathbf{A})$ satisfies

$$j \in L_p(\mathbf{A}) \iff a, b, c, d \in \Pi^2 \mathcal{W}. \tag{6.6.3}$$

First suppose $a, b, c, d \in \Pi^2\mathcal{W}$. The theory of higher ramification groups implies that $y \equiv y^{\gamma} \pmod{\Pi\mathcal{W}}$ for every $y \in \mathcal{W}$. In particular if $x \in \mathcal{W}$ then $x^{\gamma^3} = x^{\gamma} + \Pi r$ for some $r \in \mathcal{W}$. Similarly, if we write $b = s\Pi^2$ and $c = t\Pi^2$ with $s, t \in \mathcal{W}$, then $b^{\gamma} = c$ implies $s + t \in \Pi\mathcal{W}$. Therefore

$$j(x) = \rho_2(s+t)\Pi^2 \cdot x^{\gamma} + \rho_2 t \Pi^3 r \in \mathcal{W}.$$

This proves that $j \in L_p(\mathbf{A})$. Conversely, assume that $j \in L_p(\mathbf{A})$. Then for every $y \in \mathcal{W}$

$$by + cy^{\gamma^2} \in \Pi^3\mathcal{W}.$$

Taking $y = \Pi$ shows that $b - c \in \Pi^2\mathcal{W}$, while taking $y = 1$ shows that $b + c \in \Pi^3\mathcal{W}$. As $p \neq 2$ we deduce first that $b, c \in \Pi^2\mathcal{W}$, and then that $a, b, c, d \in \Pi^2\mathcal{W}$ by the relations

$$\mathrm{ord}_{\mathcal{W}}(a) = \mathrm{ord}_{\mathcal{W}}(b) = \mathrm{ord}_{\mathcal{W}}(c) = \mathrm{ord}_{\mathcal{W}}(d)$$

computed earlier. This completes the proof of (6.6.3).

Now start with any $E_p^{\sharp}$-module generator $j \in V_p(\mathbf{A})$. As a, b, c, and d each generate the same ideal of $\mathcal{W}$, (6.1.14) shows that one may multiply j by an element of $E_p^{\sharp}$ in order to assume that a, b, c, and d each generate $\Pi^2\mathcal{W}$. As in the proof of Proposition 6.4.2, $x \mapsto x \bullet j$ defines an isomorphism $E_p^{\sharp} \to V_p(\mathbf{A})$, and (6.6.3) and (6.1.14) show that $x \bullet j \in L_p(\mathbf{A}) \iff x \in \mathcal{O}_{E^{\sharp},p}$. If we define $\beta_p(\mathbf{A}) = Q^{\sharp}_{\mathbf{A}}(j)$ then Proposition 6.1.2 allows us to compute the valuation of $\beta_p(\mathbf{A})$ at $\mathfrak{p}^{\sharp}$. □

Proposition 6.6.3. *For some $\beta_p(\mathbf{T}) \in F_p^{\sharp}$ satisfying*

$$\mathrm{ord}_{\mathfrak{p}^{\sharp}}(\beta_p(\mathbf{T})) = \begin{cases} 0 & \text{if } \mathfrak{p}^{\sharp} \text{ is inert in } E^{\sharp} \\ -1 & \text{if } \mathfrak{p}^{\sharp} \text{ is ramified in } E^{\sharp} \end{cases}$$

there is an $E_p^{\sharp}$-linear isomorphism of $F_p^{\sharp}$-quadratic spaces

$$\big(V_p(\mathbf{T}), Q^{\sharp}_{\mathbf{T}}\big) \cong \big(E_p^{\sharp}, \beta_p(\mathbf{T}) x x^{\dagger}\big)$$

identifying $L_p(\mathbf{T}) \cong \mathbb{Z}_p + \mathfrak{D}_{E^{\sharp}/F^{\sharp}} \mathcal{O}_{E^{\sharp},p}$.

Proof. First suppose that $\mathbb{E}/\mathbb{Q}_p$ is not totally ramified. As in the proof of Proposition 6.4.3, every $j \in V_p(\mathbf{T})$ satisfies $j \cdot \mathcal{T} \subset \mathcal{T}$ if and only if $a, b, c, d \in \mathcal{O}_{\mathbb{E}}$ and satisfy the congruence $a \equiv c \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}$. If $\mathbb{E}/\mathbb{Q}_p$ is unramified then the congruence is automatically satisfied. If $\mathbb{E}/\mathbb{Q}_p$ is ramified (but $\mathbb{E}_0/\mathbb{Q}_p$ is unramified) then we may pick a $c \in \mathcal{O}_{\mathbb{E}}^\times$ satisfying $c^\gamma \equiv -c \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}$ and set $a = -c^\gamma$. Then $a \equiv c \pmod{\Delta_{\mathbb{E}}\mathcal{W}}$, and if we define $d = -a^\gamma$ and $b = d^\gamma$ then (6.1.13) defines an element $j \in L_p(\mathbf{T})$ with $a, b, c, d \in \mathcal{O}_{\mathbb{E}}^\times$. The proof concludes as in the proof of Proposition 6.4.3: $x \mapsto x \bullet j$ defines an isomorphism $E_p^\sharp \to V_p(\mathbf{T})$ satisfying $x \bullet j \in L_p(\mathbf{T})$ if and only if $x \in \mathcal{O}_{E^\sharp,p}$ and $\phi_{23}(x) \equiv \phi_{12}(x) \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}$. Taking $\beta_p(\mathbf{T}) = Q_{\mathbf{T}}^\sharp(j)$ completes the proof in this case.

If $\mathbb{E}/\mathbb{Q}_p$ is totally ramified then the proof of Proposition 6.6.2 shows that any $j \in V_p(\mathbf{T})$ satisfies

$$j \in L_p(\mathbf{T}) \iff a, b, c, d \in \Pi^2\mathcal{W},$$

and the proof concludes exactly as with Proposition 6.6.2. Note that in this case $\mathbb{Z}_p + \mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p} = \mathcal{O}_{E^\sharp,p}$. □

Suppose $\alpha \in (F^\sharp)^\times$ and $p \in \mathrm{Sppt}(\alpha)$, and recall the quantity $\nu_{\mathfrak{p}^\sharp}(\alpha)$ of Proposition 5.2.3. Assume $\mathfrak{p}^\sharp \in \mathrm{Diff}(\alpha, \mathbf{T})$, so that $W_{\alpha,p}(0, \mathbf{T}) = 0$ by (4.3.3).

Proposition 6.6.4.

(1) *If $L_p(\mathbf{A})$ represents α then $W_{\alpha,p}(0, \mathbf{A}) \neq 0$ and*

$$\frac{W'_{\alpha,p}(0, \mathbf{T})}{W_{\alpha,p}(0, \mathbf{A})} = -\frac{\nu_{\mathfrak{p}^\sharp}(\alpha)}{2} \cdot \log(\mathrm{Nm}(\mathfrak{q})).$$

(2) *If $L_p(\mathbf{A})$ does not represent α then $W_{\alpha,p}(0, \mathbf{A})$ and $W'_{\alpha,p}(0, \mathbf{T})$ are both* 0.

Proof. Abbreviate $\mathfrak{p} = \mathfrak{p}^\sharp$. There are three cases: (i) p is inert in $E^\sharp$, (ii) p is totally ramified in $E^\sharp$, and (iii) p is inert in $F^\sharp$ and $\mathfrak{p}$ is ramified in $E^\sharp$.

In the first two cases, $\mathbb{Z}_p + \mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p} = \mathcal{O}_{E^\sharp,p}$, and so

$$W_{\alpha,p}(s, \mathbf{T}) = W_{\alpha,\mathfrak{p}}\big(s, \Phi^0_{\beta_p(\mathbf{T})}\big)$$
$$W_{\alpha,p}(s, \mathbf{A}) = W_{\alpha,\mathfrak{p}}\big(s, \Phi^0_{\beta_p(\mathbf{A})}\big).$$

Here Φ^μ_β is in the notation of Section 4.6 with $\mathcal{E} = E^\sharp_{\mathfrak{p}}$ and $\mathcal{F} = F^\sharp_{\mathfrak{p}}$. We first treat case (ii). Let a be a generator of $\mathfrak{D}_{F^\sharp,\mathfrak{p}}$ and set $\tilde{\psi}(x) = \psi_{\mathfrak{p}}(a^{-1}x)$, so that $\tilde{\psi}$ is unramified. As $p \neq 2$, a is a uniformizing parameter of $F^\sharp_{\mathfrak{p}}$, and Propositions 6.6.2 and 6.6.3 imply that

$$\tilde{\beta}_p(\mathbf{T}) = a\beta_p(\mathbf{T}) \qquad \tilde{\beta}_p(\mathbf{A}) = a\beta_p(\mathbf{A})$$

both lie in $\mathcal{O}^\times_{F^\sharp,\mathfrak{p}}$. Lemma 4.2.2 implies

$$W_{\alpha,\mathfrak{p}}\big(s, \Phi^0_{\beta_p(\mathbf{T})}\big) = |a|^{\frac{1}{2}} W^{\tilde{\psi}}_{a\alpha,\mathfrak{p}}\big(s, \Phi^0_{\tilde{\beta}_p(\mathbf{T})}\big)$$
$$W_{\alpha,\mathfrak{p}}\big(s, \Phi^0_{\beta_p(\mathbf{A})}\big) = |a|^{\frac{1}{2}} W^{\tilde{\psi}}_{a\alpha,\mathfrak{p}}\big(s, \Phi^0_{\tilde{\beta}_p(\mathbf{A})}\big).$$

The hypothesis $\mathfrak{p} \in \mathrm{Diff}(\alpha, \mathbf{T})$ tells us that $V_p(\mathbf{T})$ does not represent α, and so Proposition 6.2.4 implies that $V_p(\mathbf{A})$ does represent α. If $L_p(\mathbf{A})$

does not represent α then Proposition 6.6.2 implies that $a\alpha \notin \mathcal{O}_{F^\sharp,\mathfrak{p}}$, and so Proposition 4.6.2 implies

$$W_{\alpha,p}(s,\mathbf{T}) = W_{\alpha,p}(s,\mathbf{A}) = 0.$$

Assume now that $L_p(\mathbf{A})$ does represent α, so that $a\alpha \in \mathcal{O}_{F^\sharp,\mathfrak{p}}$. If $\chi_\mathfrak{p}$ is the quadratic character associated to the extension $E^\sharp_\mathfrak{p}/F^\sharp_\mathfrak{p}$ then Proposition 4.6.2 implies

$$\begin{aligned} W^{*,\tilde{\psi}}_{a\alpha,\mathfrak{p}}\big(0,\Phi^0_{\tilde{\beta}_p(\mathbf{T})}\big) &= \gamma(V_{\tilde{\beta}_p(\mathbf{T})})(1+\chi_\mathfrak{p}(\alpha\beta_p(\mathbf{T})) \\ W^{*,\tilde{\psi}}_{a\alpha,\mathfrak{p}}\big(0,\Phi^0_{\tilde{\beta}_p(\mathbf{A})}\big) &= \gamma(V_{\tilde{\beta}_p(\mathbf{A})})(1+\chi_\mathfrak{p}(\alpha\beta_p(\mathbf{A})). \end{aligned}$$

As $\chi_\mathfrak{p}(\alpha\beta_p(\mathbf{A}))=1$, we deduce

$$W^{*,\tilde{\psi}}_{a\alpha,\mathfrak{p}}\big(0,\Phi^0_{\tilde{\beta}_p(\mathbf{A})}\big) \neq 0.$$

Moreover, Proposition 4.6.2 and Lemma 4.2.1 imply

$$\begin{aligned} \frac{W'_{\alpha,p}(0,\mathbf{T})}{W_{\alpha,p}(0,\mathbf{A})} &= \frac{W^{*,\prime,\tilde{\psi}}_{a\alpha,\mathfrak{p}}\big(0,\Phi^0_{\tilde{\beta}_p(\mathbf{T})}\big)}{W^{*,\tilde{\psi}}_{a\alpha,\mathfrak{p}}\big(0,\Phi^0_{\tilde{\beta}_p(\mathbf{A})}\big)} \\ &= -\frac{\mathrm{ord}_\mathfrak{p}(a\alpha)+1}{2}\cdot\log\mathrm{Nm}(\mathfrak{p}) \\ &= -\frac{\nu_\mathfrak{p}(\alpha)}{2}\cdot\log(\mathrm{Nm}(\mathfrak{q})). \end{aligned}$$

This proves all claims in case (ii). Case (i) is similar but easier, and is left to the reader.

Now assume we are in case (iii). Let ϖ be a uniformizing parameter of $E^\sharp_\mathfrak{p}$, so that $\tilde{\beta}_p(\mathbf{T}) = \varpi\varpi^\dagger\beta_p(\mathbf{T})$ is an element of $\mathcal{O}^\times_{F^\sharp,\mathfrak{p}}$. The map $x\mapsto x\varpi^{-1}$ is an isomorphism from $(E^\sharp_p,\beta_p(\mathbf{T})xx^\dagger)$ to $(E^\sharp_p,\tilde{\beta}_p(\mathbf{T})xx^\dagger)$, and identifies

$$\mathbb{Z}_p+\mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p} \cong \bigsqcup_{a\in\mathbb{Z}/p\mathbb{Z}}\left(\frac{a}{\varpi}+\mathcal{O}_{E^\sharp,\mathfrak{p}}\right).$$

Therefore

$$W^*_{\alpha,p}(s,\mathbf{T}) = \sum_{a\in\mathbb{Z}/p\mathbb{Z}} W^*_{\alpha,\mathfrak{p}}\big(s,\Phi^{\frac{a}{\varpi}}_{\tilde{\beta}_p(\mathbf{T})}\big).$$

The same argument used in the proof of Proposition 6.4.4 shows that only the $a=0$ term contributes, and so

$$W^*_{\alpha,p}(s,\mathbf{T}) = W^*_{\alpha,\mathfrak{p}}\big(s,\Phi^0_{\tilde{\beta}_p(\mathbf{T})}\big).$$

Proposition 6.6.2 shows that

$$W^*_{\alpha,p}(s,\mathbf{A}) = W^*_{\alpha,\mathfrak{p}}\big(s,\Phi^0_{\beta_p(\mathbf{A})}\big),$$

and the rest of the proof is similar to case (ii). □

6.7. **The Klein four case, part I.** Suppose that $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p) = \langle \gamma_1, \gamma_2 \rangle$ is isomorphic to the Klein four group, and that γ_1 acts on $\mathrm{Hom}(E, \mathbb{Q}_p^{\mathrm{alg}})$ and $\mathrm{Hom}(E^\sharp, \mathbb{Q}_p^{\mathrm{alg}})$ by

$$\begin{array}{cccc} \pi_1 & \pi_2 & \phi_{12} & \phi_{23} \\ \updownarrow & \updownarrow & & \!\!\!\!\!\!\!\!\nwarrow\!\!\searrow \\ \pi_4 & \pi_3 & \phi_{14} & \phi_{34} \end{array}$$

while γ_2 acts by

$$\begin{array}{cccc} \pi_1 \longleftrightarrow \pi_2 & & \phi_{12} & \phi_{23} \\ & & & \!\!\!\!\!\!\!\!\swarrow\!\!\nearrow \\ \pi_4 \longleftrightarrow \pi_3 & & \phi_{14} & \phi_{34}. \end{array}$$

If we set $\gamma_0 = \gamma_1 \circ \gamma_2$ and let $\mathbb{E}_i$ denote the fixed field of γ_i, then $\mathbb{E}_\Sigma = \mathbb{E}_2$ and

$$F_p \cong \mathbb{E}_0 \qquad\qquad E_p \cong \mathbb{E}$$
$$F_p^\sharp \cong \mathbb{Q}_p \times \mathbb{Q}_p \qquad\qquad E_p^\sharp \cong \mathbb{E}_1 \times \mathbb{E}_2.$$

If $\mathbb{E}_2/\mathbb{Q}_p$ is unramified then both $\mathbb{E}_0/\mathbb{Q}_p$ and $\mathbb{E}_1/\mathbb{Q}_p$ must be ramified. If $\mathbb{E}_2/\mathbb{Q}_p$ is ramified then Hypothesis 6.3.1 implies $p > 2$, and it follows from class field theory that exactly one of $\mathbb{E}_0/\mathbb{Q}_p$ and $\mathbb{E}_1/\mathbb{Q}_p$ is ramified over $\mathbb{Q}_p$. In any case, exactly one of $\mathbb{E}_0$, $\mathbb{E}_1$, $\mathbb{E}_2$ is unramified over $\mathbb{Q}_p$. Let $\Delta_\mathbb{E}$ be a generator of the different of $\mathbb{E}/\mathbb{Q}_p$, and let δ_i be a generator of the different of $\mathbb{E}/\mathbb{E}_i$. One of $\delta_0, \delta_1, \delta_2$ generates $\Delta_\mathbb{E}\mathcal{O}_\mathbb{E}$, and the other two are units. The fixed isomorphism (6.1.8) identifies

$$\mathcal{D} \cong \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \in \mathcal{W}^4 : \begin{array}{c} w_1 - w_2, w_3 - w_4 \in \delta_2\mathcal{W} \\ w_1 - w_4, w_2 - w_3 \in \delta_1\mathcal{W} \\ w_1 - w_3, w_2 - w_4 \in \delta_0\mathcal{W} \end{array} \right\}.$$

The condition (6.1.6) implies

$$\mathrm{ord}_\mathcal{W}(\rho_1\Delta_\mathbb{E}) = 0 \qquad \mathrm{ord}_\mathcal{W}(\rho_2\Delta_\mathbb{E}) = 0.$$

There are two primes of $F^\sharp$ above p. One is $\mathfrak{p}^\sharp$, and the other we denote by $\mathfrak{p}^\sharp_*$.

First suppose that $\mathbb{E}_0/\mathbb{Q}_p$ is unramified, so that $\delta_0\mathcal{O}_\mathbb{E} = \Delta_\mathbb{E}\mathcal{O}_\mathbb{E}$. There are unique lifts of $\gamma_0, \gamma_1, \gamma_2 \in \mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ to $\gamma_0 \in \Gamma(0)$ and $\gamma_1, \gamma_2 \in \Gamma(1)$. These lifts commute, and any two of the three generate Γ. The commuting operators $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ on $\mathcal{D}$ have the form

$$\mathcal{V}_{\gamma_0} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \gamma_0^{-1} \qquad \mathcal{V}_{\gamma_1} = \begin{pmatrix} & & & \varpi_2^{\gamma_0} \\ & & \varpi_1^{\gamma_0} & \\ & \varpi_2 & & \\ \varpi_1 & & & \end{pmatrix} \circ \gamma_1^{-1}$$

for some uniformizers $\varpi_1, \varpi_2 \in \mathcal{W}$. The condition (6.1.5) implies that ρ_1 and ρ_2 satisfy $\rho_1^{\gamma_0} = -\rho_1$ and $\rho_2^{\gamma_0} = -\rho_2$, together with

$$-p\rho_1 = (\varpi_1 \varpi_1^{\gamma_0} \rho_2)^{\gamma_1} \qquad -p\rho_2 = (\varpi_2 \varpi_2^{\gamma_0} \rho_1)^{\gamma_1}.$$

The condition that (6.1.13) commutes with $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ is equivalent to the conditions $a^{\gamma_0} = -b$ and $c^{\gamma_0} = d$, together with

$$a = -\left(\frac{pa}{\varpi_1^{\gamma_0} \varpi_2} \right)^{\gamma_1} \qquad c = -\left(\frac{pc}{\varpi_1^{\gamma_0} \varpi_2^{\gamma_0}} \right)^{\gamma_2}.$$

Now consider the triple $(\mathcal{T}, \kappa_{\mathcal{T}}, \lambda_{\mathcal{T}})$. The operators commuting $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ on $\mathcal{T}$ are equal to

$$\mathcal{V}_{\gamma_0} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \gamma_0^{-1} \qquad \mathcal{V}_{\gamma_1} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \circ \gamma_1^{-1}.$$

The condition (6.1.15) implies that ρ_1 and ρ_2 satisfy

$$\rho_1^{\gamma_0} = -\rho_1 \qquad \rho_2^{\gamma_0} = -\rho_2 \qquad \rho_2^{\gamma_1} = -\rho_1 \qquad \rho_1^{\gamma_1} = -\rho_2.$$

The condition that (6.1.13) commute with $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ is equivalent to $a, b, c, d \in \mathbb{E}$ satisfying

$$a^{\gamma_0} = -b \qquad c^{\gamma_0} = d \qquad a^{\gamma_1} = -a \qquad c^{\gamma_2} = -c.$$

Next suppose that $\mathbb{E}_1/\mathbb{Q}_p$ is unramified, so that $\delta_1 \mathcal{O}_{\mathbb{E}} = \Delta_{\mathbb{E}} \mathcal{O}_{\mathbb{E}}$. There are unique lifts of $\gamma_0, \gamma_1, \gamma_2 \in \mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ to $\gamma_1 \in \Gamma(0)$ and $\gamma_0, \gamma_2 \in \Gamma(1)$. The group Γ is generated by any two of these three commuting elements. The commuting operators $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ on $\mathcal{D}$ are equal to

$$\mathcal{V}_{\gamma_0} = \begin{pmatrix} & & \varpi_2^{\gamma_1} & \\ & & & \varpi_1^{\gamma_1} \\ \varpi_1 & & & \\ & \varpi_2 & & \end{pmatrix} \circ \gamma_0^{-1} \qquad \mathcal{V}_{\gamma_1} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \circ \gamma_1^{-1}$$

for some uniformizers $\varpi_1, \varpi_2 \in \mathcal{W}$. The condition (6.1.5) is equivalent to ρ_1 and ρ_2 satisfying $\rho_2^{\gamma_1} = -\rho_1$ and $\rho_1^{\gamma_1} = -\rho_2$, together with

$$-p\rho_1 = (\varpi_1 \varpi_2^{\gamma_1} \rho_1)^{\gamma_0} \qquad -p\rho_2 = (\varpi_1^{\gamma_1} \varpi_2 \rho_2)^{\gamma_0}.$$

The condition that (6.1.13) commutes with $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ is equivalent to the conditions

$$a^{\gamma_1} = -a \qquad b^{\gamma_1} = -b \qquad c^{\gamma_1} = -d$$

together with

$$a = \left(\frac{-pb}{\varpi_2 \varpi_2^{\gamma_1}} \right)^{\gamma_0} \qquad b = \left(\frac{-pa}{\varpi_1 \varpi_1^{\gamma_1}} \right)^{\gamma_0} \qquad c = \left(\frac{pd}{\varpi_1 \varpi_2} \right)^{\gamma_0}.$$

Now consider the triple $(\mathcal{T}, \kappa_{\mathcal{T}}, \lambda_{\mathcal{T}})$. The commuting operators $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ on $\mathcal{T}$ are equal to

$$\mathcal{V}_{\gamma_0} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \gamma_0^{-1} \qquad \mathcal{V}_{\gamma_1} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \circ \gamma_1^{-1}.$$

The condition (6.1.15) implies that ρ_1 and ρ_2 satisfy

$$\rho_2^{\gamma_1} = -\rho_1 \qquad -\rho_1 = \rho_1^{\gamma_0} \qquad \rho_1^{\gamma_1} = -\rho_2 \qquad -\rho_2 = \rho_2^{\gamma_0}.$$

The condition that (6.1.13) commutes with $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ is equivalent to $a, b, c, d \in \mathbb{E}$ satisfying

$$a^{\gamma_1} = -a \qquad b^{\gamma_1} = -b \qquad c^{\gamma_1} = -d$$

together with $b^{\gamma_0} = -a$ and $d^{\gamma_0} = c$.

Finally suppose that $\mathbb{E}_2/\mathbb{Q}_p$ is unramified, so that $\delta_2\mathcal{O}_{\mathbb{E}} = \Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}$. The elements $\gamma_0, \gamma_1, \gamma_2 \in \mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ admit unique lifts to $\gamma_2 \in \Gamma(0)$ and $\gamma_0, \gamma_1 \in \Gamma(1)$, and Γ is generated by any two of these three commuting elements. The commuting operators $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_2}$ on $\mathcal{D}$ have the form

$$\mathcal{V}_{\gamma_0} = \begin{pmatrix} & & u & \\ & & & u^{\gamma_2} \\ z & & & \\ & z^{\gamma_2} & & \end{pmatrix} \circ \gamma_0^{-1} \qquad \mathcal{V}_{\gamma_2} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \circ \gamma_2^{-1}$$

for some $z \in \mathcal{W}$ satisfying $\mathrm{ord}_{\mathcal{W}}(z) = 2$ and some unit $u \in \mathcal{W}^\times$. The condition (6.1.5) implies that ρ_1 and ρ_2 satisfy $\rho_2^{\gamma_2} = \rho_1$ and $\rho_1^{\gamma_2} = \rho_2$, together with

$$-p\rho_1 = (zu\rho_1)^{\gamma_0} \qquad -p\rho_2 = (z^{\gamma_2}u^{\gamma_2}\rho_2)^{\gamma_0}.$$

The condition that (6.1.13) commutes with $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_2}$ is equivalent to the conditions

$$a^{\gamma_2} = b \qquad c^{\gamma_2} = -c \qquad d^{\gamma_2} = -d$$

together with

$$a = \left(\frac{-pb}{z^{\gamma_2}u}\right)^{\gamma_0} \qquad c = \left(\frac{pd}{zz^{\gamma_2}}\right)^{\gamma_0} \qquad d = \left(\frac{pc}{u^{\gamma_2}u}\right)^{\gamma_0}.$$

Now consider the triple $(\mathcal{T}, \kappa_{\mathcal{T}}, \lambda_{\mathcal{T}})$. The operators $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_2}$ on $\mathcal{T}$ are equal to

$$\mathcal{V}_{\gamma_0} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \gamma_0^{-1} \qquad \mathcal{V}_{\gamma_2} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \circ \gamma_2^{-1}.$$

The condition (6.1.15) implies that ρ_1 and ρ_2 satisfy

$$\rho_2^{\gamma_2} = \rho_1 \qquad -\rho_1 = \rho_1^{\gamma_0} \qquad \rho_1^{\gamma_2} = \rho_2 \qquad -\rho_2 = \rho_2^{\gamma_0}.$$

The condition that (6.1.13) commutes with $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_2}$ is equivalent to $a, b, c, d \in \mathbb{E}$ satisfying

$$a^{\gamma_2} = b \qquad c^{\gamma_2} = -c \qquad d^{\gamma_2} = -d$$

together with $a^{\gamma_0} = -b$ and $d^{\gamma_0} = c$.

Proposition 6.7.1. *For any $j \in L_p(\mathbf{A})$, j lifts to $L_p(\mathbf{A}_m^{\mathrm{can}})$ but not to $L_p(\mathbf{A}_{m+1}^{\mathrm{can}})$ where*

$$m = \frac{\mathrm{ord}_{\mathfrak{p}^\sharp}\big(Q^\sharp_{\mathbf{A}}(j)\big) + 1}{2} \cdot \begin{cases} 1 & \text{if } \mathfrak{p}^\sharp \text{ is inert in } E^\sharp \\ 2 & \text{if } \mathfrak{p}^\sharp \text{ is ramified in } E^\sharp \end{cases}$$

Proof. If either $\mathbb{E}_0/\mathbb{Q}_p$ or $\mathbb{E}_1/\mathbb{Q}_p$ is unramified then $\mathcal{W}_\Sigma = \mathcal{W}$, $\mathfrak{p}^\sharp$ is ramified in $E^\sharp$, $p > 2$, and $\Delta_{\mathbb{E}}$ is a uniformizing parameter of $\mathcal{O}_{\mathbb{E}}$. The submodule $\mathcal{D}^1$ is free on the generators $\Delta_{\mathbb{E}} e_1$ and $\Delta_{\mathbb{E}} e_2$. The map $\mathcal{D} \to \mathcal{W}^2$ defined by (6.2.1) identifies $\mathcal{D}/\mathcal{D}^1 \cong \mathcal{W}^2$, and exactly as in the proof of Proposition 6.4.1, $\mathrm{obst}_m(j)$ vanishes if and only if $m \le \mathrm{ord}_{\mathcal{W}}(d)$. As $c/d \in \mathcal{W}^\times = 0$ and $e(\mathbb{E}/F^\sharp_{\mathfrak{p}^\sharp}) = 2$, Proposition 6.1.2 implies

$$\mathrm{ord}_{\mathcal{W}}(d) = \mathrm{ord}_{\mathfrak{p}^\sharp}\big(Q^\sharp_{\mathbf{A}}(j)\big) + 1.$$

As $e(\mathbb{E}/\mathbb{E}_\Sigma) = 1$, the claim now follows from Proposition 6.2.3.

Now suppose that $\mathbb{E}_2/\mathbb{Q}_p$ is unramified, so that $\mathcal{W}_\Sigma = W$ and $\mathfrak{p}^\sharp$ is unramified in $E^\sharp$. The submodule $\mathcal{D}^1 \subset \mathcal{D}$ is generated by $\Delta_{\mathbb{E}} e_1$, $\Delta_{\mathbb{E}} e_2$, and $e_1 + e_2$. The map $\mathcal{D} \to \mathcal{W}^2$ defined by (6.2.1) identifies

$$\mathcal{D}/\mathcal{D}^1 \cong \left\{ \begin{bmatrix} w_3 \\ w_4 \end{bmatrix} \in \mathcal{W}^2 : w_3 - w_4 \in \Delta_{\mathbb{E}} \mathcal{W} \right\}.$$

After (6.2.2) the obstruction $\mathrm{obst}(j) : \mathcal{D}^1 \to \mathcal{D}/\mathcal{D}^1$ is given by

$$\Delta_{\mathbb{E}} e_1 \mapsto \begin{bmatrix} 0 \\ \Delta_{\mathbb{E}} \rho_1 d \end{bmatrix} \qquad \Delta_{\mathbb{E}} e_2 \mapsto \begin{bmatrix} -\Delta_{\mathbb{E}} \rho_2 d \\ 0 \end{bmatrix} \qquad e_1 + e_2 \mapsto \begin{bmatrix} -\rho_2 d \\ \rho_1 d \end{bmatrix}.$$

The relation $\rho_1^{\gamma_2} = \rho_2$ implies $\rho_1 + \rho_2 \in \mathcal{W}$ (choose $\Delta_{\mathbb{E}}$ so that $\Delta_{\mathbb{E}}^{\gamma_2} = -\Delta_{\mathbb{E}}$, write $\rho_1 = x \Delta_{\mathbb{E}}^{-1}$ with $x \in \mathcal{W}^\times$ and use $x^{\gamma_2} \equiv x \pmod{\Delta_{\mathbb{E}} \mathcal{W}}$), and it follows that $\mathrm{obst}_m(j)$ vanishes if and only if

$$m \le \mathrm{ord}_{\mathcal{W}}(d) - \mathrm{ord}_{\mathcal{W}}(\Delta_{\mathbb{E}}).$$

As $\mathrm{ord}_{\mathcal{W}}(c/d) = -2$ and $e(\mathbb{E}/F^\sharp_{\mathfrak{p}^\sharp}) = 2$, Proposition 6.1.2 implies

$$\mathrm{ord}_{\mathcal{W}}(d) = \mathrm{ord}_{\mathfrak{p}^\sharp}\big(Q^\sharp_{\mathbf{A}}(j)\big) + \mathrm{ord}_{\mathcal{W}}(\Delta_{\mathbb{E}}) + 1,$$

and so $\mathrm{obst}_m(j)$ vanishes if and only if $m \le \mathrm{ord}_{\mathfrak{p}^\sharp}(Q^\sharp_{\mathbf{A}}(j)) + 1$. As $e(\mathbb{E}/\mathbb{E}_\Sigma) = 2$, the claim follows from Proposition 6.2.3. ☐

Proposition 6.7.2. *For some $\beta_p(\mathbf{A}) \in F_p^\sharp$ satisfying* $\mathrm{ord}_{\mathfrak{p}^\sharp_*}(\beta_p(\mathbf{A})) = 0$ *and*

$$\mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{A})) = \begin{cases} 1 & \text{if } \mathfrak{p}^\sharp \text{ is inert in } E^\sharp \\ 0 & \text{if } \mathfrak{p}^\sharp \text{ is ramified in } E^\sharp \end{cases}$$

there is an $E^\sharp_p$-linear isomorphism of $F^\sharp_p$-quadratic spaces

$$\big(V_p(\mathbf{A}), Q^\sharp_{\mathbf{A}}\big) \to \big(E^\sharp_p, \beta_p(\mathbf{A})xx^\dagger\big)$$

identifying $L_p(\mathbf{A}) \cong \mathcal{O}_{E^\sharp,p}$.

Proof. Arguing as in the proofs of Proposition 6.4.2 and Proposition 6.5.2, in all cases $j \in V_p(\mathbf{A})$ satisfies

$$j \in L_p(\mathbf{A}) \iff a, b, c, d \in \Delta_{\mathbb{E}}\mathcal{W}.$$

The rest of the proof proceeds as in the proof of Proposition 6.4.2. Fix any $E^\sharp_p$-module generator $j \in V_p(\mathbf{A})$. If $\mathbb{E}_0$ or $\mathbb{E}_1$ is unramified over $\mathbb{Q}_p$ (so that $\mathfrak{p}^\sharp$ is ramified in $E^\sharp$) then by the calculations above

$$\mathrm{ord}_{\mathcal{W}}(a) = \mathrm{ord}_{\mathcal{W}}(b) \qquad \mathrm{ord}_{\mathcal{W}}(c) = \mathrm{ord}_{\mathcal{W}}(d),$$

and we may multiply j by an element of $E^\sharp_p$ in order to assume that a, b, c, d each generate $\Delta_{\mathbb{E}}\mathcal{W}$. If $\mathbb{E}_2$ is unramified over $\mathbb{Q}_p$ (so that $\mathfrak{p}^\sharp$ is unramified in $E^\sharp$) then

$$\mathrm{ord}_{\mathcal{W}}(a) = \mathrm{ord}_{\mathcal{W}}(b) \qquad \mathrm{ord}_{\mathcal{W}}(c) + 2 = \mathrm{ord}_{\mathcal{W}}(d),$$

and after multiplying j by an element of $E^\sharp_p$ we may assume that a, b, c each generate $\Delta_{\mathbb{E}}\mathcal{W}$. In any case one sets $\beta_p(\mathbf{A}) = Q^\sharp_{\mathbf{A}}(j)$ and uses Proposition 6.1.2 to compute the valuation of $\beta_p(\mathbf{A})$ at $\mathfrak{p}^\sharp$ and $\mathfrak{p}^\sharp_*$. The rule $x \mapsto x \bullet j$ defines the desired isomorphism $E^\sharp_p \to V_p(\mathbf{A})$, as in the proof of Proposition 6.4.2. □

Proposition 6.7.3. *For some $\beta_p(\mathbf{T}) \in F^\sharp_p$ satisfying*

$$\mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{T})) = \mathrm{ord}_{\mathfrak{p}^\sharp_*}(\beta_p(\mathbf{T})) = \begin{cases} -1 & \text{if } \mathfrak{p}^\sharp \text{ and } \mathfrak{p}^\sharp_* \text{ are both ramified in } E^\sharp \\ 0 & \text{otherwise} \end{cases}$$

there is an $E^\sharp_p$-linear isomorphism of $F^\sharp_p$-quadratic spaces

$$\big(V_p(\mathbf{T}), Q^\sharp_{\mathbf{T}}\big) \to \big(E^\sharp_p, \beta_p(\mathbf{T})xx^\dagger\big)$$

identifying

$$L_p(\mathbf{T}) \cong \begin{cases} \mathbb{Z}_p + \mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p} & \text{if } \mathfrak{p}^\sharp \text{ and } \mathfrak{p}^\sharp_* \text{ are both ramified in } E^\sharp \\ \mathcal{O}_{E^\sharp,p} & \text{otherwise.} \end{cases}$$

Proof. The argument is similar to the proof of Proposition 6.4.3. Fix a $j \in V_p(\mathbf{T})$. If $\mathbb{E}_0/\mathbb{Q}_p$ is unramified (so that $\mathfrak{p}^\sharp$ and $\mathfrak{p}^\sharp_*$ are both ramified in $E^\sharp$) then $j \in L_p(\mathbf{T})$ if and only if $a, b, c, d \in \mathcal{O}_{\mathbb{E}}$ and $a \equiv c \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}$. If one first chooses $a, c \in \mathcal{O}_{\mathbb{E}}^\times$ satisfying $a^{\gamma_1} = -a$, $c^{\gamma_2} = -c$, and $a \equiv c \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}$, and then sets $b = -a^{\gamma_0}$ and $d = c^{\gamma_0}$, then (6.1.13) defines an element $j \in L_p(\mathbf{T})$. The map $x \mapsto x \bullet j$ defines an isomorphism $E^\sharp_p \to V_p(\mathbf{T})$, and examination of (6.1.14) shows that $x \bullet j \in L_p(\mathbf{T})$ if and only if $x \in \mathcal{O}_{E^\sharp,p}$ and

$$\phi_{23}(x)a \equiv \phi_{12}(x)c \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}.$$

The congruence is equivalent to $\phi_{23}(x) \equiv \phi_{12}(x) \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}$, from which we deduce

$$x \bullet j \in L_p(\mathbf{T}) \iff x \in \mathbb{Z}_p + \mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p}.$$

As $a, b, c, d \in \mathcal{O}_{\mathbb{E}}^\times$, Proposition 6.1.2 shows that $\beta_p(\mathbf{T}) = Q^\sharp_{\mathbf{A}}(j)$ has valuation -1 at each of $\mathfrak{p}^\sharp$ and $\mathfrak{p}^\sharp_*$, and $x \mapsto x \bullet j$ defines the desired isomorphism $E^\sharp_p \to V_p(\mathbf{T})$.

If $\mathbb{E}_1/\mathbb{Q}_p$ or $\mathbb{E}_2/\mathbb{Q}_p$ is unramified then $j \in L_p(\mathbf{T})$ if and only if $a, b, c, d \in \Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}$. One may choose a $j \in V_p(\mathbf{T})$ such that a, b, c, d each generate the ideal $\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}$, and again $x \mapsto x \bullet j$ defines the desired isomorphism $E^\sharp_p \to V_p(\mathbf{T})$. □

Suppose $\alpha \in (F^\sharp)^\times$ and $p \in \mathrm{Sppt}(\alpha)$, and recall the quantity $\nu_{\mathfrak{p}^\sharp}(\alpha)$ of Proposition 5.2.3. Assume $\mathfrak{p}^\sharp \in \mathrm{Diff}(\alpha, \mathbf{T})$, so that $W_{\alpha,p}(0, \mathbf{T}) = 0$ by (4.3.3).

Proposition 6.7.4.

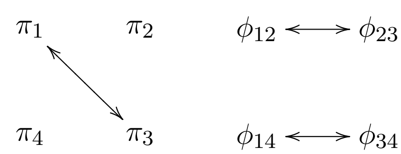
(1) *If $L_p(\mathbf{A})$ represents α then $W_{\alpha,p}(0, \mathbf{A}) \neq 0$ and*

$$\frac{W'_{\alpha,p}(0, \mathbf{T})}{W_{\alpha,p}(0, \mathbf{A})} = -\frac{\nu_{\mathfrak{p}^\sharp}(\alpha)}{2} \cdot \log(\mathrm{Nm}(\mathfrak{q})).$$

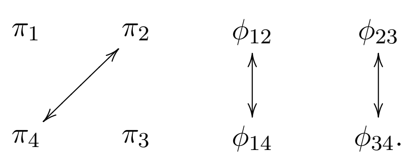
(2) *If $L_p(\mathbf{A})$ does not represent α then $W_{\alpha,p}(0, \mathbf{A})$ and $W'_{\alpha,p}(0, \mathbf{T})$ are both 0.*

Proof. (sketch) When both $\mathfrak{p}^\sharp$ and $\mathfrak{p}^\sharp_*$ are ramified in $E^\sharp$, the proof is the same as the ramified case in the proof of Proposition 6.4.4. Otherwise, the proof is the same as in the unramified case in the proof of Proposition 6.4.4. We leave the details to the reader. □

6.8. **The Klein four case, part II.** Suppose that $\mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p) = \langle \gamma_1, \gamma_2 \rangle$ is isomorphic to the Klein four group, and that γ_1 acts on $\mathrm{Hom}(E, \mathbb{Q}_p^{\mathrm{alg}})$ and $\mathrm{Hom}(E^\sharp, \mathbb{Q}_p^{\mathrm{alg}})$ by



while γ_2 acts by



Set $\gamma_0 = \gamma_1 \circ \gamma_2$, and let $\mathbb{E}_i$ denote the fixed field of $\mathbb{E}_i$. Then $\mathbb{E}_\Sigma = \mathbb{E}$ and

$$F_p \cong \mathbb{Q}_p \times \mathbb{Q}_p \qquad E_p \cong \mathbb{E}_1 \times \mathbb{E}_2$$
$$F^\sharp_p \cong \mathbb{E}_0 \qquad E^\sharp_p \cong \mathbb{E}.$$

The extension $\mathbb{E}/\mathbb{Q}_p$ is ramified, and so Hypothesis 6.3.1 implies $p > 2$. It then follows from class field theory that exactly one of $\mathbb{E}_0$, $\mathbb{E}_1$, and $\mathbb{E}_2$ is unramified over $\mathbb{Q}_p$. Let $\Delta_{\mathbb{E}}$ be a generator of the different of $\mathbb{E}/\mathbb{Q}_p$. As $\mathbb{E}/\mathbb{Q}_p$ is tamely ramified with ramification degree 2, $\Delta_{\mathbb{E}}$ is a uniformizer of $\mathbb{E}$. Let δ_i be a generator of the different of $\mathbb{E}/\mathbb{E}_i$. One of $\delta_0, \delta_1, \delta_2$ generates the ideal $\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}$, and the other two are units. The fixed isomorphism (6.1.8) identifies

$$\mathcal{D} \cong \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \in \mathcal{W}^4 : \begin{array}{l} w_2 - w_4 \in \delta_2 \mathcal{W} \\ w_1 - w_3 \in \delta_1 \mathcal{W} \\ w_1 - w_3, w_2 - w_4 \in \delta_0 \mathcal{W}. \end{array} \right\}.$$

The condition (6.1.6) implies

$$\mathrm{ord}_{\mathcal{W}}(\rho_1 \Delta_{\mathbb{E}}) = \mathrm{ord}_{\mathcal{W}}(\delta_2) \qquad \mathrm{ord}_{\mathcal{W}}(\rho_2 \Delta_{\mathbb{E}}) = \mathrm{ord}_{\mathcal{W}}(\delta_1).$$

First suppose that $\mathbb{E}_0/\mathbb{Q}_p$ is unramified, so that δ_0 is a uniformizing parameter of $\mathbb{E}$. The elements $\gamma_0, \gamma_1, \gamma_2 \in \mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ admit unique lifts $\gamma_0 \in \Gamma(0)$ and $\gamma_1, \gamma_2 \in \Gamma(1)$. The group Γ is generated by any two of these three commuting elements. The commuting operators $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ on $\mathcal{D}$ have the form

$$\mathcal{V}_{\gamma_0} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \gamma_0^{-1} \qquad \mathcal{V}_{\gamma_1} = \begin{pmatrix} & & \varpi_1^{\gamma_0} & \\ & \varpi_2^{\gamma_0} & & \\ \varpi_1 & & & \\ & & & \varpi_2 \end{pmatrix} \circ \gamma_1^{-1}$$

for some uniformizers $\varpi_1, \varpi_2 \in \mathcal{W}$. The condition (6.1.5) implies that ρ_1 and ρ_2 satisfy $\rho_1^{\gamma_0} = -\rho_1$, $\rho_2^{\gamma_0} = -\rho_2$, and

$$-p\rho_1 = (\varpi_1 \varpi_1^{\gamma_0} \rho_1)^{\gamma_1} \qquad p\rho_2 = (\varpi_2 \varpi_2^{\gamma_0} \rho_2)^{\gamma_1}.$$

The condition that (6.1.13) commutes with $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ is equivalent to $a^{\gamma_0} = -b$ and $c^{\gamma_0} = d$ together with

$$a = \left(\frac{pc}{\varpi_1^{\gamma_0} \varpi_2^{\gamma_0}} \right)^{\gamma_1} \qquad c = \left(\frac{pa}{\varpi_1 \varpi_2^{\gamma_0}} \right)^{\gamma_1}.$$

Now consider the triple $(\mathcal{T}, \kappa_{\mathcal{T}}, \lambda_{\mathcal{T}})$. The operators $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ on $\mathcal{T}$ are equal to

$$\mathcal{V}_{\gamma_0} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \gamma_0^{-1} \qquad \mathcal{V}_{\gamma_1} = \begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix} \circ \gamma_1^{-1}.$$

The condition (6.1.15) implies that ρ_1 and ρ_2 satisfy

$$\rho_1^{\gamma_0} = -\rho_1 \qquad \rho_1^{\gamma_1} = -\rho_1 \qquad \rho_2^{\gamma_0} = -\rho_2 \qquad \rho_2^{\gamma_1} = \rho_2.$$

The condition that (6.1.13) commutes with $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ is equivalent to $a, b, c, d \in \mathbb{E}$ satisfying $a^{\gamma_0} = -b$ and $c^{\gamma_0} = d$, together with $a^{\gamma_1} = c$, Next suppose that $\mathbb{E}_1/\mathbb{Q}_p$ is unramified, so that δ_1 is a uniformizing parameter of $\mathbb{E}$. The elements $\gamma_0, \gamma_1, \gamma_2 \in \mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ admit unique lifts $\gamma_1 \in \Gamma(0)$ and $\gamma_0, \gamma_2 \in \Gamma(1)$. The group Γ is generated by any two of these three commuting elements. The operators $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ on $\mathcal{D}$ have the form

$$\mathcal{V}_{\gamma_0} = \begin{pmatrix} & & \varpi^{\gamma_1} & \\ & & & u \\ \varpi & & & \\ & z & & \end{pmatrix} \circ \gamma_0^{-1} \qquad \mathcal{V}_{\gamma_1} = \begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix} \circ \gamma_1^{-1}$$

where $\varpi \in \mathcal{W}$ is a uniformizer, $u \in W^\times$, and $z \in W$ satisfies $\mathrm{ord}_{\mathcal{W}}(z) = 2$. The condition (6.1.5) implies that ρ_1 and ρ_2 satisfy $\rho_1^{\gamma_1} = -\rho_1$ and $\rho_2^{\gamma_1} = \rho_2$, together with

$$-p\rho_1 = (\varpi\varpi^{\gamma_1}\rho_1)^{\gamma_0} \qquad -p\rho_2 = (uz\rho_2)^{\gamma_0}.$$

The condition that (6.1.13) commutes with $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ is equivalent to $a^{\gamma_1} = c$ and $b^{\gamma_1} = -d$, together with

$$a = \left(\frac{-pb}{z\varpi^{\gamma_1}}\right)^{\gamma_0} \qquad b = \left(\frac{-pa}{u\varpi}\right)^{\gamma_0}.$$

Now consider the triple $(\mathcal{T}, \kappa_{\mathcal{T}}, \lambda_{\mathcal{T}})$. The operators $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ on $\mathcal{T}$ are equal to

$$\mathcal{V}_{\gamma_0} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \gamma_0^{-1} \qquad \mathcal{V}_{\gamma_1} = \begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix} \circ \gamma_1^{-1}.$$

The condition (6.1.15) implies that ρ_1 and ρ_2 satisfy $\rho_1^{\gamma_1} = -\rho_1$ and $\rho_2^{\gamma_1} = \rho_2$, together with

$$-p\rho_1 = \rho_1^{\gamma_0} \qquad -p\rho_2 = \rho_2^{\gamma_0}.$$

The condition that (6.1.13) commutes with $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ is equivalent to $a, b, c, d \in \mathbb{E}$ satisfying $a^{\gamma_1} = c$ and $b^{\gamma_1} = -d$, together with $a = -b^{\gamma_0}$.

Finally suppose $\mathbb{E}_2/\mathbb{Q}_p$ is unramified, so that δ_2 is a uniformizing parameter of $\mathbb{E}$. The elements $\gamma_0, \gamma_1, \gamma_2 \in \mathrm{Gal}(\mathbb{E}/\mathbb{Q}_p)$ admit unique lifts $\gamma_2 \in \Gamma(0)$ and $\gamma_0, \gamma_1 \in \Gamma(1)$. The group Γ is generated by any two of these three commuting elements. The commuting operators $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_2}$ on $\mathcal{D}$ have the form

$$\mathcal{V}_{\gamma_0} = \begin{pmatrix} & & u & \\ & & & \varpi^{\gamma_2} \\ z & & & \\ & \varpi & & \end{pmatrix} \circ \gamma_0^{-1} \qquad \mathcal{V}_{\gamma_2} = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{pmatrix} \circ \gamma_2^{-1}$$

where $\varpi \in \mathcal{W}$ is a uniformizer, $u \in W^\times$, and $z \in W$ satisfies $\mathrm{ord}_{\mathcal{W}}(z) = 2$. The condition (6.1.5) implies that ρ_1 and ρ_2 satisfy $\rho_1^{\gamma_2} = \rho_1$ and $\rho_2^{\gamma_2} = -\rho_2$, together with

$$-p\rho_1 = (uz\rho_1)^{\gamma_0} \qquad -p\rho_2 = (\varpi\varpi^{\gamma_2}\rho_2)^{\gamma_0}.$$

The condition that (6.1.13) commutes with $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ is equivalent to $a^{\gamma_2} = d$ and $b^{\gamma_2} = -c$ together with

$$a = \left(\frac{-pb}{u\varpi}\right)^{\gamma_0} \qquad b = \left(\frac{-pa}{z\varpi^{\gamma_2}}\right)^{\gamma_0}.$$

Now consider the triple $(\mathcal{T}, \kappa_{\mathcal{T}}, \lambda_{\mathcal{T}})$. The operators $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_2}$ on $\mathcal{T}$ are equal to

$$\mathcal{V}_{\gamma_0} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix} \circ \gamma_0^{-1} \qquad \mathcal{V}_{\gamma_2} = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{pmatrix} \circ \gamma_2^{-1}.$$

The condition (6.1.15) implies that ρ_1 and ρ_2 satisfy

$$\rho_1^{\gamma_2} = \rho_1 \qquad \rho_2^{\gamma_2} = -\rho_2 \qquad \rho_1^{\gamma_0} = -\rho_1 \qquad \rho_2^{\gamma_0} = -\rho_2.$$

The condition that (6.1.13) commutes with $\mathcal{V}_{\gamma_0}$ and $\mathcal{V}_{\gamma_1}$ is equivalent to $a^{\gamma_2} = d$ and $b^{\gamma_2} = -c$, together with $a = -b^{\gamma_0}$.

Proposition 6.8.1. *For any $j \in L_p(\mathbf{A})$, j lifts to $L_p(\mathbf{A}_m^{\mathrm{can}})$ but not to $L_p(\mathbf{A}_{m+1}^{\mathrm{can}})$ where*

$$m = \frac{\mathrm{ord}_{\mathfrak{p}^\sharp}\big(Q_{\mathbf{A}}^\sharp(j)\big) + \mathrm{ord}_{\mathfrak{p}^\sharp}(\mathfrak{D}_{F^\sharp}) + 1}{2} \cdot \begin{cases} 1 & \text{if } \mathfrak{p}^\sharp \text{ is inert in } E^\sharp \\ 2 & \text{if } \mathfrak{p}^\sharp \text{ is ramified in } E^\sharp \end{cases}$$

Proof. First suppose $\mathbb{E}_0/\mathbb{Q}_p$ is unramified, so that $\mathfrak{p}^\sharp$ is ramified in $E^\sharp$, $\delta_1, \delta_2 \in \mathcal{O}_{\mathbb{E}}^\times$ and $\delta_0\mathcal{O}_{\mathbb{E}} = \Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}$. The submodule $\mathcal{D}^1$ is free on the generators $\Delta_{\mathbb{E}}e_1$ and $\Delta_{\mathbb{E}}e_2$, and the map $\mathcal{D} \to \mathcal{W}^2$ defined by (6.2.1) identifies $\mathcal{D}/\mathcal{D}^1 \cong \mathcal{W}^2$. After (6.2.2) the obstruction $\mathrm{obst}(j) : \mathcal{D}^1 \to \mathcal{D}/\mathcal{D}^1$ is given by

$$\Delta_{\mathbb{E}}e_1 \mapsto \begin{bmatrix} 0 \\ \Delta_{\mathbb{E}}\rho_1 d \end{bmatrix} \qquad \Delta_{\mathbb{E}}e_2 \mapsto \begin{bmatrix} -\Delta_{\mathbb{E}}\rho_2 d \\ 0 \end{bmatrix},$$

and so $\mathrm{obst}_m(j)$ vanishes if and only if $m \le \mathrm{ord}_{\mathcal{W}}(d)$. As $c/d \in \mathcal{W}^\times$ and $e(\mathbb{E}/F_{\mathfrak{p}^\sharp}^\sharp) = 2$, Proposition 6.1.2 implies

$$\mathrm{ord}_{\mathcal{W}}(d) = \mathrm{ord}_{\mathfrak{p}^\sharp}\big(Q_{\mathbf{A}}^\sharp(j)\big) + 1.$$

The claim now follows from Proposition 6.2.3.

Now suppose that $\mathbb{E}_1/\mathbb{Q}_p$ is unramified, so that $\mathfrak{p}^\sharp$ is ramified in $E^\sharp$, $\delta_0, \delta_2 \in \mathcal{O}_{\mathbb{E}}^\times$ and $\delta_1\mathcal{O}_{\mathbb{E}} = \Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}$. The submodule $\mathcal{D}^1$ is generated by $\Delta_{\mathbb{E}}e_1$ and e_2. The map $\mathcal{D} \to \mathcal{W}^2$ defined by (6.2.1) identifies $\mathcal{D}/\mathcal{D}^1$ with $\mathcal{W}^2$, and the obstruction $\mathrm{obst}(j) : \mathcal{D}^1 \to \mathcal{D}/\mathcal{D}^1$ is given by

$$\Delta_{\mathbb{E}}e_1 \mapsto \begin{bmatrix} 0 \\ \Delta_{\mathbb{E}}\rho_1 d \end{bmatrix} \qquad e_2 \mapsto \begin{bmatrix} -\rho_2 d \\ 0 \end{bmatrix},$$

and so $\mathrm{obst}_m(j)$ vanishes if and only if $m \le \mathrm{ord}_{\mathcal{W}}(d)$. Earlier calculations show that $\mathrm{ord}_{\mathcal{W}}(c/d) = -1$, and so $e(\mathbb{E}/F_{\mathfrak{p}^\sharp}^\sharp) = 1$ together with Proposition

6.1.2 imply

$$2 \cdot \mathrm{ord}_{\mathcal{W}}(d) - \mathrm{ord}_{\mathcal{W}}(\Delta_{\mathbb{E}}) - 1 = \mathrm{ord}_{\mathfrak{p}^\sharp}\big(Q^\sharp_{\mathbf{A}}(j)\big).$$

Again the claim follows from Proposition 6.2.3. The case of $\mathbb{E}_2/\mathbb{Q}_p$ unramified is entirely similar. □

Proposition 6.8.2. *For some $\beta_p(\mathbf{A}) \in F^\sharp_p$ satisfying $\mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{A})) = 0$ there is an $E^\sharp_p$-linear isomorphism of $F^\sharp_p$-quadratic spaces*

$$\big(V_p(\mathbf{A}), Q^\sharp_{\mathbf{A}}\big) \to \big(E^\sharp_p, \beta_p(\mathbf{A})xx^\dagger\big)$$

identifying $L_p(\mathbf{A}) \cong \mathcal{O}_{E^\sharp,p}$.

Proof. First suppose that $\mathbb{E}_0/\mathbb{Q}_p$ is unramified, so that $\mathfrak{p}^\sharp$ is ramified in $E^\sharp$. Arguing as in the proof of Proposition 6.4.2, each $j \in V_p(\mathbf{A})$ satisfies $j \in L_p(\mathbf{A})$ if and only if $a, b, c, d \in \Delta_{\mathbb{E}}\mathcal{W}$. If we pick any $E^\sharp_p$-module generator $j \in V_p(\mathbf{A})$, then using (6.1.14) and

$$\mathrm{ord}_{\mathcal{W}}(a) = \mathrm{ord}_{\mathcal{W}}(b) = \mathrm{ord}_{\mathcal{W}}(c) = \mathrm{ord}_{\mathcal{W}}(d)$$

we may multiply j by an element of $E^\sharp_p$ in order to assume that a, b, c, d each generate $\Delta_{\mathbb{E}}\mathcal{W}$. Setting $\beta_p(\mathbf{A}) = Q^\sharp_{\mathbf{A}}(j)$, Proposition 6.1.2 implies

$$2 \cdot \mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{A})) = \mathrm{ord}_{\mathcal{W}}(\rho_1\rho_2\Delta^2_{\mathbb{E}}) = 0$$

As in the proof of Proposition 6.4.2, $x \mapsto x \bullet j$ defines the desired isomorphism $E^\sharp_p \to V_p(\mathbf{A})$.

If $\mathbb{E}_1/\mathbb{Q}_p$ is unramified then $\mathfrak{p}^\sharp$ is unramified in $E^\sharp$ and each $j \in V_p(\mathbf{A})$ satisfies $j \in L_p(\mathbf{A})$ if and only if $a, b, c, d \in \mathcal{W}$. Using

$$1 + \mathrm{ord}_{\mathcal{W}}(a) = \mathrm{ord}_{\mathcal{W}}(b) = 1 + \mathrm{ord}_{\mathcal{W}}(c) = \mathrm{ord}_{\mathcal{W}}(d)$$

We may construct a $j \in L_p(\mathbf{A})$ such that $a, c \in \mathcal{W}^\times$ and b, d generate the maximal ideal of $\mathcal{W}$. Setting $\beta_p(\mathbf{A}) = Q^\sharp_{\mathbf{A}}(j)$, Proposition 6.1.2 implies

$$\mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{A})) = \mathrm{ord}_{\mathcal{W}}(\rho_1\rho_2) + 1 = 0,$$

and again $x \mapsto x \bullet j$ defines the desired isomorphism $E^\sharp_p \to V_p(\mathbf{A})$. The case of $\mathbb{E}_2/\mathbb{Q}_p$ unramified is similar. □

Proposition 6.8.3. *For some $\beta_p(\mathbf{T}) \in F^\sharp_p$ satisfying $\mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{T})) = -1$ there is an $E^\sharp_p$-linear isomorphism of $F^\sharp_p$-quadratic spaces*

$$\big(V_p(\mathbf{T}), Q^\sharp_{\mathbf{T}}\big) \to \big(E^\sharp_p, \beta_p(\mathbf{T})xx^\dagger\big)$$

identifying

$$L_p(\mathbf{T}) \cong \mathbb{Z}_p + \mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p}.$$

Proof. First suppose that $\mathbb{E}_0/\mathbb{Q}_p$ is unramified, so that $\mathfrak{p}^\sharp$ is ramified in $E^\sharp$. Arguing as in the proofs of Proposition 6.4.2 and Proposition 6.4.3, each $j \in V_p(\mathbf{T})$ satisfies $j \in L_p(\mathbf{T})$ if and only if $a, b, c, d \in \mathcal{O}_{\mathbb{E}}$ and $a \equiv c \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}$. We may construct a $j \in L_p(\mathbf{T})$ in such a way that $a, b, c, d \in$

$\mathcal{O}_{\mathbb{E}}^{\times}$, and then (6.1.14) shows that $x \bullet j \in L_p(\mathbf{T})$ if and only if $x \in \mathcal{O}_{E^\sharp,p}$ and $\phi_{12}(x)c \equiv \phi_{23}(x)a \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}$. This congruence is equivalent to $\phi_{12}(x) \equiv \phi_{23}(x) \pmod{\Delta_{\mathbb{E}}\mathcal{O}_{\mathbb{E}}}$, which is equivalent to $x \in \mathbb{Z}_p + \mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p}$. Setting $\beta_p(\mathbf{T}) = Q^\sharp_{\mathbf{T}}(j)$ and applying Proposition 6.1.2 gives

$$2 \cdot \mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{T})) = \mathrm{ord}_{\mathcal{W}}(\rho_1\rho_2) = -2.$$

Thus $x \mapsto x \bullet j$ defines the desired isomorphism $E_p^\sharp \to V_p(\mathbf{T})$.

If either $\mathbb{E}_1/\mathbb{Q}_p$ or $\mathbb{E}_2/\mathbb{Q}_p$ is unramified then $\mathfrak{p}^\sharp$ is unramified in $E^\sharp$, and each $j \in V_p(\mathbf{T})$ satisfies $j \in L_p(\mathbf{T})$ if and only if $a, b, c, d \in \mathcal{O}_{\mathbb{E}}$. We may choose $j \in L_p(\mathbf{T})$ in such a way that $a, b, c, d \in \mathcal{O}_{\mathbb{E}}^{\times}$. Setting $\beta_p(\mathbf{T}) = Q^\sharp_{\mathbf{T}}(j)$, Proposition 6.1.2 implies

$$\mathrm{ord}_{\mathfrak{p}^\sharp}(\beta_p(\mathbf{T})) = \mathrm{ord}_{\mathcal{W}}(\rho_1\rho_2) = -1,$$

and again $x \mapsto x \bullet j$ defines the desired isomorphism $E_p^\sharp \to V_p(\mathbf{T})$. Note that in this case $\mathbb{Z}_p + \mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p} = \mathcal{O}_{E^\sharp,p}$. □

Suppose $\alpha \in (F^\sharp)^\times$ and $p \in \mathrm{Sppt}(\alpha)$, and recall the quantity $\nu_{\mathfrak{p}^\sharp}(\alpha)$ of Proposition 5.2.3. Assume $\mathfrak{p}^\sharp \in \mathrm{Diff}(\alpha, \mathbf{T})$, so that $W_{\alpha,p}(0, \mathbf{T}) = 0$ by (4.3.3).

Proposition 6.8.4.

(1) *If $L_p(\mathbf{A})$ represents α then $W_{\alpha,p}(0, \mathbf{A}) \neq 0$ and*

$$\frac{W'_{\alpha,p}(0, \mathbf{T})}{W_{\alpha,p}(0, \mathbf{A})} = -\frac{\nu_{\mathfrak{p}^\sharp}(\alpha)}{2} \cdot \log(\mathrm{Nm}(\mathfrak{q})).$$

(2) *If $L_p(\mathbf{A})$ does not represent α then $W_{\alpha,p}(0, \mathbf{A})$ and $W'_{\alpha,p}(0, \mathbf{T})$ are both* 0.

Proof. (sketch) If $\mathfrak{p}^\sharp$ is inert in $E^\sharp$ then $F_p^\sharp/\mathbb{Q}_p$ is ramified, and

$$\mathbb{Z}_p + \mathfrak{D}_{E^\sharp/F^\sharp}\mathcal{O}_{E^\sharp,p} = \mathcal{O}_{E^\sharp,p}.$$

In this case, the proof is the same as that of Proposition 6.6.4 case (ii). If $\mathfrak{p}^\sharp$ is ramified in $E^\sharp$, then $F_p^\sharp/\mathbb{Q}_p$ is unramified over $\mathbb{Q}_p$. In this case, the proof is the same as that of Proposition 6.6.4 case (iii). □

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